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# Numerical evidence for universality in the excited instability spectrum of magnetically charged Reissner–Nordström black holes

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**Abstract** It is well known that the SU(2) Reissner–Nordström black-hole solutions of the Einstein–Yang–Mills theory are characterized by an infinite set of unstable (imaginary) eigenvalues $\{\omega_n(T_{\text{BH}})\}_{n=0}^{n=\infty}$ (here $T_{\text{BH}}$ is the black-hole temperature). In this paper we analyze the excited instability spectrum of these magnetically charged black holes. The numerical results suggest the existence of a universal behavior for these black-hole excited eigenvalues. In particular, we show that unstable eigenvalues in the regime $\omega_n \ll T_{\text{BH}}$ are characterized, to a very good degree of accuracy, by the simple universal relation $\omega_n(r_+ - r_-) = \text{constant}$, where $r_\pm$ are the horizon radii of the black hole.

## 1 Introduction

The familiar U(1) Reissner–Nordström spacetime is well known to describe a *stable* black-hole solution of the coupled Einstein–Maxwell equations [1,2] and the coupled Einstein–Maxwell-scalar equations [3–5]. Yasskin [6] has proved that the Einstein–Yang–Mills theory also admits an explicit black-hole solution which is described by the magnetically charged SU(2) Reissner–Nordström spacetime. However, the SU(2) Reissner–Nordström black-hole solution of the coupled Einstein–Yang–Mills equations is known to be *unstable* [7–10]. In fact, it was proved in [11,12] that the magnetically charged Reissner–Nordström black-hole spacetime is characterized by an *infinite* family of unstable (growing in time) perturbation modes.

The recent numerical work of Rinne [13] has revealed that these unstable SU(2) Reissner–Nordström black-hole spacetimes play the role of approximate[1] codimension-two intermediate attractors (that is, nonlinear critical solutions [14]) in the dynamical gravitational collapse of the Yang–Mills field.[2] In particular, this interesting numerical study [13] has explicitly demonstrated that, during a near-critical evolution of the Yang–Mills field, the time spent in the vicinity of an unstable SU(2) Reissner–Nordström black-hole solution is characterized by the critical scaling law[3]

$$\tau = \text{const} - \gamma \ln|p - p^*|. \tag{1}$$

Interestingly, the critical exponents of the scaling law (1) are directly related to the characteristic instability eigenvalues of the corresponding SU(2) Reissner–Nordström black holes [13]:

$$\gamma = 1/\omega_{\text{instability}}. \tag{2}$$

It is therefore of physical interest to explore the instability spectrum $\{\omega_n\}_{n=0}^{n=\infty}$ of the SU(2) Reissner–Nordström black holes. Indeed, Rinne [13] has recently computed numerically the characteristic unstable eigenvalues of these magnetically charged black-hole solutions of the Einstein–Yang–Mills theory.[4]

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[1] As emphasized in [13], the magnetically charged Reissner–Nordström black-hole spacetime is only an approximate intermediate attractor because it is characterized by an infinite set of unstable (growing in time) modes.

[2] This fact refers to type I and type III critical behaviors, see [13] for details.

[3] Here $|p - p^*|$ is a measure for the distance of the initial data from the threshold (critical) solution [14].

[4] As emphasized above, the magnetically charged Reissner–Nordström black-hole solution of the Einstein–Yang–Mills theory is characterized by an infinite family of unstable perturbation modes [11,12]. Reference [13] provides, for the first time, detailed numerical results for the first three instability eigenvalues.

In the present paper we shall analyze these numerically computed black-hole eigenvalues in an attempt to identify a possible hidden pattern which characterizes the black-hole instability spectrum. As we shall show below, the numerical results indeed suggest the existence of a universal behavior for these black-hole unstable eigenvalues.

## 2 Description of the system

The Reissner–Nordström black-hole solution of the Einstein–Yang–Mills theory with unit magnetic charge is described by the line element [6]

$$\mathrm{d}s^2=-\Big(1-\frac{2m}{r}\Big)\mathrm{d}t^2+\Big(1-\frac{2m}{r}\Big)^{-1}\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2), \tag{3}$$

where the mass function $m=m(r)$ is given by[5]

$$m(r)=M-\frac{1}{2r}. \tag{4}$$

The black-hole temperature is given by

$$T_{\mathrm{BH}}=\frac{r_+-r_-}{4\pi r_+^2}, \tag{5}$$

where

$$r_\pm=M\pm\sqrt{M^2-1} \tag{6}$$

are the (outer and inner) horizons of the black hole.

Linearized perturbations $\xi(r)e^{-i\omega t}$[6] of the magnetically charged black-hole spacetime are governed by the Schrödinger-like wave equation [15]

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\omega^2+\frac{1}{r^2}\Big[1-\frac{2m(r)}{r}\Big]\Big\}\xi=0, \tag{7}$$

where the "tortoise" radial coordinate $x$ is defined by the relation[7]

$$\mathrm{d}x/\mathrm{d}r=[1-2m(r)/r]^{-1}. \tag{8}$$

Well-behaved (spatially bounded) perturbation modes are characterized by the boundary conditions

$$\xi(x\to-\infty)\sim \mathrm{e}^{|\omega|x}\to 0 \tag{9}$$

and

$$\xi(x\to\infty)\sim xe^{-|\omega|x}\to 0, \tag{10}$$

where $\omega=i|\omega|$. As shown in [9,11,12], these boundary conditions single out a discrete set of unstable ($\Im\omega>0$) black-hole eigenvalues $\{\omega_n(r_+)\}_{n=0}^{n=\infty}$.

## 3 Numerical evidence for universality in the excited instability spectrum

Most recently, Rinne [13] computed numerically the first three instability eigenvalues which characterize the SU(2) Reissner–Nordström black-hole solutions of the coupled Einstein–Yang–Mills equations. We have examined these numerically computed eigenvalues in an attempt to reveal a possible hidden pattern which characterizes the black-hole instability spectrum.

In Table 1 we present the first excited instability eigenvalues $\{\omega_1(r_+)\}$ of the magnetically charged SU(2) Reissner–Nordström black holes. In particular, we display the dimensionless ratio $\omega_1(r_+)/\pi T_{\mathrm{BH}}$, where the black-hole temperature $T_{\mathrm{BH}}$ is given by (5). We also display the ratio between the dimensionless quantity $\omega_1(r_+)\times(r_+-r_-)$ for generic SU(2) Reissner–Nordström black holes and the corresponding quantity $\omega_1(r_+=10)\times(10-1/10)$ for the weakly

**Table 1** The instability eigenvalues of SU(2) Reissner–Nordström black holes. The data shown refers to the first excited eigenvalues $\{\omega_1(r_+)\}$ of these magnetically charged black holes. We display the dimensionless ratio $\omega_1(r_+)/\pi T_{\mathrm{BH}}$, where $T_{\mathrm{BH}}$ is the black-hole temperature. Also shown is the ratio between the dimensionless quantity $\omega_1(r_+)\times(r_+-r_-)$ for generic SU(2) Reissner–Nordström black holes and the corresponding quantity $\omega_1(r_+=10)\times(10-1/10)$ for the weakly magnetized Reissner–Nordström black hole with $r_+=10$ (see footnote 8). One finds that the instability eigenvalues in the regime $\omega_1(r_+)/\pi T_{\mathrm{BH}}\lesssim 0.1$ are characterized, to a good degree of accuracy, by the universal relation $\omega_1(r_+-r_-)=\text{constant}$

| $r_+$ | $\omega_1(r_+)/\pi T_{\mathrm{BH}}$ | $\frac{\omega_1(r_+)\times(r_+-r_-)}{\omega_1(r_+=10)\times(10-1/10)}$ |
|---|---|---|
| 9.0 | $9.86\times10^{-2}$ | 0.999 |
| 8.0 | $9.91\times10^{-2}$ | 0.999 |
| 7.0 | $9.99\times10^{-2}$ | 0.998 |
| 6.0 | $1.01\times10^{-1}$ | 0.996 |
| 5.0 | $1.04\times10^{-1}$ | 0.993 |
| 4.0 | $1.08\times10^{-1}$ | 0.987 |
| 3.0 | $1.18\times10^{-1}$ | 0.973 |
| 2.0 | $1.58\times10^{-1}$ | 0.925 |
| 1.5 | $2.57\times10^{-1}$ | 0.824 |
| 1.2 | $6.04\times10^{-1}$ | 0.586 |

[5] We use natural units in which $G=c=\hbar=1$.

[6] Note that unstable (growing in time) modes are characterized by $\Im\omega>0$.

[7] Note that the near-horizon limit $r\to r_+$ corresponds to $x\to-\infty$, whereas the large-$r$ limit $r\to\infty$ corresponds to $x\to\infty$.

**Table 2** The instability eigenvalues of SU(2) Reissner–Nordström black holes. The data shown refers to the second excited eigenvalues $\{\omega_2(r_+)\}$ of these magnetically charged black holes. We display the dimensionless ratio $\omega_2(r_+)/\pi T_{\rm BH}$, where $T_{\rm BH}$ is the black-hole temperature. Also shown is the ratio between the dimensionless quantity $\omega_2(r_+)\times(r_+-r_-)$ for generic SU(2) Reissner–Nordström black holes and the corresponding quantity $\omega_2(r_+=10)\times(10-1/10)$ for the weakly magnetized Reissner–Nordström black hole with $r_+=10$ (see footnote 8). One finds that the instability eigenvalues in the regime $\omega_2(r_+)/T_{\rm BH}\ll 1$ are characterized, to a good degree of accuracy, by the universal relation $\omega_2(r_+-r_-)=\text{constant}$

| $r_+$ | $\omega_2(r_+)/\pi T_{\rm BH}$ | $\frac{\omega_2(r_+)\times(r_+-r_-)}{\omega_2(r_+=10)\times(10-1/10)}$ |
|---|---|---|
| 9.0 | $2.90\times10^{-3}$ | 1.011 |
| 8.0 | $2.95\times10^{-3}$ | 1.021 |
| 7.0 | $3.00\times10^{-3}$ | 1.029 |
| 6.0 | $3.06\times10^{-3}$ | 1.034 |
| 5.0 | $3.15\times10^{-3}$ | 1.038 |
| 4.0 | $3.31\times10^{-3}$ | 1.040 |
| 3.0 | $3.68\times10^{-3}$ | 1.041 |
| 2.0 | $5.17\times10^{-3}$ | 1.039 |
| 1.5 | $9.37\times10^{-3}$ | 1.034 |
| 1.2 | $3.03\times10^{-2}$ | 1.011 |

magnetized Reissner–Nordström black hole with $r_+=10$.[8] Remarkably, the numerical data presented in Table 1 reveals that the black-hole instability eigenvalues in the regime $\omega_1(r_+)/T_{\rm BH}\ll 1$ are characterized, to a good degree of accuracy, by the universal relation[9]

$$\omega_1(r_+-r_-)=\lambda_1;\quad \lambda_1=\text{constant}. \tag{11}$$

In order to support this intriguing finding, we display in Table 2 the second excited instability eigenvalues $\{\omega_2(r_+)\}$ of the SU(2) Reissner–Nordström black holes. Remarkably, the numerical data presented in Table 2 provide compelling evidence for the validity of the suggested universal behavior of the black-hole instability eigenvalues in the regime $\omega_2(r_+)/T_{\rm BH}\ll 1$. In particular, one finds[10]

$$\omega_2(r_+-r_-)=\lambda_2;\quad \lambda_2=\text{constant}. \tag{12}$$

## 4 Summary

The U(1) Reissner–Nordström black holes are known to be stable within the framework of the coupled Einstein–Maxwell theory [1–5]. This stability property of the black holes manifests itself in the form of an infinite spectrum of damped quasi-normal resonances [16,17]. To the best of our knowledge, for generic U(1) Reissner–Nordström black holes, there is no simple universal formula which describes the infinite family of these damped black-hole quasi-normal resonances.

On the other hand, the SU(2) Reissner–Nordström black holes are known to be unstable within the framework of the coupled Einstein–Yang–Mills theory [7–10]. This instability property of the magnetically charged black holes manifests itself in the form of an infinite spectrum of exponentially growing black-hole resonances [11,12]. In this paper we have provided compelling numerical evidence that the infinite family of these unstable black-hole resonances can be described, to a very good degree of accuracy, by the simple *universal* formula

$$\omega_n(r_+-r_-)=\text{constant}_n\quad\text{for}\quad \omega_n\ll T_{\rm BH}. \tag{13}$$

We believe that it would be highly interesting to find an analytical explanation for this numerically suggested universal behavior.

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[8] The weakly-magnetized SU(2) Reissner–Nordström black hole with horizon radius $r_+=10$ is the largest black-hole solution studied numerically in [13].

[9] It is worth emphasizing that, in the regime $\omega_1(r_+)/T_{\rm BH}\ll 1$, the value of $\lambda_1$ is almost independent of the black-hole horizon radius $r_+$.

[10] It is worth emphasizing that, in the regime $\omega_2(r_+)/T_{\rm BH}\ll 1$, the value of $\lambda_2$ is almost independent of the black-hole horizon radius $r_+$.