

WIMT in Gullstränd–Painlevé and Reissner–Nordström metrics: induced stable gravito-magnetic monopoles

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Abstract The aim of this work is to apply Weitzböck Induced Matter Theory (WIMT) to Gullstränd–Painlevé and Reissner–Nordström metrics in the framework of WIMT. This is a newly developed method that extends Induced Matter Theory from a curved 5D manifold using the Weitzböck’s geometry, using the fact that the Riemann–Weitzenböck curvature tensor is always null. We obtain the presence of currents whose interpretation can lead to the presence of stable gravito-magnetic monopoles.

Introduction

In a previous paper [1] we incorporated the Weitzböck’s geometry (using its characteristic connections) into the treatment of extended Induced Matter Theory (IMT) [2,3]. This theory is based on the assumption that ordinary matter and physical fields, which we can observe in our 4D universe, can be geometrically obtained from a 5D space–time which is at least Ricci-flat (in the sense of Levi-Civita connections). We are interested in the cases where the extra dimension is non-compact,¹ and we define a physical vacuum supported by the Ricci-flatness condition [4]. This theory is founded in the Campbell–Magaard embedding theorem [5–8] as a par-

ticular case in which ${}^{(5D)}R_{ab} = \lambda {}^{(5D)}g_{ab}$,² with $\lambda = 0$. We must remark that the $\lambda = 0$ condition implies Ricci-flatness and this makes it possible to define a 5D apparent vacuum that determines the equations of motion for the fields of the theory. Such equations are given by ${}^{(5D)}R_{ab} = 0$. The aim of WIMT is to use an alternative description, based on the Weitzenböck geometry, to apply a IMT-like formalism for any 5D space–time, even if this is not flat in the Riemannian sense with Levy-Civita (LC) connections. The central idea of WIMT is to convert the geometrical description of the problem from a non-flat (but torsion-less) Riemannian with LC connections, into a Weitzenböck’s geometrical description with a Riemann–Weitzenböck null tensor,³ but with non-zero torsion. In this case we can use the IMT tools to induce 4D effective space–time dynamics making a foliation over the Riemann–Weitzenböck flat 5D space–time (this is the vacuum in the Weitzenböck sense).⁴ Once this choice has been made, we can recover the Riemannian description by doing a transformation over the induced tensors (now obtained in terms of Weitzenböck connections) according to ${}^{(LC)}\Gamma_{bc}^a = {}^{(W)}\Gamma_{bc}^a + K_{bc}^a$. Here ${}^{(LC)}\Gamma_{bc}^a$ is the symbol denoting the LC connection (the usual Christoffel

³ What we refer to as the Riemann–Weitzenböck tensor is the Riemann curvature tensor expressed in terms of Weitzenböck connections. In a coordinate (holonomic) basis the Riemann–Weitzenböck curvature takes the form

$${}^{(W)}R_{bcd}^a = {}^{(W)}\Gamma_{dc,b}^a - {}^{(W)}\Gamma_{db,c}^a + {}^{(W)}\Gamma_{dc}^n {}^{(W)}\Gamma_{nb}^a - {}^{(W)}\Gamma_{db}^n {}^{(W)}\Gamma_{nc}^a.$$

The label ${}^{(W)}$ identifies the geometric objects expressed in the Weitzenböck representation and the ${}^{(W)}\Gamma_{dc,b}^a$ terms are the ordinary partial derivatives of the Weitzenböck connection ${}^{(W)}\Gamma_{dc}^a$. We assume the Einstein summation convention for repeated indices.

⁴ We set the Weitzenböck vacuum as a 5D space–time with null Riemann curvature tensor in Weitzenböck geometry,

$${}^{(W)}R_{bcd}^a = 0.$$

¹ In contrast with the Kaluza–Klein Theory (KK) in which it is assumed that the extra dimension is compact, cyclic, and having a small radius.

² In our convention the indices “ a, b, c, \dots, h ” and “ A, B, C, \dots, H ” are related to 5D space–time and run from 1 to 5. Greek indices “ $\alpha, \beta, \gamma, \dots$ ” are related to 4D space–time and run from 1 to 4, and indices “ i, j, k, \dots ” are only spatial and run from 2 to 4. We choose our first index related to time coordinate and the last index in 5D space–time is due to the extra coordinate associated with the extra dimension.

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symbols of the second kind), ${}^{(W)}\Gamma_{bc}^a$ denotes the Weitzenböck connections and K_{bc}^a is the contortion, which depends on the non-metricity and torsion. The shape of the contortion is presented in the next section.

Part of the motivation for the present work comes from two principal ideas: (i) WIMT provides a successful description for the formation of a spherically symmetric body of finite size, and (ii) the Weitzenböck torsion present in WIMT is a possible geometrical source for the existence of magnetic monopoles.

With respect to the first assertion we can say that already the WIMT was used to address the problem of the formation of a massive compact object [1] as a remnant of a collapsing space–time. This object can be thought of as a black hole. The second one has been worked out using WIMT in the framework of the gravito-magnetic formalism in a variety of cosmological scenarios [9, 10], with the result that currents of gravito-magnetic monopoles are strongly linked to the Weitzenböck torsion. With the support of these results, it is generally expected that the density of gravito-magnetic monopoles and charges decay in an accelerated universe due to the strong increasing of the scale factor. Hence, the study of a stationary black hole appears to be an excellent issue to complete our approach to a spherically symmetric object and to try to appreciate the existence of stable gravito-magnetic charges using WIMT. We choose to work in the 5D extension of the Gullstrand–Painlevé (GP) metric, because with this metric it is very simple to get the associated basis of pentads and to operate with them. Additionally, but no less important, the GP metric is widely used in the study of BH and their evolution [11–14], so that it is an interesting topic that deserves further study.

1 WIMT

We shall introduce some basic concepts of the Weitzenböck geometry and we expose some results that we developed in previous works, because these will be tools for obtaining some important results of this article.

1.1 Weitzenböck geometry and vielbeins

In order to revisit the WIMT we shall use some elements of Weitzenböck geometry [15, 16]. The Weitzenböck connection is

$${}^{(W)}\Gamma_{ab}^c = \bar{e}_N^c \bar{e}_b(e_a^N); \tag{1}$$

we assume the transformation

$$\bar{e}_a = e_a^A \vec{E}_A, \tag{2}$$

which represents a transformation from an element \vec{E}_A (belonging to $\{\vec{E}_A\}$, an orthonormal Lorentzian basis of the 5D tangent space), into \bar{e}_a (belonging to $\{\bar{e}_a\}$, also a basis of 5D tangent space but in general not orthonormal). The elements e_a^A that perform the transformation are known as vielbeins [17, 18] and fulfill the relations $e_a^A \bar{e}_A^b = \delta_a^b$ and $e_a^A \bar{e}_B^a = \delta_B^A$. The vielbeins provide us with a prescription to transform bases and therefore any tensorial element. For example,

$$g_{ab} = e_a^A e_b^B \eta_{AB}, \tag{3}$$

where η_{AB} is a 5D Minkowski-like metric with signature $(+, -, -, -, -)$, $\{\vec{E}_A\}$ is an orthonormal Lorentzian basis and g_{ab} is the metric tensor expressed in terms of the basis $\{\bar{e}_a\}$. This clarifies that $e_a^A e_b^B$ is an object with five “legs” (corresponding to the five components of the space–time), in the $\{\vec{E}_A\}$ representation for tensorial elements (upper case indices) and the other five “legs” in the $\{\bar{e}_a\}$ representation (lower case indices). For a general tensor $T^{A_1 \dots A_n}_{B_1 \dots B_m}$, which is n times tangent (or contravariant) and m times cotangent (or covariant), we obtain

$$T^{a_1 \dots a_n}_{b_1 \dots b_m} = \bar{e}_{A_1}^{a_1} \dots \bar{e}_{A_n}^{a_n} e_{b_1}^{B_1} \dots e_{b_m}^{B_m} T^{A_1 \dots A_n}_{B_1 \dots B_m}. \tag{4}$$

An important property of the Weitzenböck connection ${}^{(W)}\Gamma_{ab}^c = \bar{e}_N^c \bar{e}_b(e_a^N)$ is that

$$\begin{aligned} {}^{(W)}\nabla_{\bar{e}_b}(\vec{E}_A) &= {}^{(W)}\nabla_{\bar{e}_b}(e_a^A \bar{e}_a) \\ &= e_a^A \left\{ {}^{(W)}\Gamma_{ab}^c - e_N^c \bar{e}_b(e_a^N) \right\} \bar{e}_c = 0. \\ [{}^{(W)}\nabla_{\bar{e}_b}(\vec{E}_A)]^c &= [\vec{E}_A]^c{}_{;b} = \bar{e}_{A;b}^c \end{aligned} \tag{5}$$

This means that the covariant Weitzenböck derivative of the vielbein is null. The expression (1) characterizes the Weitzenböck connection with lower case indices as a transformation of a trivial null connection with upper case indices. About the trivial upper case index connection we must be more formal and construct the following expression:

$$\begin{aligned} {}^{(W)}\Gamma_{BA}^C \vec{E}_C &= \nabla_{\vec{E}_A}(\vec{E}_B) = \bar{e}_A^a \nabla_{\bar{e}_a}(\bar{e}_B^b \bar{e}_b) \\ &= \bar{e}_A^a \bar{e}_a(\bar{e}_B^b) \bar{e}_b + \bar{e}_A^a \bar{e}_B^b \underbrace{\nabla_{\bar{e}_a}(\bar{e}_b)} \\ {}^{(W)}\Gamma_{ba}^c \bar{e}_c &= \bar{e}_N^c \bar{e}_a(e_b^N) \bar{e}_c \\ &= \bar{e}_A^a \{ \bar{e}_a(\bar{e}_B^b) - \bar{e}_B^b \bar{e}_a^N \bar{e}_a(\bar{e}_B^c) \} \bar{e}_b = 0, \end{aligned} \tag{6}$$

where we have used $e_b^N \bar{e}_B^b = \delta_B^N$, $e_b^N \bar{e}_N^c = \delta_b^c$, and consequently $\bar{e}_a(e_b^N) \bar{e}_N^c = -e_b^N \bar{e}_a(\bar{e}_N^c)$. The elements \vec{E}_C are linearly independent so that ${}^{(W)}\Gamma_{AB}^C = 0$. In this way the Weitzenböck connection represents the extension of the derivative operator with trivial zero connections in the orthonormal basis to the new basis. In the most general case

the Riemann curvature tensor in the Weitzenböck geometry takes the following shape in terms of the upper case index connection:

$$\begin{aligned}
 {}^{(W)}R_{BCD}^A &= \vec{E}_B({}^{(W)}\Gamma_{DC}^A) - \vec{E}_C({}^{(W)}\Gamma_{DB}^A) \\
 &+ {}^{(W)}\Gamma_{DC}^N {}^{(W)}\Gamma_{NB}^A - {}^{(W)}\Gamma_{DB}^N {}^{(W)}\Gamma_{NC}^A \\
 &- C_{CB}^N {}^{(W)}\Gamma_{DN}^A = 0,
 \end{aligned}
 \tag{7}$$

due to each upper case index connection being null: ${}^{(W)}\Gamma_{MO}^N = 0$, as we can see in (6). The structure coefficients in the expression (7) are defined by $[\vec{E}_B, \vec{E}_A] = C_{AB}^C \vec{E}_C$. Therefore, the following expression is trivial:

$${}^{(W)}R_{bcd}^a = \vec{e}_A^a e_b^B e_c^C e_d^D {}^{(W)}R_{BCD}^A = 0.
 \tag{8}$$

This relationship is the key of WIMT. It provides us with a tool to define the 5D Weitzenböck vacuum and make use of the usual philosophy of IMT,⁵ even in the absence of the 5D Riemannian (LC) vacuum.⁶

Finally, the relationships between the LC and Weitzenböck connections are

$${}^{(W)}\Gamma_{bc}^a = {}^{(LC)}\Gamma_{bc}^a - {}^{(W)}K_{bc}^a,
 \tag{9}$$

where the Weitzenböck contortion ${}^{(W)}K_{bc}^a$ is

$$\begin{aligned}
 {}^{(W)}K_{bc}^a &= \frac{g^{ma}}{2} \{g_{bm;c} + g_{mc;b} - g_{bc;m}\} \\
 &+ \frac{g^{ma}}{2} \{{}^{(W)}T_{cm}^n g_{bn} + {}^{(W)}T_{bm}^n g_{nc} - {}^{(W)}T_{cb}^n g_{nm}\}.
 \end{aligned}
 \tag{10}$$

In the last expression $g_{ab;c} = {}^{(W)}Q_{abc}$ is the Weitzenböck non-metricity. In the absence of such a non-metricity we recover the usual shape for contortion, which is totally given in terms of the torsion tensor [20,21]. In the present case we make a start with an orthonormal basis $\{\vec{E}_A\}$. The Weitzenböck non-metricity is

$$\begin{aligned}
 {}^{(W)}Q_{ABC} &= \eta_{AB;C} = \vec{E}_C(\eta_{AB}) - \eta_{NB} {}^{(W)}\Gamma_{AC}^N \\
 &- \eta_{AN} {}^{(W)}\Gamma_{BC}^N,
 \end{aligned}$$

which reduces to ${}^{(W)}Q_{ABC} = \vec{E}_C(\eta_{AB})$, because of ${}^{(W)}\Gamma_{BC}^A = 0$. We must remember that each η_{AB} is constant and then

$${}^{(W)}Q_{ABC} = \vec{E}_C(\eta_{AB}) = 0.$$

The last expression is general but in the next section we shall study some cases in which this is specially easy to see. We shall work with a holonomic coordinate basis $\{\vec{e}_a\}$, so that

$${}^{(W)}Q_{ABC} = \vec{E}_C(\eta_{AB}) = \vec{e}_C^c \vec{e}_c(\eta_{AB}) = \vec{e}_C^c \eta_{AB,c} = 0,$$

⁵ The central ideas of IMT are very didactically developed in [19].

⁶ We want to clarify what we mean by ‘‘Riemannian (LC) vacuum’’: it means that the Einstein tensor written in terms of LC connections is zero. This is trivial when the Riemann curvature tensor is null.

because of $\eta_{AB,c} = 0$. In the last expression $\eta_{AB,c} = \frac{\partial \eta_{AB}}{\partial x^c}$, where the coordinate x^c is related to the c th element of the coordinate basis \vec{e}_c . The Weitzenböck non-metricity in lower case indices is given by ${}^{(W)}Q_{abc} = e_a^A e_b^B e_c^C {}^{(W)}Q_{ABC} = 0$. Because of all this, in the present case we must set

$${}^{(W)}K_{bc}^a = \frac{g^{ma}}{2} \{{}^{(W)}T_{cm}^n g_{bn} + {}^{(W)}T_{bm}^n g_{nc} - {}^{(W)}T_{cb}^n g_{nm}\},$$

which is the usual well-known contortion tensor [20,21]. This tensor transforms according to ${}^{(W)}K_{BC}^A = e_a^A \vec{e}_B^b \vec{e}_C^c {}^{(W)}K_{bc}^a$.

1.2 Some results in WIMT

We consider a 5D space–time with coordinates $\phi(p) = (t, x, y, z, \psi)_p$ dotted by an inner product described by $\text{Diag}(\eta_{AB}) = (+1, -1, -1, -1, -1)$, expressed in a non-holonomic basis. The action on the 5D apparent vacuum is given by the gravito-electromagnetic fields

$$S = \int d^5x \sqrt{|\eta|} \left[\frac{R}{16\pi G} - \frac{1}{4} F_{AB} F^{AB} - \frac{\lambda}{2} (A_{;B}^B)^2 \right],
 \tag{11}$$

where the penta-vector potential is $\vec{A} = (\phi, \mathbf{A}, \Psi)$, ϕ is the electric potential, $\mathbf{A} = (A^x, A^y, A^z)$ is the vector potential, and Ψ is the gravitational scalar potential. If we assume the Lorenz gauge $A_{;B}^B = 0$, the equations of motion will be reduced to

$$\square A^K = \eta^{BC} A_{;BC}^K = 0.
 \tag{12}$$

In general, we find that the 5D Faraday tensors expressed in terms of the LC and Weitzenböck versions are given, respectively, by

$$\begin{aligned}
 {}^{(LC)}F_{AB} &= \vec{E}_A(A_B) - \vec{E}_B(A_A) + A_N C_{AB}^N, \\
 {}^{(W)}F_{AB} &= \vec{E}_A(A_B) - \vec{E}_B(A_A) + A_N ({}^{(W)}T_{BA}^N + C_{AB}^N), \\
 {}^{(LC)}F^{NB} &= {}^{(W)}F^{NB} + \eta^{RN} A^P K_{PR}^B - \eta^{RB} A^P K_{PR}^N.
 \end{aligned}
 \tag{13}$$

The 5D-current of gravito-magnetic monopoles is obtained (still in 5D space–time) from the relation ${}^{(*)}J_{AB} = {}^{(*)}\mathfrak{F}_{ABC}{}^{;C}$. This is motivated by the symmetry of Maxwell equations.⁷ Thus, $*$ must be LC or W according to the corresponding derivative operator. Furthermore, \mathfrak{F}_{ABC} is the dual tensor of

⁷ In a 4D space–time, this symmetry is manifest in the expressions

$$\begin{aligned}
 F_{\mu\nu}{}^{;v} &= {}^{(e)}J_\mu, \\
 \mathfrak{F}_{\mu\nu}{}^{;v} &= {}^{(m)}J_\mu.
 \end{aligned}
 \tag{14}$$

Here, the labels e and m , identify, respectively, the electrical and magnetic quantities according to ${}^{(e)}\vec{J} = (\rho_e, {}^{(e)}J_1, {}^{(e)}J_2, {}^{(e)}J_3)$ and ${}^{(m)}\vec{J} = (\rho_m, {}^{(m)}J_1, {}^{(m)}J_2, {}^{(m)}J_3)$. Furthermore, $F_{\mu\nu}$ is the Faraday tensor and $\mathfrak{F}_{\mu\nu}$ is its dual [22,23].

F_{AB} .⁸ The resulting currents, 2-forms in the 5D space–time, are

$$(LC) \underline{J} \underset{\rightarrow}{=} = \frac{\sqrt{|\eta|}}{2} \varepsilon_{ABCDE} \eta^{CF} \eta^{DG} \eta^{EH} \times \left[(LC) \underline{d} \left((LC) \underline{F} \underset{\rightarrow}{=} \right) \right]_{[FGH]} \underline{E}^A \otimes \underline{E}^B, \quad (15)$$

$$\begin{aligned} \left[(LC) \underline{d} \left((LC) \underline{F} \underset{\rightarrow}{=} \right) \right]_{[ABC]} &= {}^{(W)} T_{AC}^N \underline{\vec{E}}_N(A_B) \\ &+ {}^{(W)} T_{BA}^N \underline{\vec{E}}_N(A_C) + {}^{(W)} T_{CB}^N \underline{\vec{E}}_N(A_A) + {}^{(W)} T_{AC}^n \underline{\vec{E}}_B(A_n) \\ &+ {}^{(W)} T_{CB}^n \underline{\vec{E}}_A(A_n) + {}^{(W)} T_{BA}^n \underline{\vec{E}}_C(A_n) + \underline{\vec{E}}_A ({}^{(W)} T_{CD}^n) A_n \\ &+ \underline{\vec{E}}_B ({}^{(W)} T_{AC}^n) A_n + \underline{\vec{E}}_C ({}^{(W)} T_{AB}^n) A_n. \end{aligned} \quad (16)$$

The Riemannian (LC) current of the gravito-magnetic monopole originates in the Weitzenböck torsion ${}^{(W)} T_{BC}^A$.⁹ It must be noticed that once we choose the Lorenz gauge condition ${}^{(LC)} A_{;N}^N = 0$, the same condition is preserved in the Weitzenböck representation: ${}^{(W)} A_{;n}^n = 0$.

Once we have established the 5D-currents it is easy to obtain the 4D-induced currents by applying a constant foliation on the fifth extra dimension: $\psi = \psi_0$. The effective resulting 4D gravito-magnetic current is the 1-form given by

$${}^{(*)4D} \underline{J}(\cdot) = \left. {}^{(*)5D} \underline{J}(\underline{\vec{n}}, \cdot) \right|_{\psi=\psi_0}, \quad (19)$$

so that $*$ must be LC or W and $\underline{\vec{n}}$ is a vector pointing to the extra direction. The vector $\underline{\vec{n}}$ is normalized and orthogonal to the 4D tangent sub-space associated with the effective 4D

⁸ The dual tensor of a 2-form (characterized in this case by ${}^{(*)} F_{AB}$) in 5D manifold is a 3-form defined by

$${}^{(*)} \mathfrak{F}_{ABC} = \frac{1}{2} \varepsilon_{ABCDE} {}^{(*)} F^{DE}.$$

Here, $*$ must be LC or W and ε_{ABCDE} is the antisymmetric Levi-Civita symbol. We have made use of the fact that $\sqrt{|\eta|} = 1$, $\eta = \text{Det}([\eta]_{AB})$ being the determinant of the metric tensor in the upper case index representation.

⁹ We focus our attention on the Riemannian (LC) current, but it is formally easy to relate the Riemannian (LC) and Weitzenböck currents of the gravito-magnetic monopole, with the expressions

$$(LC) J_{AB} - {}^{(W)} J_{AB} = \frac{\sqrt{|\eta|}}{2} \varepsilon_{ABCDE} \frac{1}{4} M^{[CDE]}, \quad (17)$$

where $M^{[CDE]} = \eta^{CF} \eta^{DG} \eta^{EH} M_{[FGH]}$, and

$$\begin{aligned} M_{[FGH]} &= (A_M {}^{(W)} T_{[FG]}^M)_{;H} - 2 {}^{(W)} T_{[FH]}^N {}^{(W)} T_{N[G]}^M A_M \\ &- 2 {}^{(W)} T_{[GH]}^N {}^{(W)} T_{F[N]}^M A_M - {}^{(W)} T_{[FH]}^N \underline{\vec{E}}_N(A_{[G]}) \\ &+ {}^{(W)} T_{[GH]}^N \underline{\vec{E}}_N(A_{[F]}) + {}^{(W)} T_{[FH]}^N \underline{\vec{E}}_G(A_N) \\ &- {}^{(W)} T_{[GH]}^N \underline{\vec{E}}_F(A_N). \end{aligned} \quad (18)$$

sub-manifold. Such orthogonality is realized with respect to the complete 5D metric. All expressions must be evaluated in $\psi = \psi_0$, so that $dx^5|_{\psi=\psi_0} = d\psi|_{\psi=\psi_0} = 0$ and hence ${}^{(*)4D} \underline{J}$ must be a cotangent four-vector (i.e., it is a 1-form in the cotangent 4D space–time).

2 5D Gullstränd–Painlevé (GP) metric

The GP coordinates are commonly presented as a set of coordinates obtained by transformation of the Schwarzschild, or Reissner–Nordström (RN) ones, in which the static time is replaced by a GP time $t \rightarrow \tilde{t}(t, r)$, preserving the spherical symmetry of the problem [24,25]. We shall use a slightly different method by using the GP cartesian metric and using the vielbein related to an orthonormal basis. In this case we obtain the GP cartesian metric from a normalized one, using a certain vielbein, prescribed in this section. The GP cartesian metric must be obtained from the Schwarzschild or RN BH by doing a space–time transformation. We choose to start with an orthonormal basis $\{\underline{\vec{E}}_A\}$, in which

$$\underline{\vec{E}}_A = \underline{\vec{e}}_A^a \underline{\vec{e}}_a, \quad (20)$$

where $\{\underline{\vec{e}}_a\}$ is some coordinate basis. The vielbein is constructed according to

$$[\underline{\vec{e}}_M^n] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\beta^x & 1 & 0 & 0 & 0 \\ -\beta^y & 0 & 1 & 0 & 0 \\ -\beta^z & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (21)$$

and its inverse is

$$[e_m^M] = \begin{pmatrix} 1 & \beta^x & \beta^y & \beta^z & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (22)$$

in which $\beta^i = \beta x^i$, such that x^i is the i th coordinate associated to the coordinate basis $\{\underline{\vec{e}}_a\}$, $\beta = \sqrt{\frac{2(M - \frac{Q^2}{2r})}{r^3}}$ is a function, M is a mass parameter, and Q is an electric charge parameter. If $Q = 0$ we deal with a Schwarzschild BH, but in the case $Q \neq 0$, we are in presence of a Reissner–Nordström BH. We must remark that $M - \frac{Q^2}{2r}$ is the gravito-electric charge of the BH, and

$$\beta^i \beta_i = \beta^l \beta_l = -(\beta^2 x^2 + \beta^2 y^2 + \beta^2 z^2) = -\beta^2 r^2,$$

so that, by the definition of β , we have

$$\beta^2 = 2 \left(M - \frac{Q^2}{2r} \right) \frac{1}{r^3} = -\frac{1}{r^2} \left(\frac{2M}{r} - \frac{Q^2}{r^2} \right).$$

Hence

$$\beta^i \beta_i = - \left(\frac{2M}{r} - \frac{Q^2}{r^2} \right).$$

The charge of the BH is dominant at small distances, but the mass dominates at long distances. The metric is

$$dS^2 = \underbrace{\eta_{MN} e_m^M e_n^N}_{g_{mn}} dx^m dx^n = \underbrace{\eta_{MN} e_m^M}_{E^M} \underbrace{e_n^N}_{E^N} dx^n, \\ = dt^2 - \delta_{ij} (dx^i - \beta^i dt)(dx^j - \beta^j dt) - d\psi^2. \quad (23)$$

In the last expression $E^N = E_n^N dx^n$ is an element belonging to a basis for the cotangent space, which fulfills the duality relation $E^N(\vec{E}_M) = \delta_M^N$. Furthermore, g_{mn} is the corresponding GP metric in cartesian 5D GP coordinates, such that $x^1 = t, x^2 = x, x^3 = y, x^4 = z,$ and $x^5 = \psi$.¹⁰

Now we must calculate the structure coefficients for the basis (20). According to the definition $[\vec{E}_B, \vec{E}_A] = C_{AB}^C \vec{E}_C$ [27], we obtain

$$C_{AB}^C = \vec{e}_B^m \frac{\partial \vec{e}_A^j}{\partial x^m} e_l^C - \vec{e}_A^m \frac{\partial \vec{e}_B^j}{\partial x^m} e_l^C. \quad (24)$$

The non-zero elements are $C_{21}^1 = -C_{12}^1, C_{32}^3 = -C_{23}^3 = -\frac{\sqrt{1+\beta^i \beta_i}}{\sqrt{x^2+y^2+z^2}} = -\frac{\sqrt{f(r)}}{r}, C_{42}^4 = -C_{24}^4 = -\frac{\sqrt{1+\beta^i \beta_i}}{\sqrt{x^2+y^2+z^2}} = -\frac{\sqrt{f(r)}}{r},$ and $C_{43}^4 = -C_{24}^4 = -\frac{z}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} = -\frac{\cos(\theta)}{r \sin(\theta)},$ expressed in GP, or in the usual BH coordinates.

Here, $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$ for a Reissner–Nordström BH. In the case with $Q = 0$ one obtains a Schwarzschild BH. On the other hand, if we take the expression (24), will be easily shown that $C_{ab}^c = 0,$ which means that $\{\vec{e}_a\}$ is a coordinate or holonomic basis for 5D tangent space. It is very interesting to notice that in the upper case index representation, the torsion tensor is numerically equal to the structure coefficients, due to the fact that ${}^{(W)}T_{BC}^A = {}^{(W)}\Gamma_{CB}^A - {}^{(W)}\Gamma_{BC}^A + C_{BC}^A.$ Since the Weitzenböck connection extends the trivial null connection in the upper case index basis, we obtain ${}^{(W)}\Gamma_{CB}^A = 0,$ and hence

$${}^{(W)}T_{BC}^A = C_{BC}^A = -C_{CB}^A, \quad (25)$$

which is surprising because ${}^{(W)}T_{BC}^A$ is an intrinsic geometrical property of the manifold and C_{BC}^A is a coefficient depending on a circumstantially selected basis. In fact, Eq. (25) is not strange and is due to the selection of the Weitzenböck connection and particularly to the geometrical properties to

extend a derivative operator linked to a trivial null connection in the upper case index, to a nontrivial connection in the lower case index basis.

3 Gravitomagnetic monopoles from WIMT in GP and RN metrics

If we take the structure coefficients (24) and we employ the non-zero elements in (25), we obtain the Weitzenböck torsion and we could use it in the expressions (15) and (16), in order to write the 5D LC 2-form currents in upper case index basis. After it, we can express such 2-form in the lower case index representation with the transformation (4) adapted to a 2 times cotangent object [as we were doing in (3) for the metric tensor], with the vielbein (22). Notice that we have a 5D LC 2-form current expressed in terms of a Weitzenböck torsion with lower case indices and vielbeins. Hence, we can make a constant foliation getting the effective 4D current from (19). Finally we obtain the expression for the effective 4D current, which in the usual BH coordinates takes the form

$${}^{(LC)(m)}J_1 |_{\psi=\psi_0} = \rho_m = \\ = K \left\{ -\frac{\sqrt{f(r)}}{r} \frac{\partial A_3}{\partial \theta} + \frac{\sqrt{f(r)}}{r \sin(\theta)} \frac{\partial A_2}{\partial \varphi} - \frac{\cos(\theta)}{r^2 \sin^2(\theta)} \frac{\partial A_1}{\partial \varphi} \right\}, \quad (26)$$

which is the induced static gravitomagnetic monopole charge density. For the spatial effective 4D currents of gravitomagnetic monopoles, we obtain

$$J_i = 0, \quad i = 2, 3, 4, \quad (27)$$

which is consistent with the magnetic charge conservation in a scenario in which the monopoles are static. Now we must study some important invariants of the problem. We define

$${}^{(m)}J(\overset{(e)}{\vec{J}}, \overset{(e)}{\vec{U}}) = {}^{(e)}J^A {}^{(m)}J_{AB} U^B = {}^{(m)}J_A {}^{(e)}J^A, \quad (28)$$

$${}^{(m)}J := {}^{(m)}\vec{J}(\overset{(e)}{\vec{U}}) = {}^{(m)}J_A U^A, \quad (29)$$

$${}^{(e)}J := {}^{(e)}\vec{J}(\overset{(e)}{\vec{U}}) = {}^{(e)}J_A U^A, \quad (30)$$

$${}^{(gem)}J^2 := ({}^{(e)}\vec{J} \wedge {}^{(m)}\vec{J})({}^{(e)}\vec{J} \wedge {}^{(m)}\vec{J}) \\ = {}^{(gem)}J_{AB} {}^{(gem)}J^{AB}. \quad (31)$$

These expressions must be constructed as a function of the LC or Weitzenböck currents. The U^B are the components of the penta-velocities of the observers, expressed in the RN non-coordinate basis. Hence, all the invariants (28), (29) (30), and (31), defined in 5D, are constructed taking into account

¹⁰ For a most extensive treatment the reader could see the notes of Andrew Hamilton (developed in 4D) [26].

these penta-velocities and the gravito-electric and gravito-magnetic currents. From the analysis of the invariant (28) we see that the current characterized by the expression ${}^{(m)}J_A := {}^{(m)}J_{AB} U^B$ comes from the contraction between the (2-form) magnetic current \vec{J} and the (one-vector) penta-velocity \vec{U} . Hence, the complete invariant (28) must be interpreted as an inner product ${}^{(m)}\vec{J} \cdot {}^{(e)}\vec{J} = {}^{(m)}J_A {}^{(e)}J^A$, which means that both involved currents must be orthogonal when (28) is zero. On the other hand, the invariant (31) is a new 2-form, which contains information as regards gravity, electricity, and magnetism. It is due to the wedge product of the (1-form) magnetic current \vec{J} and the (1-form) electric current \vec{J} . We call such a new 2-form ${}^{(gem)}J$, in which the superscript ${}^{(gem)}$ refers to the gravito-electro-magnetic nature of the current. In general, ${}^{(gem)}J^{AB} = {}^{(e)}\vec{J} \wedge {}^{(m)}\vec{J}$ must not be null. The expression (31) is equivalent to the inner product of two 2-forms, which are expressed by $[{}^{(e)}\vec{J} \wedge {}^{(m)}\vec{J}]_{AB} [{}^{(e)}\vec{J} \wedge {}^{(m)}\vec{J}]^{AB} = {}^{(gem)}J^2$.

The invariants ${}^{(e)}J$ and ${}^{(m)}J$ have direct counterparts in the 4D space-time, defined by

$${}^{(m)}J|_{4D} = {}^{(m)}J_A U^A|_{fol}, \tag{32}$$

$${}^{(e)}J|_{4D} = {}^{(e)}J_A U^A|_{fol}. \tag{33}$$

On the other hand, the 4D counterpart of ${}^{(m)}J_A {}^{(e)}J^A$ is

$${}^{(m)}J_A {}^{(e)}J^A|_{4D} = {}^{(m)}J_A {}^{(e)}J^A|_{fol}. \tag{34}$$

The invariant ${}^{(m)}J_A {}^{(e)}J^A = 0$ only makes sense because we are supposing that the five-vectors ${}^{(m)}J$ and ${}^{(e)}J$ are orthogonal on the 5D Weitzenböck vacuum. All the previously defined invariants have a clear meaning, which can easily be explored in a static or stationary case, as the case of a RN BH, which we shall study. Then we must show that Eqs. (28–34) reduce to

$${}^{(m)}J_A {}^{(e)}J^A|_{fol} = \rho_m \rho_M U^1 + \rho_e \rho_m U^5|_{fol} = 0, \tag{35}$$

$${}^{(m)}J_A U^A|_{fol} = \rho_m (U^1 U^5 - U^5 U^1)|_{fol} = 0, \tag{36}$$

$${}^{(e)}J_A U^A|_{fol} = \rho_e U^1 + \rho_M U^5|_{fol} = 0. \tag{37}$$

In a static 5D space-time in which the observer is moving with arbitrary penta-velocity $\vec{U} = U^A \vec{E}_A$, their components U^A can be $U^1, U^i, U^5 \neq 0$. Hence, one obtains

$${}^{(m)}J_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\rho_m \\ 0 & 0 & 0 & 0 & -{}^{(m)}J_2 \\ 0 & 0 & 0 & 0 & -{}^{(m)}J_3 \\ 0 & 0 & 0 & 0 & -{}^{(m)}J_4 \\ \rho_m & {}^{(m)}J_2 & {}^{(m)}J_3 & {}^{(m)}J_4 & 0 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & -\rho_m \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \rho_m & 0 & 0 & 0 & 0 \end{pmatrix},$$

in which we have made use of the fact that ${}^{(m)}J_i = 0$, so that

$${}^{(m)}J_A = (-\rho_m U^5, 0, 0, 0, \rho_m U^1).$$

In a non-static scenario results that ${}^{(m)}J_5 = 0$. On the other hand, using the fact that ${}^{(e)}J_i = 0$, we obtain

$${}^{(e)}J_A = (\rho_e, 0, 0, 0, {}^{(e)}J_5),$$

where in general ${}^{(e)}J_5 \neq 0$. All makes sense when we apply our extended 5D Maxwell equations for ${}^{(e)}J_5$, so that $F_{5A}{}^A = {}^{(e)}J_5$, which in the extended version is

$${}^{(e)}J_5 = -\square\Psi + (A^A_A)_{;5}.$$

If we apply the 5D Lorenz gauge ($A^A_A = 0$), the last expression evaluated on the foliation reduces to

$${}^{(e)}J_5|_{fol} = -\square\Psi|_{fol}. \tag{38}$$

The expression (38) must be interpreted as an effective equation of the form $\rho_M = -\square\Psi$, in which Ψ is the gravitational potential for the 4D BH. In this case ${}^{(e)}J_5$ must be ρ_M ; the effective density of the gravitational mass. We must remark that in the present case the operator \square does not have a time related covariant derivative part, due to the static nature of the problem here considered (we maintain the complete notation for clarity). We must set

$${}^{(e)}J_A = (\rho_e, 0, 0, 0, \rho_M). \tag{39}$$

Equation (39) justifies the notation used in Eqs. (35) and (37).

We shall study the case with $U^i = 0$ and $U^5 \neq 0$. For such a scenario we assume that the observer is moving with a penta-velocity $\vec{U} = U^A \vec{E}_A$, with $U^A \neq 0$. Then we must set U^5 to an arbitrary non-zero value. The penta-velocity is defined on a 5D TM and is a penta-vector. The notation used is:

- The 5D M is the 5D manifold.
- The 5D TM is tangent space associated to 5D M.
- 4D M is the 4D manifold induced by foliation $\psi = \psi_0$ and, of course, a sub-manifold of the 5D M.
- The 4D TM is the tangent space associated to 4D M and, of course, a sub-space of the 5D TM.

When we apply a constant foliation $\psi = \psi_0$ on the extra coordinate, the penta-velocity turns into a four-vector by projecting it onto the 4D TM. Then $U^1 \vec{E}_1 + U^i \vec{E}_i + U^5 \vec{E}_5 \rightarrow u^1 \vec{e}_1 + u^i \vec{e}_i |_{\psi=\psi_0}$, so that $\{\vec{E}_1, \vec{E}_i, \vec{E}_5\}$ is a basis of the 5D TM, and $\{\vec{e}_1, \vec{e}_i\}$ is a basis of the induced 4D TM. Due to the static nature of the studied scenario the invariants (35–37) are not dependent on u^i and we must choose the value of u^i preserving the validity of the geodesic equations: $\frac{du^\alpha}{ds} + {}^{(W)}\Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0$, with the normalization condition $g_{\alpha\beta} u^\alpha u^\beta = 1$. Here, $g_{\alpha\beta}$ is the effective induced metric with the foliation $\psi = \psi_0$ and we must take the RN, or the GP ordinary 4D form, depending on the employed coordinates.

The use of WIMT warrants that the 5D space–time is a vacuum in the sense of the Weitzenböck geometry, even if $\rho_m, \rho_e, \rho_M \neq 0$. At this point we choose to establish some values for the 5D invariants, in order to obtain a realistic physical behavior in the effective space–time. Hence, we set ${}^{(m)}J_A {}^{(e)}J^A = 0$ to find

$$\frac{\rho_M}{\rho_e} = -\frac{U^5}{U^1}. \tag{40}$$

On the another hand, from ${}^{(e)}J = 0$, we obtain

$$\frac{\rho_e}{\rho_M} = -\frac{U^5}{U^1}, \tag{41}$$

so that, equating (40) and (41), we obtain

$$\frac{U^5}{U^1} = \frac{\rho_M}{\rho_e} = \pm 1, \tag{42}$$

which provides us with the quantization law between the mass density and the electric density charge. In order to obtain the quantization of charge we must use Eqs. (31) and (42), with ${}^{(gem)}J^2 = -n^2$. The resulting expression, which provides the quantization between the electric and magnetic density charges, is

$$\rho_e \rho_m = \frac{n}{2}. \tag{43}$$

Notice that all these expressions were obtained in the ONB representation. A very important fact is that the choices ${}^{(m)}J_A {}^{(e)}J^A = 0$ and ${}^{(gem)}J^2 = -n^2$ imply that the duality of charges ρ_e and ρ_m is mutually exclusive, with the constriction (43). This means that one only can “see” the electric ρ_e or the magnetic charge $\rho_m = \frac{n}{2\rho_e}$. This is because Eq. (36) indicates that ${}^{(m)}\mathbf{J}$ and ${}^{(e)}\mathbf{J}$ are orthogonal. Therefore, our model with these choices is incompatible with a dyonic Reissner–Nordström BH, except for $n = 0$. In order to make our theory compatible with a dyonic Reissner–Nordström BH one should consider ${}^{(m)}J_A {}^{(e)}J^A \neq 0$ and ${}^{(gem)}J^2 = -n^2$, but this issue will be studied in a future work.

We shall return to the expression (26), in order to see the consistency of such an equation for the 4D Wu–Yang potentials, which describe a localized magnetic monopole with charge q_m [28–30],

$$\vec{A}^{(N)} = q_m \frac{(1 - \cos(\theta))}{r \sin(\theta)} \vec{e}_\varphi = \frac{q_m}{r} A_3^{(N)} \vec{e}_\varphi, \tag{44}$$

$$\vec{A}^{(S)} = -q_m \frac{(1 + \cos(\theta))}{r \sin(\theta)} \vec{e}_\varphi = \frac{q_m}{r} A_3^{(S)} \vec{e}_\varphi. \tag{45}$$

Here, labels (N) and (S) indicate the North or South hemisphere, on which are valid the potentials. The magnetic monopole density in the RN metric is

$$\rho_m^{(*)} = -\frac{K}{r} \frac{\partial A_3^{(*)}}{\partial \theta}, \tag{46}$$

in which the asterisk (*) stands for the labels (N) or (S), depending on what hemisphere we are dealing with. For simplicity we choose to integrate the last expression in the orthonormalized metric, in which the associated volume element is $dV = \frac{1}{\sqrt{f(r)}} r^2 \sin(\theta) d\theta d\varphi dr$ due to $|g| = 1$. In the orthonormalized basis $g_{\alpha\beta} = \eta_{\alpha\beta} = \pm 1$. Then $|g| = 1$ and the basis of the cotangent space is $\vec{E}_1 = \sqrt{f(r)} dt, \vec{E}_2 = \frac{1}{\sqrt{f(r)}} dr, \vec{E}_3 = r d\theta, \vec{E}_4 = r \sin(\theta) d\varphi$. Therefore

$$dV = \frac{1}{\sqrt{f(r)}} r^2 \sin(\theta) d\theta d\varphi dr.$$

On the other hand, the current is

$$\begin{aligned} {}^{(m)}\vec{J} &= \underbrace{{}^{(m)}J(\text{ON})_0}_{\text{orthonormalized basis}} \frac{1}{\sqrt{f(r)}} \frac{\vec{\partial}}{\partial t} \Rightarrow \\ {}^{(m)}J(\text{RN})_0 &= \frac{1}{\sqrt{f(r)}} {}^{(m)}J(\text{ON})_0. \\ &\text{RNbasis,} \end{aligned}$$

Therefore, by integrating on the North and South hemispheres we find that the magnetic charge in the BH is given by

$$q_m^{(*)} = -K \int \int \int_{(*)} \frac{\partial A_3^{(*)}}{\partial \theta} \sin(\theta) d\theta d\varphi dr,$$

in which $q_m^{(*)}$ is the total magnetic charge on the corresponding hemisphere. Of course, the total BH magnetic charge will be $q_m = q_m^{(N)} + q_m^{(S)}$, and the integral on the volume for the total magnetic charge of the BH takes the form

$$\begin{aligned}
 q_m &= \int \int \int^{(N)} \rho_m^{(N)} dV + \int \int \int^{(S)} \rho_m^{(S)} dV \\
 &= -K \left(\int \int \int^{(N)} \frac{\partial A_3^{(N)}}{\partial \theta} \sin(\theta) d\theta d\varphi dr \right. \\
 &\quad \left. + \int \int \int^{(S)} \frac{\partial A_3^{(S)}}{\partial \theta} \sin(\theta) d\theta d\varphi dr \right), \tag{47}
 \end{aligned}$$

which leads to

$$\int \int \int_{\text{BH}} \rho_m dV = -2\pi K r_{\text{int}} q_m, \tag{48}$$

$r_{\text{int}} = M - \sqrt{M^2 - Q^2}$ being the interior radius of the RN BH. We assume that $Q < M$, so that for consistence we shall require that $K = -\frac{1}{2\pi r_{\text{int}}}$. Then to the effective RN BH, in which our WIMT induces a gravito-magnetic current, belongs a density of monopole charge ρ_m , which is not uniform ($\rho_m \neq \frac{q_m}{\frac{4}{3}\pi r_{\text{int}}^3}$), but compatible with a total magnetic charge q_m , over the BH, which is the origin of the Wu–Yang potentials.

4 Final remarks

We have employed WIMT over the 5D GP and RN metrics in order to obtain the Weitzenböck torsion in the ONB representation, which must be transformed with the vielbein (21) and (22) to the GP representation. The matrix which represents the vielbein has an antisymmetric part, with which we obtained an effective non-zero gravito-magnetic current originating in the Weitzenböck torsion. The effective current in the RN representation gives rise to a non-zero magnetic monopole density without any time dependence. We have confirmed that it is compatible with the Wu–Yang potentials for a localized magnetic charge of magnitude q_m . In our case the magnetic density ρ_m must be seen as a magnetic charge distribution over the BH, where the θ -dependence (which is a manifestation of an anisotropic distribution of charge), represents the total magnetic charge, q_m , of the BH. We analyze various invariants which make it possible to interpret the physical magnitudes of our problem (ρ_m, ρ_M, ρ_e) as elements coming from such invariants; this highlights the correspondence between such magnitudes and the geometry of the space–time. In particular, the quantization condition arises as an effective manifestation of the current expression between the components of the penta-velocity of the observer. Finally, we have obtained an induced gravito-magnetic monopole in the framework of WIMP, which is compatible with the Wu–Yang fields, with the same magnitude of charge and quantization of charge included. This is a very important result that manifests the power of our formalism.

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5 Appendix A: Connecting ONB, GP, and RN

Here we try to order the different representations of space–time and the relationship between them:

- **ONB representation:** in 5D the orthonormal basis of 5D TM is $\{\vec{E}_A\}$, which is non-holonomic with structure coefficients given in (24) and the subsequent expressions. In this case the matrix of the metric tensor is η_{AB} with $\vec{E}_A \cdot \vec{E}_B = \eta_{AB} = \pm 1$. With a constant foliation we must obtain $\eta_{\alpha\beta} = \pm 1$, for the basis $\{\vec{E}_\alpha\}$. In this context the Weitzenböck torsion is specially easy to obtain according to (25), and we can transform it to other representations using the corresponding vielbein.
- **GP representation:** The 5D GP cartesian metric (23) is linked to the coordinate basis $\vec{e}_a = \frac{\partial}{\partial x^a}$ associated to x^a , with $x^1 = t_{GP}, x^2 = x, x^3 = y, x^4 = z, x^5 = l$, in which $t_{GP} = t + h(r)$, where $h(r)$ is an arbitrary (C^∞) function. Using a constant foliation we must obtain the familiar 4D GP cartesian metric [26].
- **RN representation:** The usual 5D RN metric associated to an effective 4D charged black hole which in 5D is characterized by a length element $dS^2 = f(r)dt^2 - f(r)^{-1}dr^2 - d\Omega^2 - dl^2$, with $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$ expressed for the coordinate basis $\{\vec{e}'_{a'} = \frac{\partial}{\partial x^{a'}}\}$, associated to $x^{a'}$, where $x'^1 = t_{RN}, x'^2 = r, x'^3 = \theta, x'^4 = \varphi, x'^5 = l$. The 4D RN metric must be obtained by a constant foliation.
- **Tensorial objects:** in the ONB representation must be transformed to the GP representation using the vielbein (21) and (22), and vice versa. To obtain the vielbein connecting GP and RN, we must use the fact that both bases are coordinate (holonomic). Then $\vec{e}'_{a'} = e^a_{a'} \vec{e}_a$ with $e^a_{a'} = \frac{\partial x^a}{\partial x^{a'}}$ and vice versa. The non-zero vielbeins relating the ONB with RN, according to $\vec{E}_A = e^a_A \vec{e}_a$, are $e^1_{A=1} = \frac{1}{\sqrt{f(r)}}$, $e^2_{A=2} = \sqrt{f(r)}$, $e^3_{A=3} = \frac{1}{r}$, $e^4_{A=4} = \frac{1}{r \sin(\theta)}$, $e^5_{A=5} = 1$, and therefore they comply with the expressions $e^a_A e^A_{b'} = \delta^a_{b'}$ and $e^a_A e^B_{a'} = \delta^B_A$.

- **Constant foliation:** must be viewed as the application of a vielbein $e_\alpha^a = \delta_\alpha^a$, $e_\alpha^5 = 0$ to the tensor objects plus a specialization to a certain value for $\psi = \psi_0$. Then, for example, $F_b^a \rightarrow F_\beta^\alpha = e_a^\alpha e_\beta^b F_b^a |_{\psi=\psi_0}$. This idea must be extended to another more general kind of foliation (dynamic foliation) varying the vielbein as required.

Using the previous concepts and formulas we must relate ONB, GP, and RN representations and the effective 4D space–time, tensor objects, and invariants.

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