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On the (1 + 3) threading of spacetime with respect to an arbitrary timelike vector field

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Abstract We develop a new approach on the $(1+3)$ threading of spacetime (M, g) with respect to a congruence of curves defined by an arbitrary timelike vector field. The study is based on spatial tensor fields and on the Riemannian spatial connection $\nabla^\star$, which behave as 3D geometric objects. We obtain new formulas for local components of the Ricci tensor field of (M, g) with respect to the threading frame field, in terms of the Ricci tensor field of $\nabla^\star$ and of kinematic quantities. Also, new expressions for time covariant derivatives of kinematic quantities are stated. In particular, a new form of Raychaudhuri's equation enables us to prove Lemma 6.3, which completes a well-known lemma used in the proof of the Penrose–Hawking singularity theorems. Finally, we apply the new $(1+3)$ formalism to the study of the dynamics of a Kerr–Newman black hole.

1 Introduction

The $(1+3)$ threading of spacetime by a congruence of curves determined by a unit timelike vector field ξ (4-velocity) is by now a well-established theory which studies the geometry, dynamics, and observational properties of some well-known cosmological models. Most of the important results on this theory, and an exhaustive list of references wherein these results have been published, can be found in the excellent monograph of Ellis et al. [1].

Our work on this matter is motivated by the simple remark that it is difficult to apply the above theory to the metrics of general form presented in (2.8). This is because, in this case, $\xi = \partial/\partial x^0$ is not a unit vector field and thus it should be normalized. But this process leads to complicated formulas for kinematic quantities and Ricci tensor field, which of course makes difficult their study. The question is: Are there important cosmological models whose metrics have the general form (2.8). The answer is in the affirmative and it is based on the following two examples. First, the study of the cosmological perturbations of the FLRW universes is developed with respect to the metric (cf. (10.12) of [1])

$$\mathrm{d}s^2 = a^2\{-(1+2\phi)(\mathrm{d}x^0)^2 + 2(B_{|i} - S_i)\mathrm{d}x^0\mathrm{d}x^i + [(1-2\psi)\gamma_{ij} + 2E_{i|j} + 2F_{i|j} + h_{ij}]\mathrm{d}x^i\mathrm{d}x^j\}. \quad (1.1)$$

We recall that the metric of a Kerr–Newman black hole is given by (cf. (12.3.1) in [2])

$$\begin{aligned}\mathrm{d}s^2 = &-\left(\frac{\Delta - a^2(\sin x^2)^2}{\Sigma}\right)(\mathrm{d}x^0)^2\\ &-\frac{2a((x^1)^2 + a^2 - \Delta)(\sin x^2)^2}{\Sigma}\mathrm{d}x^0\mathrm{d}x^3\\ &+\left[\frac{((x^1)^2+a^2)^2 - \Delta a^2(\sin x^2)^2}{\Sigma}\right](\sin x^2)^2(\mathrm{d}x^3)^2\\ &+\frac{\Sigma}{\Delta}(\mathrm{d}x^1)^2 + \Sigma(\mathrm{d}x^2)^2, \end{aligned} \quad (1.2)$$

where we put

$$\Delta = (x^1)^2 + a^2 + e^2 - 2mx^1, \quad \Sigma = (x^1)^2 + a^2(\cos x^2)^2.$$

The metric (1.1) was intensively studied with respect to the $(1+3)$ threading of almost FLRW universes. From Chapters 10 and 11 of [1] we can see that the study is not an easy one in literature. Also, as far we know, very little has been done with respect to the $(1+3)$ threading theory for the metric (1.2) (cf. [2,3]).

In this paper we present a new approach on the $(1+3)$ threading of spacetime with respect to a congruence of curves defined by an arbitrary timelike vector field ξ. We develop a method that is based on the following concepts:

(i) Threading frame and coframe fields.
(ii) Spatial tensor fields.
(iii) Riemannian spatial connection.

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The threading frame and coframe fields are naturally constructed from the coordinate fields [cf. (2.3) and (2.4)], and have a great role throughout the paper. The spatial tensor fields have been used in earlier literature, but here we work only with their 3D local components with respect to the above special frames [cf. (3.1)]. This brings a substantial simplification into the study of such general metrics. Finally, the Riemannian spatial connection $\nabla^{\star}$ [cf. (3.12)] is a metric linear connection on the spatial distribution, which introduces both the spatial and the time covariant derivatives. It is important to note that throughout the paper, all geometric objects and equations involved into the study, are expressed in terms of spatial tensor fields and their spatial or time covariant derivatives. As the metrics (1.1) and (1.2) fall into the class of the general metrics given by (2.8), the $(1+3)$ threading theory developed here can easily be applied to their study.

Now, we outline the content of the paper. In Sect. 2 we consider the orthogonal decomposition (2.1) of the tangent bundle of the spacetime (M, g), and construct the threading frame and coframe fields $\{\partial/\partial x^0, \delta/\delta x^i\}$ and $\{\delta x^0, \mathrm{d}x^i\}$, respectively [cf. (2.3) and (2.4)]. Also, we consider the Riemannian metric h on the spatial distribution SM given by its 3D local components [cf. (2.15)]. In Sect. 3 we introduce the notion of spatial tensor field via 3D local components [cf. (3.1)], and show that the vorticity, expansion and shear tensor fields ω_{ij}, θ_{ij} and σ_{ij} given by (3.5a) and (3.8), are indeed spatial tensor fields. Also, we define the Riemannian spatial connection $\nabla^{\star}$ on SM [cf. (3.11)] and express the Levi-Civita connection ∇ in terms of the local coefficients of $\nabla^{\star}$ and the above kinematic quantities [cf. (3.17)]. A comparison between the concepts defined in this paper and the corresponding ones from earlier literature is done in Sect. 4. In particular, for a unit timelike vector field we obtain (4.14) for kinematic quantities, and we deduce that they do not depend on the Levi-Civita connection of the spacetime. In Sect. 5 we express both the curvature tensor field and the Ricci tensor field of (M, g) by spatial tensor fields and their spatial and time covariant derivatives [cf. (5.3), (5.5a), (5.11), (5.12a)]. Next, in Sect. 6 we obtain Raychaudhuri's equation (6.1) with respect to an arbitrary timelike vector field, which for a congruence of timelike geodesics takes the forms (6.5) or (6.28). It is important to note that (6.28) is the main ingredient used in the proof of Lemma 6.3, which should be considered as a completion of Lemma 6.2, which has been the key in the proof of Penrose–Hawking singularity theorems. Also, we express the non-zero local components of the electric Weyl curvature tensor field in terms of spatial tensor fields [cf. (6.20)], and deduce new formulas for time covariant derivatives of the kinematic quantities [cf. (6.9), (6.12), (6.24), (6.25)]. Finally, the last three sections are devoted to the study of a Kerr–Newman black hole via the new approach on the $(1+3)$ threading of spacetime developed in the paper. In particular, we characterize spatial geodesics and obtain the 3D force identity [cf. (9.10)].

2 Threading frame and coframe fields

Let (M, g) be a 4D spacetime, and ξ be a timelike vector field that is globally defined on M. Note that ξ is not necessarily a unit timelike vector field, as it was considered in the early literature. The timelike congruence determined by ξ is tangent to the fibers of the line bundle VM, which we call the *time distribution*. Also, we consider the *spatial distribution* SM, which is complementary orthogonal to VM in TM, that is, we have

$$\mathrm{TM} = \mathrm{VM} \oplus \mathrm{SM}. \tag{2.1}$$

Throughout the paper we use the ranges of indices: $i, j, k, \ldots \in \{1, 2, 3\}$ and $a, b, c, \ldots \in \{0, 1, 2, 3\}$. Also, for any vector bundle E over M denote by $\Gamma(E)$ the $\mathcal{F}(M)$-module of smooth sections of E, where $\mathcal{F}(M)$ is the algebra of smooth functions on M.

The foliation by curves that is tangent to VM, induces a special coordinate system (x^a) such that $\xi = \partial/\partial x^0$. If $(\tilde{x}^a)$ is another coordinate system, then we have

$$\tilde{x}^i = \tilde{x}^i(x^1, x^2, x^3); \quad \tilde{x}^0 = x^0 + f(x^1, x^2, x^3), \tag{2.2}$$

since $\partial/\partial x^0$ and $\partial/\partial \tilde{x}^0$ represent the same vector field ξ, and hence $\partial \tilde{x}^0/\partial x^0 = 1$. From (2.1) we deduce that for each $\partial/\partial x^i$ there exist a unique $\delta/\delta x^i \in \Gamma(\mathrm{SM})$ and a unique function A_i, such that

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - A_i \frac{\partial}{\partial x^0}. \tag{2.3}$$

This enables us to consider the *threading frame field* $\{\partial/\partial x^0, \delta/\delta x^i\}$, and the *threading coframe field* $\{\delta x^0, \mathrm{d}x^i\}$, where we put

$$\delta x^0 = \mathrm{d}x^0 + A_i \mathrm{d}x^i. \tag{2.4}$$

Now, by direct calculations using (2.1)–(2.3), we obtain

$$\text{(a)}\ \frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\delta}{\delta \tilde{x}^k}, \quad \text{(b)}\ \delta \tilde{x}^0 = \delta x^0, \quad \text{(c)}\ A_i = \widetilde{A}_k \frac{\partial \tilde{x}^k}{\partial x^i} + \frac{\partial f}{\partial x^i}. \tag{2.5}$$

Note that $\{\delta/\delta x^i\}$ are transformed exactly as $\{\partial/\partial x^i\}$ on a 3D manifold, while $\{A_i\}$, in general, do not satisfy some 3D tensorial transformations.

Next, we consider the 1-form $\xi^{\star}$ given by

$$\xi^{\star}(X) = g(X, \xi), \quad \forall X \in \Gamma(\mathrm{TM}). \tag{2.6}$$

The local components of $\xi^{\star}$ with respect to the natural frame field $\{\partial/\partial x^i\}$ are given by

$$\text{(a)}\ \xi_i = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^0}\right), \quad \text{(b)}\ \xi_0 = g\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0}\right) = -\Phi^2, \tag{2.7}$$

where Φ is a non-zero function on M which is independent of x^0. The above condition on Φ is not restrictive for our theory, because most of the important cosmological models satisfy it.

According to (2.7), the line element of g is expressed as follows:

$$\mathrm{d}s^2 = -\Phi^2(\mathrm{d}x^0)^2 + 2\xi_i \mathrm{d}x^i \mathrm{d}x^0 + g_{ij}\mathrm{d}x^i \mathrm{d}x^j, \tag{2.8}$$

where we put

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right). \tag{2.9}$$

Taking into account that

$$g\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^0}\right) = 0, \tag{2.10}$$

and using (2.3) and (2.7), we obtain

$$A_i = -\Phi^{-2}\xi_i, \tag{2.11}$$

and therefore

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + \Phi^{-2}\xi_i \frac{\partial}{\partial x^0}. \tag{2.12}$$

Multiplying (2.5c) by Φ^2, and using (2.11), we obtain

$$\xi_i = \tilde{\xi}_k \frac{\partial \tilde{x}^k}{\partial x^i} - \Phi^2 \frac{\partial f}{\partial x^i}. \tag{2.13}$$

Hence the ξ_i *do not define a* 3D *1-form on* M.

Now, denote by h the Riemannian metric induced by g on SM, and put

$$h_{ij} = h\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = g\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right). \tag{2.14}$$

Then by using (2.14), (2.3), (2.11), (2.9), and (2.7), we infer that

$$h_{ij} = g_{ij} + \Phi^{-2}\xi_i\xi_j. \tag{2.15}$$

Thus $\mathrm{d}s^2$ from (2.8) is expressed in terms of the threading coframe field $\{\delta x^0, \mathrm{d}x^i\}$ as follows:

$$\mathrm{d}s^2 = -\Phi^2(\delta x^0)^2 + h_{ij}\mathrm{d}x^i \mathrm{d}x^j. \tag{2.16}$$

Note that h_{ij} and the entries h^{ij} of the inverse of the matrix $[h_{ij}]$ are transformed exactly like 3D tensor fields, that is, we have

$$\text{(a)}\ h_{ij} = \widetilde{h}_{kh}\frac{\partial \widetilde{x}^k}{\partial x^i}\frac{\partial \widetilde{x}^h}{\partial x^j}, \quad \text{(b)}\ \widetilde{h}^{kh} = h^{ij}\frac{\partial \widetilde{x}^k}{\partial x^i}\frac{\partial \widetilde{x}^h}{\partial x^j}. \tag{2.17}$$

3 Kinematic quantities as spatial tensor fields on (M, g)

The purpose of this section is to define the vorticity tensor field, expansion tensor field, expansion scalar and shear tensor field, as spatial tensor fields on the spacetime (M, g). First, we give the following definition.

A *spatial tensor field* T of type (p, q) on M is locally given by 3^{p+q} locally defined functions $T^{j_1\ldots j_p}_{i_1\ldots i_q}(x)$, satisfying

$$T^{j_1\ldots j_p}_{i_1\ldots i_q}\frac{\partial \tilde{x}^{k_1}}{\partial x^{j_1}}\cdots\frac{\partial \tilde{x}^{k_p}}{\partial x^{j_p}} = \tilde{T}^{k_1\ldots k_p}_{h_1\ldots h_q}\frac{\partial \tilde{x}^{h_1}}{\partial x^{i_1}}\cdots\frac{\partial \tilde{x}^{h_q}}{\partial x^{i_q}}, \tag{3.1}$$

with respect to the transformations (2.2). In other words, the local components of a spatial tensor field on M should satisfy the same transformations as the local components of a tensor field on a three-dimensional manifold. From (2.17) we see that h_{ij} (resp. h^{ij}) define a spatial tensor field of type (0,2) [resp. (2,0)] on M.

By using (2.5a) and taking into account that

$$\frac{\partial \Phi}{\partial x^0} = 0, \tag{3.2}$$

we deduce that

$$c_i = \Phi^{-1}\frac{\delta \Phi}{\delta x^i} = \Phi^{-1}\frac{\partial \Phi}{\partial x^i} \tag{3.3}$$

define a spatial tensor field of type (0,1). Next, by direct calculations using (2.3), (2.11), and (3.3), we deduce that

$$\text{(a)}\ \left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right] = 2\omega_{ij}\frac{\partial}{\partial x^0}, \quad \text{(b)}\left[\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i}\right] = a_i\frac{\partial}{\partial x^0}, \tag{3.4}$$

where we put

$$\text{(a)}\ \omega_{ij} = \frac{1}{2}\left\{\frac{\delta A_j}{\delta x^i} - \frac{\delta A_i}{\delta x^j}\right\} = \Phi^{-2}\left\{c_i\xi_j - c_j\xi_i + \frac{1}{2}\left(\frac{\delta \xi_i}{\delta x^j} - \frac{\delta \xi_j}{\delta x^i}\right)\right\}, \tag{3.5}$$

$$\text{(b)}\ a_i = -\frac{\partial A_i}{\partial x^0} = \Phi^{-2}\frac{\partial \xi_i}{\partial x^0}.$$

Now, applying $\delta/\delta x^j$ and $\partial/\partial x^0$ to (2.5c) and by using (2.5a) and (2.5b), we infer that

$$\begin{aligned}&\text{(a)}\ \frac{\delta A_i}{\delta x^j} = \frac{\delta \widetilde{A}_k}{\delta \widetilde{x}^h}\frac{\partial \widetilde{x}^h}{\partial x^j}\frac{\partial \widetilde{x}^k}{\partial x^i} + \widetilde{A}_k\frac{\partial^2 \widetilde{x}^k}{\partial x^i \partial x^j} + \frac{\partial^2 f}{\partial x^i \partial x^j},\\ &\text{(b)}\ \frac{\partial A_i}{\partial x^0} = \frac{\partial \widetilde{A}_k}{\partial \widetilde{x}^0}\frac{\partial \widetilde{x}^k}{\partial x^i}.\end{aligned} \tag{3.6}$$

Then, by using (3.6) into (3.5), we obtain

$$\text{(a)}\ \omega_{ij} = \widetilde{\omega}_{kh}\frac{\partial \widetilde{x}^k}{\partial x^i}\frac{\partial \widetilde{x}^h}{\partial x^j}, \quad \text{(b)}\ a_i = \widetilde{a}_k\frac{\partial \widetilde{x}^k}{\partial x^i}. \tag{3.7}$$

Hence ω_{ij} and a_i define spatial tensor fields of type (0.2) and (0,1), respectively. We call $\omega = (\omega_{ij})$ the *vorticity tensor field* for the timelike congruence defined by ξ on M. From (3.4a) we see that the spatial distribution SM is integrable if and only if ω_{ij} vanish identically on M.

Next, we define

$$\text{(a)}\ \Theta_{ij} = \frac{1}{2}\frac{\partial h_{ij}}{\partial x^0}, \quad \text{(b)}\ \Theta = h^{ij}\Theta_{ij}, \quad (c)\ \sigma_{ij} = \Theta_{ij} - \frac{1}{3}\Theta h_{ij}. \tag{3.8}$$

Then we take derivatives with respect to x^0 in (2.17a) and obtain

$$\Theta_{ij} = \widetilde{\Theta}_{kh}\frac{\partial \widetilde{x}^k}{\partial x^i}\frac{\partial \widetilde{x}^h}{\partial x^j},$$

that is, the Θ_{ij} define a spatial tensor field of type (0,2). We call it the *expansion tensor field*. Clearly Θ from (3.8b) is a function, and σ_{ij} from (3.8c) define a trace-free spatial tensor field of type (0,2). We call Θ the *expansion scalar* and σ_{ij} the *shear tensor field* for the congruence. According to the terminology from earlier literature, we call $\{\omega_{ij}, \Theta_{ij}, \Theta, \sigma_{ij}\}$ given by (3.5a) and (3.8), the *kinematic quantities* with respect to the congruence of curves defined by the timelike vector field $\xi = \partial/\partial x^0$.

Raising and lowering Latin indices is done by using h^{ij} and h_{ij}, as follows:

$$\text{(a) } \omega_j^k = h^{ki}\omega_{ij}, \quad \text{(b) } \omega^{kh} = h^{ki}h^{hj}\omega_{ij}. \tag{3.9}$$

In order to define covariant derivatives of the above kinematic quantities, we consider the Levi-Civita connection ∇ on (M, g) given by (cf. [4, p. 61])

$$\begin{aligned}2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad + g([X,Y],Z) - g([Y,Z],X) \\ &\quad + g([Z,X],Y),\end{aligned} \tag{3.10}$$

for all $X, Y, Z \in \Gamma(\mathrm{TM})$. We define the linear connection $\nabla^\star$ on the spatial distribution as the spatial projection of ∇ on SM, that is, we have

$$\text{(a) } \nabla^\star_X sY = s\nabla_X sY, \quad \forall\ X, Y \in (\Gamma(\mathrm{SM}), \tag{3.11}$$

where s is the projection morphism of TM on SM with respect to (2.1). Note that $\nabla^\star$ is a metric linear connection on SM. We call it the *Riemannian spatial connection*.

Remark 3.1 The Riemannian spatial connection $\nabla^\star$ is different from the three-dimensional operator $\bar{\nabla}$, which has been used in the earlier literature (cf. (4.19) of [1]). $\nabla^\star$ is a linear connection on SM and therefore defines covariant derivatives of any spatial tensor field with respect to vector fields on M. On the contrary, $\bar{\nabla}$ is an operator which acts on tensor fields on M, but, in general, it does not define a linear connection on M.

Locally, we put

$$\text{(a) } \nabla^\star_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} = \Gamma^{\star\ k}_{i\ j}\frac{\delta}{\delta x^k}, \quad \text{(b) } \nabla^\star_{\frac{\partial}{\partial x^0}}\frac{\delta}{\delta x^i} = \Gamma^{\star\ k}_{i\ 0}\frac{\delta}{\delta x^k}. \tag{3.12}$$

We take $X = \delta/\delta x^j$, $Y = \delta/\delta x^i$ and $Z = \delta/\delta x^h$ in (3.10) and by using (3.11), (3.12a), (2.14), and (3.4a), we obtain

$$\Gamma^{\star\ k}_{i\ j} = \frac{1}{2}h^{kh}\left\{\frac{\delta h_{hj}}{\delta x^i} + \frac{\delta h_{hi}}{\delta x^j} - \frac{\delta h_{ij}}{\delta x^h}\right\}. \tag{3.13}$$

Similarly, we deduce that

$$\Gamma^{\star\ k}_{i\ 0} = \Theta_i^k + \Phi^2\omega_i^k. \tag{3.14}$$

Now, consider a spatial tensor field T of type (p, q). Then $\nabla^\star_{\frac{\delta}{\delta x^k}} T$ and $\nabla^\star_{\frac{\partial}{\partial x^0}} T$ are spatial tensor of type $(p, q+1)$ and (p, q), respectively. As an example, we consider $T = (T_j^i)$, and obtain

$$\begin{aligned}&\text{(a) } T^i_{j|k} = \frac{\delta T^i_j}{\delta x^k} + T^h_j\Gamma^{\star\ i}_{h\ k} - T^i_h\Gamma^{\star\ h}_{j\ k},\\ &\text{(b) } T^i_{j|0} = \frac{\partial T^i_j}{\partial x^0} + T^h_j\Gamma^{\star\ i}_{h\ 0} - T^i_h\Gamma^{\star\ h}_{j\ 0}.\end{aligned} \tag{3.15}$$

We call (3.15a) [resp. (3.15b)] the *spatial* (resp. *time*) *covariant derivative* of T. As $\nabla^\star$ is a metric connection on SM, we have

$$\text{(a) } h_{ij|k} = 0, \quad \text{(b) } h^{ij}_{\ |k} = 0, \quad \text{(c) } h_{ij|0} = 0, \quad \text{(d) } h^{ij}_{\ |0} = 0. \tag{3.16}$$

Finally, by using (3.10), the above spatial tensor fields and the local coefficients of $\nabla^\star$, we express the Levi-Civita connection ∇ on (M, g) as follows:

$$\begin{aligned}&\text{(a) } \nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} = \Gamma^{\star\ k}_{i\ j}\frac{\delta}{\delta x^k} + (\omega_{ij} + \Phi^{-2}\Theta_{ij})\frac{\partial}{\partial x^0},\\ &\text{(b) } \nabla_{\frac{\partial}{\partial x^0}}\frac{\delta}{\delta x^i} = (\Theta_i^k + \Phi^2\omega_i^k)\frac{\delta}{\delta x^k} + (a_i + c_i)\frac{\partial}{\partial x^0},\\ &\text{(c) } \nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial x^0} = (\Theta_i^k + \Phi^2\omega_i^k)\frac{\delta}{\delta x^k} + c_i\frac{\partial}{\partial x^0},\\ &\text{(d) } \nabla_{\frac{\partial}{\partial x^0}}\frac{\partial}{\partial x^0} = \Phi^2(a^k + c^k)\frac{\delta}{\delta x^k}.\end{aligned} \tag{3.17}$$

4 Comparison with concepts from earlier literature

In the previous section we introduced the kinematic quantities on a spacetime (M, g) with respect to the congruence that is tangent to an arbitrary timelike vector field ξ. If in particular, ξ is a unit timelike vector field, the configuration of the spacetime with respect to the congruence of timelike curves determined by ξ is known in the literature as (1 + 3) *threading of spacetime* (cf. [1,5–14]). In this section we show that for $\Phi^2 = 1$ in (3.5) and (3.8) (that is, ξ is a unit vector field), we obtain the well-known kinematic quantities from earlier literature.

First, by using (2.6) and taking into account that ∇ is a metric connection, we obtain

$$(\nabla_X\xi^\star)(Y) = g(Y, \nabla_X\xi), \quad \forall\ X, Y \in \Gamma(\mathrm{TM}). \tag{4.1}$$

Consider the threading frame $\{\partial/\partial x^0, \delta/\delta x^i\}$ and using (3.17), we infer that

$$\begin{aligned}&\text{(a) } \left(\nabla_{\frac{\delta}{\delta x^j}}\xi^\star\right)\left(\frac{\delta}{\delta x^i}\right) = \Theta_{ij} + \Phi^2\omega_{ij},\\ &\text{(b) } \left(\nabla_{\frac{\partial}{\partial x^0}}\xi^\star\right)\left(\frac{\delta}{\delta x^i}\right) = \Phi^2(a_i + c_i),\\ &\text{(c) } \left(\nabla_{\frac{\delta}{\delta x^i}}\xi^\star\right)\left(\frac{\partial}{\partial x^0}\right) = -\Phi^2 c_i,\\ &\text{(d) } \left(\nabla_{\frac{\partial}{\partial x^0}}\xi^\star\right)\left(\frac{\partial}{\partial x^0}\right) = 0.\end{aligned} \tag{4.2}$$

Next, taking into account (2.12) and (2.7b), we express the natural frame field as follows:

$$\frac{\partial}{\partial x^a} = \delta^i_a \frac{\delta}{\delta x^i} - \Phi^{-2}\xi_a \frac{\partial}{\partial x^0}, \quad a \in \{0, 1, 2, 3\}. \tag{4.3}$$

Consider the *covariant acceleration vector field*

$$\dot{\xi}_a = \left(\nabla_{\frac{\partial}{\partial x^0}} \xi^\star\right)\left(\frac{\partial}{\partial x^a}\right), \tag{4.4}$$

and using (4.2b) and (4.2d), we obtain

$$\dot{\xi}_a = \Phi^2 \delta^i_a b_i, \tag{4.5}$$

where we put

$$b_i = a_i + c_i. \tag{4.6}$$

Thus, we deduce that the congruence defined by ξ *is a congruence of timelike geodesics if, and only if, we have*

$$b_i = 0, \quad \forall\, i \in \{1, 2, 3\}. \tag{4.7}$$

For this reason we call b_i the *geodesic spatial covector field* of the congruence.

Now, by direct calculations, using (4.3) and (4.2), we infer that

$$\begin{aligned} &(a)\ \left(\nabla_{\frac{\partial}{\partial x^b}} \xi^\star\right)\left(\frac{\delta}{\delta x^i}\right) = \delta^j_b(\Theta_{ij} + \Phi^2 \omega_{ij}) - b_i \xi_b, \\ &(b)\ \left(\nabla_{\frac{\partial}{\partial x^b}} \xi^\star\right)\left(\frac{\partial}{\partial x^0}\right) = -\Phi^2 \delta^j_b c_j. \end{aligned} \tag{4.8}$$

Taking into account (4.3), (4.8), and (4.5), we find

$$\nabla_b \xi_a = -\Phi^{-2}\xi_b \dot{\xi}_a + \delta^i_a \delta^j_b(\Theta_{ij} + \Phi^2\omega_{ij}) + \xi_a \delta^j_b c_j. \tag{4.9}$$

Now, we suppose that ξ is a unit vector field. According to Eq. (4.38) from [1, p. 85], we have

$$\nabla_b \xi_a = -\xi_b \dot{\xi}_a + \sigma_{ab} + \frac{1}{3}\Theta h_{ab} + \omega_{ab}, \tag{4.10}$$

where σ_{ab}, θ, h_{ab} and ω_{ab} are quantities defined in the earlier literature. On the other hand, in this case we have $\Phi^2 = 1$, and from (4.9) we obtain

$$\nabla_b \xi_a = -\xi_b \dot{\xi}_a + \delta^i_a \delta^j_b(\Theta_{ij} + \omega_{ij}), \tag{4.11}$$

since $c_j = 0$, for all $j \in \{1, 2, 3\}$.

Comparing the symmetric and skew-symmetric parts in (4.10) and (4.11) we deduce that

$$\begin{aligned} &(a)\ \sigma_{ab} = \delta^i_a \delta^j_b \Theta_{ij} - \tfrac{1}{3}\Theta h_{ab}, \\ &(b)\ \omega_{ab} = \delta^i_a \delta^j_b \omega_{ij}. \end{aligned} \tag{4.12}$$

Finally, comparing (4.12a) with (4.31) from [1, p. 81], we obtain

$$\Theta_{ab} = \delta^i_a \delta^j_b \Theta_{ij}. \tag{4.13}$$

According to (3.8), (3.5a), (4.12), and (4.13), we conclude that in case ξ is a unit vector field, the only possible non-zero local components of expansion, shear and vorticity tensor fields from earlier literature are the following:

$$\begin{aligned} &(a)\ \Theta_{ij} = \tfrac{1}{2}\tfrac{\partial h_{ij}}{\partial x^0}, \quad (b)\ \sigma_{ij} = \tfrac{1}{2}\tfrac{\partial h_{ij}}{\partial x^0} - \tfrac{1}{3}\Theta h_{ij}, \\ &(c)\ \omega_{ij} = \tfrac{1}{2}\left\{\tfrac{\delta \xi_i}{\delta x^j} - \tfrac{\delta \xi_j}{\delta x^i}\right\}. \end{aligned} \tag{4.14}$$

As far as we know, Eqs. (4.14) do not appear in earlier literature. Due to them we can state that *the expansion, shear, and vorticity tensor fields do not depend on the Levi-Civita connection of the spacetime* (M, g). Of course, due to (3.5a) and (3.8), this conclusion is still valid for the general case of a congruence defined by an arbitrary timelike vector field ξ.

We close this section with an interesting property of vorticity tensor field. Suppose that we have a congruence of geodesics defined by ξ. According to (4.7), (4.6), (3.3), and (3.5b), we have

$$\frac{\partial \xi_i}{\partial x^0} + \Phi \frac{\partial \Phi}{\partial x^i} = 0. \tag{4.15}$$

By using (4.15), (3.2), and (3.3), we infer that

$$(a)\ \frac{\partial^2 \xi_i}{(\partial x^0)^2} = 0, \ (b)\ c_i \frac{\partial \xi_j}{\partial x^0} = c_j \frac{\partial \xi_i}{\partial x^0}, \ (c)\ \frac{\partial^2 \xi_i}{\partial x^j \partial x^0} = \frac{\partial^2 \xi_j}{\partial x^i \partial x^0}. \tag{4.16}$$

We take the derivative with respect to x^0 in (3.5a) and by (3.2), (3.3), and (4.16), we obtain

$$\frac{\partial \omega_{ij}}{\partial x^0} = 0. \tag{4.17}$$

Thus, we can state that *the vorticity tensor field for a congruence of timelike geodesics of a spacetime is independent of time*. In particular, this is true for a congruence of geodesics with respect to a unit vector field ξ. However, we did not see this result in earlier literature. This is because the formulas (3.5a), (3.8), and (4.14) we deduced for the kinematic quantities are much simpler than the ones by means of Levi-Civita connection.

5 Curvature and Ricci tensor fields of a spacetime via spatial tensor fields

In this section we show that the curvature tensor field of (M, g) is completely determined by three spatial tensor fields R_{ijkh}, R_{i0kh} and R_{i0k0} [cf. (5.3), (5.5a)]. A similar result we obtain for the Ricci tensor of (M, g) [cf. (5.11), (5.12a)]. Note that all these spatial tensor fields are expressed in terms of the curvature and Ricci tensor fields of the of the Riemannian spatial connection, and of all kinematic quantities introduced in Sect. 3.

In the following, R denotes both the curvature tensor fields of (M, g) of type (0,4) and of type (1,3), given by

$$\begin{aligned}&\text{(a) } R(X, Y, Z, U) = g(R(X, Y, U), Z),\\ &\text{(b) } R(X, Y, U) = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U,\end{aligned} \tag{5.1}$$

for all $X, Y, Z, U \in \Gamma(\mathrm{TM})$. Then the curvature tensor field of (M, g) is completely determined by its local components

$$\begin{aligned}&\text{(a) } R_{ijkh} = R\left(\tfrac{\delta}{\delta x^h}, \tfrac{\delta}{\delta x^k}, \tfrac{\delta}{\delta x^j}, \tfrac{\delta}{\delta x^i}\right),\\ &\text{(b) } R_{i0kh} = R\left(\tfrac{\delta}{\delta x^h}, \tfrac{\delta}{\delta x^k}, \tfrac{\partial}{\partial x^0}, \tfrac{\delta}{\delta x^i}\right),\\ &\text{(c) } R_{i0k0} = R\left(\tfrac{\partial}{\partial x^0}, \tfrac{\delta}{\delta x^k}, \tfrac{\partial}{\partial x^0}, \tfrac{\delta}{\delta x^i}\right).\end{aligned} \tag{5.2}$$

By direct calculations, using (5.2), (5.1), (3.17), (3.4), (3.3), and (4.6), we obtain

$$\begin{aligned}\text{(a) } R_{ijkh} &= R^\star_{ijkh} + \omega_{ik}\Theta_{jh} - \omega_{ih}\Theta_{jk} + \Phi^{-2}(\Theta_{ik}\Theta_{jh} - \Theta_{ih}\Theta_{jk})\\ &\quad + \Phi^2(\omega_{ik}\omega_{jh} - \omega_{ih}\omega_{jk}) + \Theta_{ik}\omega_{jh} - \Theta_{ih}\omega_{jk},\\ \text{(b) } R_{i0kh} &= \Theta_{ih|_k} - \Theta_{ik|_h} + \Theta_{ik}c_h - \Theta_{ih}c_k\\ &\quad + \Phi^2\{\omega_{ih|_k} - \omega_{ik|_h} + \omega_{ih}c_k - \omega_{ik}c_h + 2\omega_{kh}b_i\},\\ \text{(c) } R_{i0k0} &= \Phi^2\{b_i b_k + b_{i|_k} + \omega_{kh}\Theta^h_i - \omega_{ih}\Theta^h_k - \omega_{ik|_0} - \Phi^2\omega_{ih}\omega^h_k\}\\ &\quad - \Theta_{ik|_0} - \Theta_{ih}\Theta^h_k,\end{aligned} \tag{5.3}$$

where $R^\star_{ijkh}$ are the local components of the curvature tensor field of the Riemannian spatial connection defined as in (5.2a) and given by

$$R^\star_{ijkh} = h_{jl}\left\{\frac{\delta\Gamma^{\star l}_{i\ k}}{\delta x^h} - \frac{\delta\Gamma^{\star l}_{i\ h}}{\delta x^k} + \Gamma^{\star n}_{i\ k}\Gamma^{\star l}_{n\ h} - \Gamma^{\star n}_{i\ h}\Gamma^{\star l}_{n\ k} - 2\omega_{kh}(\Theta^l_i + \Phi^2\omega^l_i)\right\}. \tag{5.4}$$

Taking the symmetric and skew-symmetric parts, in (5.3c) we deduce that

$$\begin{aligned}\text{(a) } R_{i0k0} &= \Phi^2\left\{b_i b_k + \tfrac{1}{2}(b_{i|_k} + b_{k|_i}) - \Phi^2\omega_{ih}\omega^h_k\right\}\\ &\quad - \Theta_{ik|_0} - \Theta_{ih}\Theta^h_k,\\ \text{(b) } \omega_{ik|_0} &= \omega_{kh}\Theta^h_i - \omega_{ih}\Theta^h_k + \tfrac{1}{2}(b_{i|_k} - b_{k|_i}).\end{aligned} \tag{5.5}$$

Now, consider an orthonormal frame field $\{E_k, \Phi^{-1}\frac{\partial}{\partial x^0}\}$ on M and put

$$E_k = E^i_k \frac{\delta}{\delta x^i}. \tag{5.6}$$

We deduce that

$$h^{ij} = \sum_{k=1}^{3} E^i_k E^j_k. \tag{5.7}$$

According to [4, p. 87], the Ricci tensor of (M, g) is given by

$$\mathrm{Ric}(X, Y) = \sum_{k=1}^{3} R(E_k, X, E_k, Y) - \Phi^{-2} R\left(\frac{\partial}{\partial x^0}, X, \frac{\partial}{\partial x^0}, Y\right). \tag{5.8}$$

By using (5.8), (5.6), (5.7), and (5.2), we obtain

$$\begin{aligned}&\text{(a) } R_{ik} = h^{jh}R_{ijkh} - \Phi^{-2}R_{i0k0}, \quad \text{(b) } R_{i0} = h^{jh}R_{j0hi},\\ &\text{(c) } R_{00} = h^{jh}R_{j0h0},\end{aligned} \tag{5.9}$$

where we put

$$\begin{aligned}&\text{(a) } R_{ik} = \mathrm{Ric}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^i}\right), \quad \text{(b) } R_{i0} = \mathrm{Ric}\left(\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i}\right),\\ &\text{(c) } R_{00} = \mathrm{Ric}\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0}\right).\end{aligned} \tag{5.10}$$

By using (5.3a), (5.3b), and (5.5a) into (5.9), we deduce that

$$\begin{aligned}\text{(a) } R_{ik} &= R^{\star h}_{i\ kh} + \Phi^{-2}(\Theta_{ik|_0} + \Theta\Theta_{ik}) - b_i b_k - \tfrac{1}{2}(b_{i|_k} + b_{k|_i})\\ &\quad + \Theta\omega_{ik} + \omega_{kh}\Theta^h_i - \omega_{ih}\Theta^h_k,\\ \text{(b) } R_{i0} &= \Theta^k_{i|_k} - \Theta_{|_i} + \Theta c_i - \Theta^k_i c_k\\ &\quad + \Phi^2\{\omega^{\ k}_{i\ |_k} + \omega^k_i c_k + 2\omega^k_i b_k\},\\ \text{(c) } R_{00} &= \Phi^2\{b_k b^k + b^k_{\ |_k} + \Phi^2\omega_{kh}\omega^{kh}\} - \Theta_{|_0} - \Theta_{kh}\Theta^{kh},\end{aligned} \tag{5.11}$$

where $\Theta^k_{i|_k}$, $\omega^{\ k}_{i\ |_k}$ and $b^k_{\ |_k}$ are *spatial divergences* given by formulas deduced from (3.15a). Now, we take symmetric and skew-symmetric parts in (5.11a) and obtain

$$\begin{aligned}&\text{(a) } R_{ik} = R^\star_{ik} + \Phi^{-2}(\Theta_{ik|_0} + \Theta\Theta_{ik}) - b_i b_k - \frac{1}{2}(b_{i|_k} + b_{k|_i}),\\ &\text{(b) } \frac{1}{2}(R^{\star h}_{i\ kh} - R^{\star h}_{k\ ih}) = \omega_{ih}\Theta^h_k - \omega_{kh}\Theta^h_i - \Theta\omega_{ik},\end{aligned} \tag{5.12}$$

where we put

$$R^\star_{ik} = \frac{1}{2}(R^{\star h}_{i\ kh} + R^{\star h}_{k\ ih}). \tag{5.13}$$

We call $R^\star_{ik}$ the *spatial Ricci tensor* of the spacetime (M, g).

From (5.12b) we see that if the spatial distribution is integrable, then we have

$$R^{\star h}_{i\ kh} = R^{\star h}_{k\ ih}. \tag{5.14}$$

In this case, we have

$$R^\star_{ik} = R^{\star h}_{i\ kh}. \tag{5.15}$$

Also, note that because in this particular case the vorticity vanishes identically, from (5.3a), (5.3b), (5.5a), (5.11a), (5.11b) and (5.12a), we deduce that the curvature and Ricci tensors on (M, g) are expressed as follows:

$$\begin{aligned}&\text{(a) } R_{ijkh} = R^\star_{ijkh} + \Phi^{-2}(\Theta_{ik}\Theta_{jh} - \Theta_{ih}\Theta_{jk}),\\ &\text{(b) } R_{i0kh} = \Theta_{ih|_k} - \Theta_{ik|_h} + \Theta_{ik}c_h - \Theta_{ih}c_k,\\ &\text{(c) } R_{i0k0} = \Phi^2\left\{b_i b_k + \tfrac{1}{2}(b_{i|_k} + b_{k|_i})\right\} - \Theta_{ik|_0} - \Theta_{ih}\Theta^h_k,\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}&\text{(a) } R_{ik} = R^\star_{ik} + \Phi^{-2}(\Theta_{ik|_0} + \Theta\Theta_{ik}) - b_i b_k - \tfrac{1}{2}(b_{i|_k} + b_{k|_i}),\\ &\text{(b) } R_{i0} = \Theta^k_{i|_k} - \Theta_{|_i} + \Theta c_i - \Theta^k_i c_k,\\ &\text{(c) } R_{00} = \Phi^2\{b_h b^h + b^h_{\ |_h}\} - \Theta_{|_0} - \Theta_{kh}\Theta^{kh},\end{aligned} \tag{5.17}$$

where $R^\star_{ik}$ is given by (5.15).

6 Raychaudhuri's equation and time covariant derivatives of kinematic quantities

First, by putting (3.8b) and (3.8c) into (5.11c), we infer

$$\Theta_{|0} = \Phi^4\omega^2 - \sigma^2 - \frac{1}{3}\Theta^2 + \Phi^2(b^h_{\ |h} + b^2) - R_{00}, \tag{6.1}$$

where we put

$$\text{(a)}\ \omega^2 = \omega_{kh}\omega^{kh}, \quad \text{(b)}\ \sigma^2 = \sigma_{kh}\sigma^{kh}, \quad \text{(c)}\ b^2 = b_h b^h. \tag{6.2}$$

In particular, if ξ is a unit timelike vector field, that is,

$$\Phi^2 = 1, \tag{6.3}$$

(6.1) becomes

$$\Theta_{|0} = \omega^2 - \sigma^2 - \frac{1}{3}\Theta^2 + b^h_{\ |h} + b^2 - R_{00}, \tag{6.4}$$

which is Raychaudhuri's equation expressed in terms of local components of spatial tensor fields introduced in the present paper. Thus, we are entitled to call (6.1) the *generalized Raychaudhuri equation* with respect to a congruence defined by an arbitrary timelike vector field ξ.

According to (4.7), in the case of a timelike congruence of geodesics, (6.1) and (6.4) become

$$\Theta_{|0} = \Phi^4\omega^2 - \sigma^2 - \frac{1}{3}\Theta^2 - R_{00}, \tag{6.5}$$

and

$$\Theta_{|0} = \omega^2 - \sigma^2 - \frac{1}{3}\Theta^2 - R_{00}, \tag{6.6}$$

respectively.

Remark 6.1 Formally, (6.6) looks like (9.2.11) in [2], but we should note that ω, σ and Θ from (6.6) are calculated via their 3D spatial components [see (3.5a), (3.8), and (6.2)], while in [2] they are calculated in terms of the 4D local components with respect to the natural frame field $\{\partial/\partial x^a\}$.

Next, observe that (5.5b) gives a formula for the time covariant derivative of vorticity tensor field. By using (3.8b) and (3.8c) into (5.5b) we find

$$\omega_{ik|0} = \omega_{kh}\sigma^h_i - \omega_{ih}\sigma^h_k - \frac{2}{3}\Theta\omega_{ik} + \frac{1}{2}(b_{i|k} - b_{k|i}). \tag{6.7}$$

Now, from (5.5a) we deduce that the time covariant derivative of the expansion tensor field is given by

$$\Theta_{ik|0} = \Phi^2\left\{b_i b_k + \frac{1}{2}(b_{i|k} + b_{k|i}) - \Phi^2\omega_{ih}\omega^h_k\right\} - \Theta_{ih}\Theta^h_k - R_{i0k0}. \tag{6.8}$$

Another formula in terms of Ricci tensors is deduced from (5.12a):

$$\Theta_{ik|0} = -\Theta\Theta_{ik} + \Phi^2\left\{b_i b_k + \frac{1}{2}(b_{i|k} + b_{k|i}) + R_{ik} - R^\star_{ik}\right\}. \tag{6.9}$$

Taking the time covariant derivative in (3.8c), and using (3.16c), (6.8), and (6.1), we infer that

$$\sigma_{ik|0} = \Phi^2\Big\{b_i b_k + \frac{1}{2}(b_{i|k} + b_{k|i}) - \frac{1}{3}(b^h_{\ |h} + b^2)h_{ik} - \Phi^2(\omega_{ih}\omega^h_k + \frac{1}{3}\omega^2 h_{ik})\Big\} + \frac{1}{3}\sigma^2 h_{ik} - \sigma_{ih}\sigma^h_k - \frac{2}{3}\Theta\sigma_{ik} - \widetilde{R}_{i0k0}, \tag{6.10}$$

where $\widetilde{R}_{i0k0}$ is the trace-free part of the spatial tensor field R_{i0k0}, given by

$$\widetilde{R}_{i0k0} = R_{i0k0} - \frac{1}{3}R_{00}h_{ik}. \tag{6.11}$$

In a similar way, but using (6.9) instead of (6.8), we obtain

$$\sigma_{ik|0} = -\Theta\sigma_{ik} + \frac{1}{3}\left(\sigma^2 - \frac{2}{3}\Theta^2 + R_{00}\right)h_{ik} + \Phi^2\Big\{b_i b_k + \frac{1}{2}(b_{i|k} + b_{k|i}) - \frac{1}{3}(b^h_{\ |h} + b^2 + \Phi^2\omega^2)h_{ik} + R_{ik} - R^\star_{ik}\Big\}. \tag{6.12}$$

Now consider the Weyl tensor field in (M, g), given by

$$C_{abcd} = \bar{R}_{abcd} + \frac{1}{2}\{g_{ad}\bar{R}_{bc} + g_{bc}\bar{R}_{ad} - g_{ac}\bar{R}_{bd} - g_{bd}\bar{R}_{ac}\} + \frac{1}{6}\mathbf{R}\{g_{ac}g_{bd} - g_{ad}g_{bc}\}, \tag{6.13}$$

where we put

$$\begin{aligned} &\text{(a)}\ C_{abcd} = C\left(\tfrac{\partial}{\partial x^d}, \tfrac{\partial}{\partial x^c}, \tfrac{\partial}{\partial x^b}, \tfrac{\partial}{\partial x^a}\right), \\ &\text{(b)}\ \bar{R}_{abcd} = R\left(\tfrac{\partial}{\partial x^d}, \tfrac{\partial}{\partial x^c}, \tfrac{\partial}{\partial x^b}, \tfrac{\partial}{\partial x^a}\right), \\ &\text{(c)}\ \bar{R}_{ab} = \text{Ric}\left(\tfrac{\partial}{\partial x^b}, \tfrac{\partial}{\partial x^a}\right), \end{aligned} \tag{6.14}$$

and $\mathbf{R}$ is the scalar curvature of (M, g). We consider the *electric Weyl curvature tensor field* $E = (E_{ac})$, given by

$$E_{ac} = E\left(\frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^a}\right) = C_{abcd}\xi^b\xi^d, \tag{6.15}$$

and taking into account that $\xi = \partial/\partial x^0$, we obtain

$$E_{ac} = C_{a0c0} = C\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^a}\right). \tag{6.16}$$

Then by direct calculations, using (6.16), (6.13), (2.7), (2.9), and (2.15), we deduce that the only possible non-zero local components of E with respect to the natural frame field are

$$E_{ik} = \bar{R}_{i0k0} + \frac{1}{2}\{\xi_i\bar{R}_{k0} + \xi_k\bar{R}_{i0} - g_{ik}\bar{R}_{00} + \Phi^2\bar{R}_{ik}\} - \frac{1}{6}\mathbf{R}\Phi^2 h_{ik}. \tag{6.17}$$

Note that due to (2.3) and (6.16), we have

$$E_{ik} = C\left(\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^k}, \frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i}\right) = E\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^i}\right). \tag{6.18}$$

By using (2.5a), from (6.18) we infer that the E_{ik} define a spatial tensor field of type (0,2). Using (2.12) in (6.14b) and (6.14c), we obtain

$$\text{(a)}\ \bar{R}_{i0k0} = R_{i0ko},\quad \text{(b)}\ \bar{R}_{i0} = R_{i0} - \Phi^{-2}\xi_i R_{00},\quad \text{(c)}\ \bar{R}_{00} = R_{00}, \tag{6.19}$$
$$\text{(d)}\ \bar{R}_{ik} = R_{ik} - \Phi^{-2}\{\xi_i R_{k0} + \xi_k R_{i0} - \Phi^{-2}\xi_i\xi_k R_{00}\},$$

where R_{i0k0} and $\{R_{ik}, R_{i0}, R_{00}\}$ are given by (5.2c) and (5.10), respectively. Taking into account of (6.19) into (6.17) and using (2.15), we express E_{ik} in terms of spatial tensor fields, as follows:

$$E_{ik} = R_{i0k0} + \frac{1}{2}\left\{\Phi^2 R_{ik} - \left(R_{00} + \frac{1}{3}\mathbf{R}\Phi^2\right) h_{ik}\right\}. \tag{6.20}$$

The scalar curvature $\mathbf{R}$ of (M, g) is given by

$$\begin{aligned}\mathbf{R} &= \sum_{k=1}^{3}\mathrm{Ric}(E_k, E_k) - \Phi^{-2}\mathrm{Ric}\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0}\right)\\ &= h^{jh}R_{jh} - \Phi^{-2}R_{00}.\end{aligned} \tag{6.21}$$

We replace $\mathbf{R}$ from (6.21) into (6.20) and taking into account (6.11), we deduce that

$$E_{ik} = \widetilde{R}_{i0k0} + \frac{1}{2}\Phi^2\widetilde{R}_{ik}, \tag{6.22}$$

where $\widetilde{R}_{ik}$ is the trace-free part of R_{ik}, that is, we have

$$\widetilde{R}_{ik} = R_{ik} - \frac{1}{3}h^{jh}R_{jh}h_{ik}. \tag{6.23}$$

Finally, by using (6.11) and (6.22) into (6.8) and (6.10), we obtain

$$\begin{aligned}\Theta_{ik|0} &= \Phi^2\left\{b_i b_k + \frac{1}{2}(b_{i|_k} + b_{k|_i}) - \Phi^2\omega_{ih}\omega^h_k\right\} - \Theta_{ih}\Theta^h_k\\ &\quad -E_{ik} + \frac{1}{2}\Phi^2\widetilde{R}_{ik} - \frac{1}{3}R_{00}h_{ik}\end{aligned} \tag{6.24}$$

and

$$\begin{aligned}\sigma_{ik|0} &= \Phi^2\Bigg\{b_i b_k + \frac{1}{2}(b_{i|_k} + b_{k|_i}) - \frac{1}{3}(b^h_{\ |_h} + b^2)h_{ik}\\ &\quad -\Phi^2\left(\omega_{ih}\omega^h_k + \frac{1}{3}\omega^2 h_{ik}\right)\Bigg\} + \frac{1}{3}\sigma^2 h_{ik} - \sigma_{ih}\sigma^h_k - \frac{2}{3}\Theta\sigma_{ik}\\ &\quad -E_{ik} + \frac{1}{2}\Phi^2\widetilde{R}_{ik},\end{aligned} \tag{6.25}$$

respectively.

It is interesting to note that the generalized Raychaudhuri equation (6.1) can be expressed by using the scalar curvature $\mathbf{R}$ of (M, g) and the *spatial scalar curvature* $\mathbf{R}^\star$ of $\nabla^\star$ given by

$$\mathbf{R}^\star = h^{jh}R^\star_{jh}. \tag{6.26}$$

Indeed, contracting (6.9) by h^{ik} and using (3.16d), (3.8b), (6.21), and (6.26) we deduce that

$$\Theta_{|0} = -\Theta^2 + \Phi^2\{b^2 + b^h_{\ |_h} + \mathbf{R} - \mathbf{R}^\star\} + R_{00}. \tag{6.27}$$

In particular, if ξ is a unit vector field that defines a timelike congruence of geodesics [see (6.3) and (4.7)], then (6.27) becomes

$$\Theta_{|0} = -\Theta^2 + \mathbf{R} - \mathbf{R}^\star + R_{00}. \tag{6.28}$$

This is a *new form of Raychaudhuri's equation* (6.6) *for a congruence of timelike geodesics*. It is well known that (6.6) is the key equation used in the proof of the Penrose–Hawking singularity theorems. More precisely, one proved the following lemma.

Lemma 6.2 (See Lemma 9.2.1 in [2]) *Let ξ be the tangent field of a hypersurface orthogonal timelike geodesic congruence. Suppose the following conditions are satisfied:*

(i) $\mathrm{Ric}(\xi, \xi) \geq 0$*, which is the case if Einstein's equations hold in the spacetime and the strong energy condition is satisfied by the matter.*
(ii) *The expansion Θ takes the negative value Θ_0 at a point on a geodesic in the congruence corresponding to the proper time $\tau = 0$.*

Then Θ goes to $-\infty$ along that geodesic within the proper time $\tau \leq \frac{3}{|\Theta_0|}$.

Note that in the above lemma, that ξ is a tangent field of a hypersurface orthogonal timelike geodesic congruence means that SM is an integrable distribution.

Now, by using the new form (6.28) of Raychaudhuri's equation we can complete Lemma 6.2 with the following lemma.

Lemma 6.3 *Let the congruence of timelike geodesics satisfying the conditions from Lemma 6.2. Then we have the following assertions:*

(a) *If $\mathbf{R} \geq \mathbf{R}^\star$, then the proper time τ must be in the interval* $[1/|\Theta_0|, 3/|\Theta_0|]$.
(b) *If $\mathbf{R} < \mathbf{R}^\star$, then the following cases occur:*
   (b_1) *If* $\mathrm{Ric}(\xi, \xi) \geq \mathbf{R}^\star - \mathbf{R}$, *then τ must be in the interval* $[1/|\Theta_0|, 3/|\Theta_0|]$.
   (b_2) *If* $\mathrm{Ric}(\xi, \xi) < \mathbf{R}^\star - \mathbf{R}$, *then τ must be in the interval* $[0, 1/|\Theta_0|)$.

Proof Suppose (a) is satisfied, and by using (i) in (6.28), we obtain

$$\Theta_{|0} + \Theta^2 \geq 0,$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\Theta}\right) \leq 1. \tag{6.29}$$

Integrating (6.29) on $[0, \tau]$, we infer that

$$\frac{1}{\Theta} \leq \frac{1}{\Theta_0} + \tau. \tag{6.30}$$

As $1/\Theta$ must pass through zero, from (6.30) we deduce that

$$\tau \geq \frac{1}{|\Theta_0|}. \tag{6.31}$$

Combining with the result from Lemma 6.2, we conclude that τ must be in the interval $[1/|\Theta_0|, 3/|\Theta_0|]$. In a similar way one proved the assertion (b_1). Finally, by using the condition from (b_2) into (6.28), we obtain

$$\Theta_{|0} + \Theta^2 < 0,$$

which is equivalent to

$$\frac{d}{d\tau}\left(\frac{1}{\Theta}\right) > 1. \tag{6.32}$$

Integrating (6.32) on $[0, \tau]$, we deduce that

$$\frac{1}{\Theta} > \frac{1}{\Theta_0} + \tau. \tag{6.33}$$

As $1/\Theta$ must pass through zero, we conclude that $\tau \in [0, 1/|\Theta_0|)$. This completes the proof of the lemma.

7 Kinematic quantities for Kerr–Newman black holes

The new point of view developed here on the $(1+3)$ threading of spacetime is applied in this section to the charged Kerr black hole (also called Kerr–Newman black hole). We show that the curvature and Ricci tensor fields of (M, g) are simply expressed in terms of the curvature and Ricci tensor fields of the Riemannian spatial connection, via the kinematic quantities.

Now, according to the notations used in Sects. 2 and 3, for the metric of a Kerr–Newman black hole given by (1.2), we have

$$\begin{aligned} &(a)\ \Phi^2 = \tfrac{\Delta - a^2(\sin x^2)^2}{\Sigma} = 1 - \tfrac{2mx^1 - e^2}{\Sigma},\\ &(b)\ \xi_1 = \xi_2 = 0, \quad xi_3 = \tfrac{(e^2 - 2mx^1)a(\sin x^2)^2}{\Sigma},\\ &(c)\ a_i = 0, \quad \forall\, i \in \{1, 2, 3\}. \end{aligned} \tag{7.1}$$

The spatial distribution SM of (M, g) is locally spanned by

$$\frac{\delta}{\delta x^1} = \frac{\partial}{\partial x^1}, \quad \frac{\delta}{\delta x^2} = \frac{\partial}{\partial x^2}, \quad \frac{\delta}{\delta x^3} = \frac{\partial}{\partial x^3} + \Phi^{-2}\xi_3\frac{\partial}{\partial x^0}, \tag{7.2}$$

and it is the kernel of the 1-form

$$\delta x^0 = dx^0 - \Phi^{-2}\xi_3 dx^3. \tag{7.3}$$

By using (2.15), (1.2), (7.1a), and (7.1b), we deduce that the only non-zero local components of the Riemannian metric h on SM with respect to the threading frame field from (7.2) are the following:

$$h_{11} = \frac{\Sigma}{\Delta}, \quad h_{22} = \Sigma, \quad h_{33} = \frac{\Delta(\sin x^2)^2}{\Phi^2}. \tag{7.4}$$

Hence the line element from (1.2) becomes

$$\begin{aligned} ds^2 = &-\Phi^2(\delta x^0)^2 + \frac{\Sigma}{\Delta}(dx^1)^2 + \Sigma(dx^2)^2\\ &+\frac{\Delta(\sin x^2)^2}{\Phi^2}(dx^3)^2, \end{aligned} \tag{7.5}$$

with respect to the threading coframe field $\{\delta x^0, dx^i\}$.

Next, by using (3.3), (4.6), (7.1a), and (7.1c), we deduce that the geodesic spatial tensor field $b = (b_i)$ is given by

$$\begin{aligned} &(a)\ b_1 = c_1 = \tfrac{x^1(2mx^1 - e^2) - m\Sigma}{(\Phi\Sigma)^2},\\ &(b)\ b_2 = c_2 = \tfrac{(e^2 - 2mx^1)a^2 \sin x^2 \cos x^2}{(\Phi\Sigma)^2}, \quad (c)\ b_3 = c_3 = 0. \end{aligned} \tag{7.6}$$

Due to (4.7) and (7.6) we conclude that the curves from the congruence defined by $\xi = \partial/\partial x^0$ which sit in the surface given by the equations

$$x^1(2mx^1 - e^2) - m\Sigma = 0, \quad (e^2 - 2mx^1)a^2 \sin x^2 \cos x^2 = 0,$$

for $a \neq 0$, or in the hypersurface

$$x^1 = \frac{e^2}{m},$$

for $a = 0$, are the only geodesics of (M, g) that are tangent to ξ. Moreover, we see that such geodesics have the equations

$$\begin{aligned} &x^0 = \lambda, \quad x^1 = \tfrac{e^2}{m}, \ x^2 = \tfrac{\pi}{2}, \quad x^3 = c, \text{ or}\\ &x^0 = \lambda, \quad x^1 = \tfrac{e^2 \pm \sqrt{e^4 + 4m^2}}{2m}, \quad x^2 = 0, \quad x^3 = c, \end{aligned}$$

for $a \neq 0$, and

$$x^0 = \lambda, \quad x^1 = \frac{e^2}{m}, \quad x^2 = k, \quad x^3 = c,$$

for $a = 0$, where k and c are constants. In particular, the integral curves of ξ cannot be geodesics in the Schwarzschild spacetime.

Now, taking into account (3.8) and (7.4), we obtain

$$(a)\ \Theta_{ij} = 0, \ (b)\ \Theta = 0, \ (c)\ \sigma_{ij} = 0, \quad \forall\, i, j \in \{1, 2, 3\}. \tag{7.7}$$

Also, by using (3.5a), (7.1b), (7.2), and (7.6), we deduce that the only non-zero local components of the vorticity tensor field (ω_{ij}) are given by

$$\begin{aligned} &(a)\ \omega_{13} = \tfrac{a(\sin x^2)^2}{\Phi^4\Sigma^2}\{m\Sigma - x^1(2mx^1 - e^2)\},\\ &(b)\ \omega_{23} = \tfrac{(2mx^1 - e^2)a\Delta \sin x^2 \cos x^2}{\Phi^4\Sigma^2}. \end{aligned} \tag{7.8}$$

From (7.8) we see that the spatial distribution SM is not integrable for both Kerr–Newman and Kerr black holes. On the contrary, for the Reissner–Nordstrom and Schwarzschild solutions, the timelike vector field $\xi = \partial/\partial x^0$ is hypersurface orthogonal.

Finally, we note that the local components of the curvature and Ricci tensor fields with respect to the threading frame field have very simple expressions. Indeed, by putting (7.6) and (7.7) into (5.3a), (5.3b), and (5.5a), we obtain

$$\begin{aligned}
&(a)\ R_{ijkh} = R^{\star}_{ijkh} + \Phi^2\{\omega_{ik}\omega_{jh} - \omega_{ih}\omega_{jk}\},\\
&(b)\ R_{i0kh} = \Phi^2\{\omega_{ih|_k} - \omega_{ik|_h} + \omega_{ih}c_k - \omega_{ik}c_h + 2\omega_{kh}c_i\},\\
&(c)\ R_{i0k0} = \Phi^2\left\{c_i c_k + \tfrac{1}{2}(c_{i|_k} + c_{k|_i}) - \Phi^2\omega_{ih}\omega^h_k\right\},
\end{aligned} \tag{7.9}$$

where $R^{\star}_{ijkh}$ is the curvature tensor field of the Riemannian spatial connection. In a similar way, from (5.12a), (5.11b), and (5.11c), we infer that the local components of the Ricci tensor of (M, g) with respect to the threading frame field are given by

$$\begin{aligned}
&(a)\ R_{ik} = R^{\star}_{ik} - c_i c_k - \tfrac{1}{2}(c_{i|_k} + c_{k|_i}),\\
&(b)\ R_{i0} = \Phi^2\{\omega^h_{i|_h} + 3\omega^h_i c_h\},\\
&(c)\ R_{00} = \Phi^2\{c_h c^h + c^h_{\ |_h} + \Phi^2\omega_{kh}\omega^{kh}\},
\end{aligned} \tag{7.10}$$

where $R^{\star}_{ik}$ is the Ricci tensor of the Riemannian spatial connection. As far as we know, (7.9) and (7.10) have not been stated before in the earlier literature. They can bring more information and ideas in the study of the geometry and physics of the black holes. In particular, for Reissner–Nordstrom and Schwarzschild solutions, (7.9) and (7.10) become

$$\begin{aligned}
&(a)\ R_{ijkh} = R^{\star}_{ijkh}, \quad (b)\ R_{i0kh} = 0,\\
&(c)\ R_{i0k0} = \Phi^2\left\{c_i c_k + \tfrac{1}{2}(c_{i|_k} + c_{k|_i})\right\},
\end{aligned} \tag{7.11}$$

and

$$\begin{aligned}
&(a)\ R_{ik} = R^{\star}_{ik} - c_i c_k - \tfrac{1}{2}(c_{i|_k} + c_{k|_i}),\\
&(b)\ R_{i0} = 0,\ (c)\ R_{00} = \Phi^2\{c_h c^h + c^h_{\ |_h}\},
\end{aligned} \tag{7.12}$$

respectively.

8 Equations of motion in a Kerr black hole

In this section and in the next one, we take $e = 0$ in (1.2), that is, we consider the Kerr black hole (M, g). As is well known, the geodesic equations in a Kerr black hole have been explicitly integrated for the first time by Carter [15]. In the present section we will state a new form of the equations of motion in a Kerr black hole, and obtain information about the position of geodesics in M with respect to the spatial distribution. In particular, we show that the geodesics of (M, g) which are tangent to SM coincide with the autoparallel curves of the Riemannian spatial connection.

Let C be a smooth curve in M given by parametric equations

$$x^0 = x^0(\lambda), \quad x^i = x^i(\lambda), \quad i \in \{1, 2, 3\}, \quad \lambda \in [a, b], \tag{8.1}$$

where λ does not necessarily represent the time in (M, g). The velocity vector field $d/d\lambda$ for C is expressed in terms of the threading frame $\{\partial/\partial x^0, \delta/\delta x^i\}$ as follows:

$$\frac{d}{d\lambda} = \frac{\delta x^0}{\delta\lambda}\frac{\partial}{\partial x^0} + \frac{dx^i}{d\lambda}\frac{\delta}{\delta x^i}, \tag{8.2}$$

where we put

$$\frac{\delta x^0}{\delta\lambda} = \frac{dx^0}{d\lambda} - \Phi^{-2}\xi_3\frac{dx^3}{d\lambda}. \tag{8.3}$$

Now, by using (7.7) and (7.1c) in (3.17), we express the Levi-Civita connection ∇ on (M, g), as follows:

$$\begin{aligned}
&(a)\ \nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} = \Gamma^{\star k}_{i\ j}\frac{\delta}{\delta x^k} + \omega_{ij}\frac{\partial}{\partial x^0},\\
&(b)\ \nabla_{\frac{\partial}{\partial x^0}}\frac{\delta}{\delta x^i} = \nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial x^0} = \Phi^2\omega^k_i\frac{\delta}{\delta x^k} + c_i\frac{\partial}{\partial x^0},\\
&(c)\ \nabla_{\frac{\partial}{\partial x^0}}\frac{\partial}{\partial x^0} = \Phi^2 c^k\frac{\delta}{\delta x^k}.
\end{aligned} \tag{8.4}$$

By direct calculations using (8.4) and (8.2), we obtain

$$\begin{aligned}
\nabla_{\frac{d}{d\lambda}}\frac{d}{d\lambda} &= \left\{\frac{d^2x^k}{d\lambda^2} + \Gamma^{\star\ k}_{i\ j}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} + 2\Phi^2\frac{\delta x^0}{\delta\lambda}\omega^k_i\frac{dx^i}{d\lambda} + \left(\frac{\delta x^0}{\delta\lambda}\right)^2\Phi^2 c^k\right\}\frac{\delta}{\delta x^k}\\
&\quad + \left\{\frac{d}{d\lambda}\left(\frac{\delta x^0}{\delta\lambda}\right) + 2\frac{\delta x^0}{\delta\lambda}c_i\frac{dx^i}{d\lambda}\right\}\frac{\partial}{\partial x^0},
\end{aligned} \tag{8.5}$$

which leads to the following equations of motion:

$$\begin{aligned}
&(a)\ \frac{d^2x^k}{d\lambda^2} + \Gamma^{\star\ k}_{i\ j}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} + 2\Phi^2\frac{\delta x^0}{\delta\lambda}\omega^k_i\frac{dx^i}{d\lambda} + \left(\frac{\delta x^0}{\delta\lambda}\right)^2\Phi^2 c^k = 0,\\
&(b)\ \frac{d}{d\lambda}\left(\frac{\delta x^0}{\delta\lambda}\right) + 2\frac{\delta x^0}{\delta\lambda}c_i\frac{dx^i}{d\lambda} = 0.
\end{aligned} \tag{8.6}$$

Note that λ from (8.6) is an affine parameter for geodesics in (M, g).

A geodesic of (M, g) which is tangent at any of its points to the spatial distribution SM is called a *spatial geodesic*. Then by using (8.2), (8.3), and (8.6) we deduce that a curve C given by (8.1) is a spatial geodesic if, and only if, it is a solution of the system

$$\begin{aligned}
&(a)\ \frac{d^2x^k}{d\lambda^2} + \Gamma^{\star k}_{i\ j}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0,\\
&(b)\ \frac{\delta x^0}{\delta\lambda} = \frac{dx^0}{d\lambda} - \Phi^{-2}\xi_3\frac{dx^3}{d\lambda} = 0.
\end{aligned} \tag{8.7}$$

Now, we remark that (7.7a) implies that the Kerr spacetime has a bundle-like metric with respect to the foliation determined by ξ (cf. [16, p. 112]). Thus, we have the following interesting property:

If a geodesic of a Kerr black hole is tangent to the spatial distribution at one point, then it remains tangent to it at all later times.

Also, due to (8.7) we may state the following:

The spatial geodesics in (M, g) coincide with autoparallel curves for the Riemannian spatial connection.

Next, we suppose that C is a geodesic in (M, g) which is not spatial, that is, we have $\delta x^0/\delta\lambda \neq 0$. Without loss of generality we suppose $\delta x^0/\delta\lambda > 0$. Then by using (3.3) in

(8.6b), we obtain

$$\left(\frac{\delta x^0}{\delta\lambda}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\delta x^0}{\delta\lambda}\right)+2\Phi^{-1}\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}=0,$$

which is equivalent to

$$\frac{\delta x^0}{\delta\lambda}\Phi^2=K, \tag{8.8}$$

where K is a positive constant. By using (8.8) into (8.6a), we deduce that a geodesic of a Kerr black hole (which is not a spatial geodesic), must be a solution of the system formed by (8.8) and the equations

$$\frac{\mathrm{d}^2x^k}{\mathrm{d}\lambda^2}+\Gamma^{\star\ k}_{i\ j}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}+2K\omega^k_i\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}+K^2\Phi^{-2}c^k=0. \tag{8.9}$$

Finally, taking into account (8.7b) and (8.8), we conclude that the system of differential equations for the geodesics in a Kerr black hole can be arranged in such a way that one of the equations is of first order.

9 A 3D identity along a geodesic in a Kerr black hole

The Riemannian spatial connection given by (3.11) enables us to define a 3D force in a Kerr black hole, and to deduce what we call the 3D force identity [cf. (9.10)]. Note that this 3D force is a direct consequence of the existence of the fourth dimension (time), and emphasizes an important difference between Newtonian gravity and Einstein's general relativity.

Let C be a geodesic in (M, g), and $U(\lambda)$ be the projection of velocity $\mathrm{d}/\mathrm{d}\lambda$ on SM. Then by (8.2) we obtain

$$U(\lambda)=\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\delta}{\delta x^i}, \tag{9.1}$$

which we call the 3D *velocity* along C. Now, we consider the 3D arc-length parameter $s^\star$ on C given by

$$s^\star=\int_a^\lambda h(U(\lambda),U(\lambda))^{1/2}\mathrm{d}\lambda=\int_a^\lambda\left(h_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right)^{1/2}\mathrm{d}\lambda,$$

and we obtain

$$(\mathrm{d}s^\star)^2=h_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{9.2}$$

Hence

$$U(s^\star)=\frac{\mathrm{d}x^i}{\mathrm{d}s^\star}\frac{\delta}{\delta x^i} \tag{9.3}$$

is a unit spatial vector field, that is, we have

$$h(U(s^\star),U(s^\star))=1. \tag{9.4}$$

Since $\mathrm{d}s^\star/\mathrm{d}\lambda$ is positive, we can take $s^\star$ as a new parameter on C. We define the 3D *force* along C as the spatial vector field $F(s^\star)$ given by

$$F(s^\star)=\nabla^\star_{\frac{\mathrm{d}}{\mathrm{d}s^\star}}U(s^\star). \tag{9.5}$$

Here, $\nabla^\star$ is the Riemannian spatial connection, and $\mathrm{d}/\mathrm{d}s^\star$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}s^\star}=\frac{\delta x^0}{\delta s^\star}\frac{\partial}{\partial x^0}+\frac{\mathrm{d}x^i}{\mathrm{d}s^\star}\frac{\delta}{\delta x^i}. \tag{9.6}$$

Taking into account that $\nabla^\star$ is a metric connection on SM, and using (9.4) and (9.5), we deduce that $F(s^\star)$ is orthogonal to both $U(s^\star)$ and $U(\lambda)$. By (9.5), (9.6), and (8.7), we see that the 3D force vanishes along a spatial geodesic C if, and only if, $s^\star$ is an affine parameter on C.

Next, we put

$$F(s^\star)=F^k(s^\star)\frac{\delta}{\delta x^k},$$

and by using (9.5), (9.6), (3.12), (3.14), and (7.7a), obtain

$$F^k(s^\star)=\frac{\mathrm{d}^2x^k}{(\mathrm{d}s^\star)^2}+\Gamma^{\star\ k}_{i\ j}\frac{\mathrm{d}x^i}{\mathrm{d}s^\star}\frac{\mathrm{d}x^j}{\mathrm{d}s^\star}+\Phi^2\frac{\delta x^0}{\delta s^\star}\omega^k_i\frac{\mathrm{d}x^i}{\mathrm{d}s^\star}, \tag{9.7}$$

provided C is not a spatial geodesic. Now, by using (8.8) and taking into account that

$$\frac{\mathrm{d}^2\lambda}{(\mathrm{d}s^\star)^2}=-\frac{\mathrm{d}^2s^\star}{\mathrm{d}\lambda^2}\left(\frac{\mathrm{d}s^\star}{\mathrm{d}\lambda}\right)^{-3},$$

from (9.7) we deduce that the local components of the 3D force with respect to the affine parameter λ are given by

$$\begin{aligned}F^k(\lambda)=&\left\{\frac{\mathrm{d}^2x^k}{\mathrm{d}\lambda^2}+\Gamma^{\star\ k}_{i\ j}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}+K\omega^k_i\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\right.\\&\left.-\left(\frac{\mathrm{d}s^\star}{\mathrm{d}\lambda}\right)^{-1}\frac{\mathrm{d}^2s^\star}{\mathrm{d}\lambda^2}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}\right\}\left(\frac{\mathrm{d}s^\star}{\mathrm{d}\lambda}\right)^{-2}.\end{aligned} \tag{9.8}$$

Finally, by using (8.9) into (9.8), we infer that

$$\begin{aligned}F^k(\lambda)=&-\left\{K\omega^k_i\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}+K^2\Phi^{-2}c^k+\left(\frac{\mathrm{d}s^\star}{\mathrm{d}\lambda}\right)^{-1}\frac{\mathrm{d}^2s^\star}{\mathrm{d}\lambda^2}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}\right\}\\&\times\left(\frac{\mathrm{d}s^\star}{\mathrm{d}\lambda}\right)^{-2}.\end{aligned} \tag{9.9}$$

Taking into account that

$$h_{kh}F^k(\lambda)\frac{\mathrm{d}x^h}{\mathrm{d}\lambda}=0,$$

and using (9.9), (3.3), and (9.2), we obtain

$$K^2\Phi^{-3}\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}+\frac{\mathrm{d}s^\star}{\mathrm{d}\lambda}\frac{\mathrm{d}^2s^\star}{\mathrm{d}\lambda^2}=0. \tag{9.10}$$

Note that the identity (9.10) is a direct consequence of the existence of the 3D force F given by (9.5). For this reason we call it the 3D *force identity*. Such an identity could be useful in a study of motions in (M, g), and even for solving the equations of motion. For example, from (9.10) we deduce the following.

The 3D *arc-length parameter* $s^\star$ *is an affine parameter on the geodesic* C, *if and only if,* Φ *is constant along* C.

10 Conclusions

In the present paper we develop a theory for a (1+3) threading of spacetime (M, g) with respect to a congruence of curves determined by an arbitrary timelike vector field $\xi = \partial/\partial x^0$. The generality of the study is not the only difference between our approach and what is known in the literature for the case of the unit vector field ξ. The main differences consist in the following:

(i) Our approach is entirely developed with geometric objects expressed by their local components with respect to the threading frames $\{\partial/\partial x^0, \delta/\delta x^i\}$ and threading coframes $\{\delta x^0, \mathrm{d}x^i\}$.

(ii) The spatial distribution is not supposed to be necessarily integrable, and therefore this theory can easily be applied to the study of any cosmological model with non-zero vorticity.

(iii) All the equations and results that we state are expressed in terms of the spatial tensor fields [cf. (3.1)] and their spatial and time covariant derivatives [cf. (3.15)] induced by the Riemannian spatial connection given by (3.12).

(iv) In spite of the numerous papers published on $(1+3)$ threading of the spacetime (cf. [1,5–14]), the generalized Raychaudhuri equations (6.1), (6.5), and (6.27) are stated here for the first time in the literature.

(v) The proof of Lemma 6.3, which completes the well-known Lemma 6.2, is entirely based on a new form of Raychaudhuri's equation for a congruence of timelike geodesics [cf. (6.28)].

(vi) It is the first time in the literature that the spatial geodesics of a Kerr black hole are investigated [see (8.7) and the assertions which follow it].

(vii) The 3D force (9.5) and the 3D force identity (9.10) are new objects in the general theory of Kerr back holes, and they illustrate the differences between Newtonian gravity and Einstein's general relativity.

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