Eur. Phys. J. C (2015) 75:87
DOI 10.1140/epjc/s10052-015-3307-y

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Classical tests of General Relativity in thick branes

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Received: 4 November 2014 / Accepted: 6 February 2015 / Published online: 24 February 2015

Abstract Classical tests of General Relativity in brane-world scenarios have been investigated recently with the purpose of posing observational constraints on the parameters of some models of infinitely thin brane. Here we consider the motion of test particles in a thick brane scenario that corresponds to a regularized version of the Garriga–Tanaka solution, which describes a black hole solution in RSII model, in the weak field regime. By adapting a mechanism previously formulated in order to describe the confinement of massive tests particles in a domain wall (which simulates classically the trapping of the Dirac field in a domain wall), we study the influence of the brane thickness on the four-dimensional (4D) path of massless particles. Although the geometry is not warped and, therefore, the bound motion in the transverse direction is not decoupled from the movement in the 4D-world, we can find an explicit solution for the light deflection and the time delay, if the motion in the fifth direction is a high frequency oscillation. We verify that, owing to the transverse motion, the light deflection and the time delay depend on the energy of the light rays. This feature may lead to the phenomenon of gravitational rainbow. We also consider the problem from a semi-classical perspective, investigating the effects of the brane thickness on the motion of the zero-mode in the 4D-world.

1 Introduction

Recently we have seen a renewed interest in higher-dimensional theories mainly motivated by braneworld models that were originally considered as a promising theoretical framework for solving the hierarchy problem [1–4]. In braneworld models, our ordinary four-dimensional space-time is viewed as a submanifold isometrically embedded in an ambient space of higher dimensions. An essential feature in the braneworld scenarios is the confinement of matter and fields in the brane, while gravity may propagate in all dimensions. As a consequence, the extra dimensions can be much larger than the Planck length – the compactification scale of the original Kaluza–Klein theory (the first modern extra-dimensional theory) – without introducing irremediable conflicts between theory and the current observational data [2–4]. As a matter of fact, the so-called RSII model [5], in which the brane is embedded in five-dimensional space endowed with a negative cosmological constant, is a phenomenologically consistent model despite the extra dimension is non-compact. Indeed, the existence of a localized zero-mode of the perturbation of the metric ensures that the gravitational field reproduces the four-dimensional behavior for large distances [5].

As gravity has access to extra dimensions, experimental tests involving the gravitational field are invaluable tools for investigating the existence of hidden dimensions in braneworld theories. Therefore, the study of black holes in higher-dimension theories (for a review, see [6]) can play a significant role in revealing traces of the hidden dimensions. However, in the RSII model specifically, the task of finding an exact solution that represents a realistic black hole confined in the brane proved to be very difficult [6]. Among several attempts to find a static black hole solution in RSII model, we would like to highlight here the Garriga–Tanaka solution [7], which is expected to be the weak field regime of a black hole in RSII model, and the exact solutions known as DMPR and CFM [8,9], which represent acceptable static and spherically symmetric geometries in the brane. More recently, it was claimed that an exact and complete solution of a realistic black hole in the RSII model was finally obtained [10,11].

It follows naturally from these studies that one would have an interest in investigating the so-called classical tests of General Relativity in this scenario. There are many works concerned with the problem of posing experimental constraints on the parameters of those models, through the study of the influence of the hidden dimensions over the motion of test particles in the 4D-world [12–34]. In particular, the

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light deflection and time delay in the infinitely thin brane of the RSII model is discussed in Refs. [33,34]. However, the classical tests impose weaker bounds than those obtained in laboratory tests of the square inverse law [35], for instance. Nevertheless, as the constraints are obtained independently, by exploring different physical aspects of the system, the classical tests should be considered as important complementary tests, at least.

String theory is the original inspiration for the braneworld models and, in this context, the brane is infinitely thin [1]. However, in the field-theory framework, the brane in the RSII model, for instance, can be described by a scalar field which is found in a domain wall configuration [36]. In this case, the matter can be trapped, by means of Yukawa-type interaction with the scalar field, in the core of the brane, which now has a non-null thickness [36]. The thick brane solutions are considered as regularized versions of the infinitely thin-brane model, since they recover the corresponding thin solution in the limit when the thickness goes to zero. There are many studies on the effects of the brane thickness over physical systems [37–43]. However, as far as we know, no exact solution of a confined black hole in a thick 3-brane was found yet, although solutions for lower dimension (2-brane) are known [44]. Here we want to discuss an approximate solution of a black hole in the context of thick brane version of the RSII model. The idea is to propagate the metric defined in the core of the brane toward the bulk by using the Einstein equations coupled to a scalar field in the case of a static and axial symmetry, as outlined in Ref. [45]. This method is detailed in Sect. II, where we find explicitly the metric in terms of a power series with respect to the transverse coordinate. The solution is valid in the vicinity of the brane center and far from the black hole.

In Sect. 3, we proceed to the analysis of the classical motion of the particle in the thick brane. However, to describe its motion, first we have to implement a confinement mechanism for test particles that simulates classically the localization of the Dirac field in a domain wall. In Ref. [46] we have proposed a confinement mechanism for massive particle, which consists in modifying the Lagrangian of the particle by introducing a coupling with the scalar field. The role of the scalar field is to increase the effective mass of the particle when its movement has a transverse component. Based on that ideas, we have developed a formalism in order to study the confinement of a massless particle in the domain wall with the particular purpose of discussing the deflection of light and the time delay caused by a mass M in the thick brane. Despite the fact that metric is not warped and, therefore, the transversal and the radial motions are not decoupled, we find explicitly the influence of the extra dimension over the four-dimensional motion and, as we shall see, the results (regarding the light deflection and time delay) depend on the energy of the light rays. An interesting consequence of this dependence is the formation of a gravitational rainbow when a beam of white light is deviated by a massive body.

In Sect. 4, we investigate this problem from a semi-classical perspective, considering the quantization of the motion in the fifth direction. We study the classical motion of the so-called zero-mode (the massless bound state) in the 4D-world. In particular, we have determined the effect of the brane thickness on the bending of light rays by a body of mass M and, as we shall see, the deflection angle depends on the width in the extra-dimension direction of the wave function that describes the zero-mode state.

2 Gravity in thick branes

A non-rotating mass M (of a black hole or a star) localized in a brane should give rise to an axisymmetric, static space-time in five dimensions. In such spaces, as is well known, there are coordinates in which the metric assumes the Weyl canonical form [47]. By means of a convenient coordinate transformation, the metric can be put into a Gaussian form adapted to the brane:

$$\mathrm{d}s^2 = -\mathrm{e}^{2A(r,z)}\mathrm{d}t^2 + \mathrm{e}^{2B(r,z)}\mathrm{d}r^2 + \mathrm{e}^{2C(r,z)}\mathrm{d}\Omega^2 + \mathrm{d}z^2 \quad (1)$$

where $z = 0$ gives the localization of the brane.

In the context of thick brane scenarios in five dimensions, the brane is usually described as a domain wall generated by a certain scalar field φ. It is reasonable to expect that a body of mass M will affect the domain wall solution. However, considering the symmetry of the problem, we might assume that the new solution will depend only on coordinates r and z, i.e., $\varphi = \varphi(r, z)$.

Even in the absence of M, the Einstein equations coupled to the scalar field ($G_{\mu\nu} = \kappa T^{(\varphi)}_{\mu\nu}$ and $\Box\varphi - V'(\varphi) = 0$, where κ is the gravitational constant in five dimensions, $T^{(\varphi)}_{\mu\nu}$ is the usual energy-momentum tensor of the scalar field and $V'(\varphi)$ is the derivative of the potential $V(\varphi)$ with respect to the scalar field) are not easy to solve. However, when the potential is conveniently chosen, an exact solution of a self-gravitating domain wall can be obtained. For example, taking $V(\varphi) = \frac{\lambda}{4}(\varphi^2 - \eta^2)^2 - \frac{\beta\lambda}{3\eta^2}\varphi^2(\varphi^2 - 3\eta^2)^2$, the solution is [48]:

$$\mathrm{d}s^2 = \mathrm{e}^{2a(z)}(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2) + \mathrm{d}z^2, \quad (2)$$

$$2a(z) = -2\beta \ln\left(\cosh^2\left(\frac{z}{\varepsilon}\right)\right) - \beta \tanh^2\left(\frac{z}{\varepsilon}\right), \quad (3)$$

$$\varphi = \eta \tanh\left(\frac{z}{\varepsilon}\right), \quad (4)$$

where $\varepsilon^2 = 2/\lambda\eta^2$, $\beta = \kappa\eta^2/9$.

This solution can be interpreted as a regularized version of the RSII brane model. Indeed, in the limit when the parameter

ε (the thickness of the wall) goes to zero with the condition $2\beta/\varepsilon \equiv \ell =$ const, the RSII solution is recovered, with ℓ playing the role of the curvature radius of AdS_5 space.

When the mass M is taken into account then the technical difficulty to solve the equations increases significantly and, as far as we know, no exact solution is known. Considering this, we have to resort to perturbation methods in order to approach the problem [45]. The presence of M certainly modifies both the original metric and the scalar field. At large distances, where the weak field regime is valid, the modification can be treated as a small perturbation of the original solution. In this case, we can write

$$\begin{aligned} ds^2 &= e^{2a(z)}[-(1+f)dt^2 + (1+m)d\xi^2 \\ &\quad + r^2(1+h)d\Omega^2] + dz^2, \\ \varphi &= \eta\left[\tanh\left(\frac{z}{\varepsilon}\right) + k(r,z)\right], \end{aligned} \tag{5}$$

where f, m, h and k are functions of r and z, which measure the small deviations of the metric components.

It may happen that, due to the perturbation, the coordinate z will not be adapted to the level surface of the scalar field anymore. For instance, the center of the domain wall ($\varphi = 0$), which originally coincides with $z = 0$, is now given by the equation $z = -\varepsilon k(r, 0)$. However, the set of adapted coordinates can be restored by means of an appropriate transformation [7]. For our purpose it is convenient to use a Gaussian coordinate system adapted to the center of domain wall, because, as we shall see, our approach to determine the solution is based on the propagation of the metric from the center of the brane to the bulk by the field equation. In this sense, the metric evaluated in the center constitute a kind of initial data for the bulk solution. Thus, the great advantage of working in these coordinates is the fact that initial conditions, i.e., the value that the functions f, m, h and k and its first derivatives assume in the center of wall, can easily be established. For example, as the center corresponds to $z = 0$, we should have

$$k(r, 0) = 0. \tag{6}$$

Another important condition can be immediately deduced based on the fact that the metric should be symmetric with respect to the center of the wall. As ∂_z is the normal vector of the hypersurface $z = 0$, in the center of the domain we should have

$$\left.\frac{\partial f}{\partial z}\right|_{z=0} = \left.\frac{\partial m}{\partial z}\right|_{z=0} = \left.\frac{\partial h}{\partial z}\right|_{z=0} = 0. \tag{7}$$

These functions must satisfy other conditions, which we shall discuss later. But now let us concentrate our attention on the field equation. In the first approximation order [49], those equations are reduced to the following set:

$$\begin{aligned} {}^{(1)}R^t_t &\equiv \frac{1}{2}f_{zz} + \frac{1}{2}\left(f_{rr} + \frac{2}{r}f_r\right)e^{-2a} \\ &\quad + \frac{1}{2}a'(5f_z + m_z + 2h_z) = -\frac{2}{3}\kappa V_0' k, \end{aligned} \tag{8}$$

$$\begin{aligned} {}^{(1)}R^r_r &\equiv \frac{1}{2}m_{zz} + \left(\frac{1}{2}f_{rr} - \frac{1}{r}m_r + h_{rr} + \frac{2}{r}h_r\right)e^{-2a} \\ &\quad + \frac{1}{4}a'(2f_z + 10m_z + 4h_z) = -\frac{2}{3}\kappa V_0' k, \end{aligned} \tag{9}$$

$$\begin{aligned} {}^{(1)}R^\theta_\theta &= \frac{1}{2}h_{zz} + \frac{1}{4}\left(2h_{rr} + \frac{4}{r}h_r + \frac{2}{r}(f_r - m_r + 2h_r)\right)e^{-2a} \\ &\quad + \frac{1}{2}a'(f_z + m_z + 6h_z) - \frac{1}{r^2}e^{-2a}(m-h) = -\frac{2}{3}\kappa V_0' k, \end{aligned} \tag{10}$$

$${}^{(1)}R_{rz} \equiv \frac{1}{2}f_{zr} + h_{zr} + \frac{1}{r}(h_z - m_z) = -\kappa k_r \varphi_z^{(0)}, \tag{11}$$

$$\begin{aligned} {}^{(1)}G_{zz} &\equiv -\frac{3}{2}a'(f_z + m_z + 2h_z) \\ &\quad - \frac{1}{4}\left(2f_{rr} + 4h_{rr} + \frac{4}{r}(f_r - m_r + 3h_r)\right)e^{-2a} \\ &\quad + \frac{1}{r^2}e^{-2a}(m-h) = -\kappa(\varphi_z^{(0)}k_z - V_0' k), \end{aligned} \tag{12}$$

$$\begin{aligned} {}^{(1)}\Box\varphi - V_0'' k &\equiv k_{zz} + 4a'k_z + \frac{1}{2}\varphi_0'(f_z + m_z + 2h_z) \\ &\quad + e^{-2a}\left(k_{rr} + \frac{2}{r}k_r\right) - V_0'' k = 0, \end{aligned} \tag{13}$$

where the quantities carrying the indices (0) and (1) are calculated in the zero and in the first approximation order with respect to GM, respectively.

This system can be divided into two classes of equations: dynamic equations (8), (9), (10), and (13) – which contain the second derivative of the metric with respect to the transverse coordinate (z) and constraint equations (11) and (12). It is well known that, owing to the Bianchi identities, the constraints are propagated by the dynamical equations. This means that if the constraint equations are satisfied in a certain hypersurface, the dynamic equations ensure that the constraint equations will be satisfied in an open set of the manifold in the vicinity of that hypersurface. Thus a solution of Eqs. (11) and (12) in the hypersurface $z = 0$ can be interpreted as an admissible initial data for the bulk solution of the dynamic equation.

The dynamic equation can be solved by an iterative method. Isolating the second derivative of the metric or the scalar field on the left hand side of the equations, we can calculate them on the hypersurface $z = 0$ from the initial data. Taking the derivative of the equations with respect to z and repeating the same procedure successively, we can evaluate the derivatives of any order of the unknown functions $f, m, h,$

and k at $z = 0$. With these derivatives, we construct a power series in respect of z. According to Cauchy–Kowalewsky theorem, this series converges in some open set and represents the solution of the equations.

We can use this procedure to find an approximate solution of the problem investigated here. The first step is to choose an appropriate set of initial data. As we have already mentioned, the initial data consist of the functions $f(r, 0)$, $m(r, 0)$, $h(r, 0)$, and $k(r, 0)$ and its first derivatives defined on the hypersurface $z = 0$ that satisfy the constraint equations (11) and (12). Some of the initial conditions have been already determined in Eqs. (6) and (7). It is interesting to note that, with these choices, the constraint equation (11) is automatically satisfied.

The remaining set of the initial conditions, i.e., $f(r, 0)$, $m(r, 0)$, $h(r, 0)$ and $\frac{\partial k}{\partial z}|_{z=0}$, can be determined by using the thin-brane solution as inspiration. According to Garriga and Tanaka [7], in the limit of thin brane, the metric of a matter distribution with mass M localized in the brane is given, in the first approximation order of GM, by

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r} - \frac{4GM\ell^2}{3r^3}\right)\mathrm{d}t^2 + \left(1 + \frac{2GM}{r} + \frac{2GM\ell^2}{r^3}\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{14}$$

We have to emphasize that this metric is valid in the brane ($z = 0$) in a region far from the source ($r >> GM$).

In order to obtain a link between our thick brane solution and thin brane solution (14), we are going to admit that the induced metric in the center of the domain wall is isometric to the thin-brane's geometry. This condition implies that

$$f(r, 0) = -\left(\frac{2GM}{r} + \frac{4GM\ell^2}{3r^3}\right), \tag{15}$$

$$m(r, 0) = \frac{2GM}{r} + \frac{2GM\ell^2}{r^3}, \tag{16}$$

$$h(r, 0) = 0. \tag{17}$$

With this choices, the constraint equation (12) evaluated in $z = 0$ imposes an additional initial condition:

$$k_z|_{z=0} = 0. \tag{18}$$

Equations (6), (7), (15), (16), (17), and (18) yield a set of appropriate initial data. Now the bulk solution can be determined from the dynamical equations by the iterative method. For instance, calculating the second derivative of the unknown functions in terms of the initial data we find directly from Eqs. (8), (9), (10), and (13) that

$$f_{zz}|_{z=0} = -\left(f_{rr} + \frac{2}{r}f_r\right)_{z=0} = \frac{8GM\ell^2}{r^5}, \tag{19}$$

$$m_{zz}|_{z=0} = -\left(f_{rr} - \frac{2}{r}m_r\right)_{z=0} = \frac{4GM\ell^2}{r^5}, \tag{20}$$

$$h_{zz}|_{z=0} = \left[\frac{2}{r^2}m - \frac{1}{r}(f_r - m_r)\right]_{z=0} = -\frac{6GM\ell^2}{r^5}, \tag{21}$$

$$k^{(1)}_{zz}\Big|_{z=0} = 0. \tag{22}$$

Therefore, up to the second order in the transverse direction, the metric, in the region $r \gg GM$ and $z \ll \varepsilon$, is given by

$$\mathrm{d}s = -\mathrm{e}^{2a}\left[1 - \frac{2GM}{r}\left(1 + \frac{2\ell^2}{3r^2} - \frac{2\ell^2}{r^4}z^2\right)\right]\mathrm{d}t^2 + \mathrm{e}^{2a}\left[1 + \frac{2GM}{r}\left(1 + \frac{\ell^2}{r^2} + \frac{\ell^2}{r^4}z^2\right)\right]\mathrm{d}r^2 + \mathrm{e}^{2a}r^2\left(1 - \frac{3GM\ell^2}{r^5}z^2\right)\mathrm{d}\Omega^2 + \mathrm{d}z^2. \tag{23}$$

The scalar solution is simply $\varphi = \eta\tanh(\frac{z}{\varepsilon})$ in the same approximation order. As a matter of fact, it is interesting to emphasize that, admitting those initial data, the scalar field has no modification in any order of the expansion. We can check this by noticing that the perturbation of the scalar field (the function k) couples to the metric by means of the function $\gamma \equiv f + m + 2h$ (which is a kind of trace of the perturbation of the metric). We can determine an equation for γ by taking the combination $R^t_t + R^r_r + 2R^\theta_\theta - G_{zz}$ from the field equations. This yields

$$\gamma_{zz} + 2a'\gamma_z = 4\kappa\left(-\frac{1}{3}V'_0 k^{(1)} - \varphi^{(0)}_z k^{(1)}_z\right). \tag{24}$$

On the other hand, the scalar field equation gives

$$k_{zz} + 4a'k_z + \frac{1}{2}\varphi'_0\gamma_z + \mathrm{e}^{-2a}\nabla^2 k - V''_0 k = 0. \tag{25}$$

By using the initial data, $\gamma_z(0, r) = 0$, $k(0, r) = 0$ and $k_z(0, r) = 0$ in the above equations, we can conclude that the derivatives of γ_z and k of any order are zero when calculated in $z = 0$. Thus, there is no correction for the scalar field $\varphi = \eta\tanh(\frac{z}{\varepsilon})$ in the first order of GM, admitting those initial conditions.

3 Motion of massless test particles

Let us now consider a test particle moving in the five-dimensional spacetime whose metric is given by (23). It is well known that, in a thick brane of RSII type, geodesics are not stable, i.e., particles escape to the bulk if they suffer any transversal perturbation. So, it is necessary to provide a confinement mechanism for test particles in the brane. In

Ref. [46], we have proposed, based on the Yukawa interaction between fermions and the domain wall, a Lagrangian to describe the particle's motion in this context. As shown in that reference, the Lagrangian has the effect of increasing the effective mass of the particle, due to the interaction with the scalar field, and this modification ensures the localization of the particle.

The new Lagrangian is given by

$$L = -\sqrt{m^2 + h^2\varphi^2}\sqrt{-\tilde{g}_{AB}\dot{x}^A\dot{x}^B} \tag{26}$$

where m is the rest mass of the particle and h is the coupling constant of the interaction.

Calculating the $5D$-momentum P_A of the particle, which is obtained by taking $\partial L/\partial\dot{x}^A$, we find, using the condition $\tilde{g}_{AB}\dot{x}^A\dot{x}^B = -1$, that the effective mass of the particle is influenced by the scalar field according to the expression

$$P^A P_A = -(m^2 + h^2\varphi^2). \tag{27}$$

Of course, the usual relation is recovered turning off the interaction, i.e., taking $h = 0$. It is worthy of mention that a similar kind of Lagrangian was also employed, in a different context, to describe the interaction between test particles and dilatonic fields [50].

From the Euler–Lagrange equations, we can express the equation of motion in the following form:

$$\ddot{x}^A + \Gamma^A_{BC}\dot{x}^B\dot{x}^C = a^A, \tag{28}$$

where Γ^A_{BC} is the Levi-Civita connection and a^A is the proper acceleration of the particle due to its interaction with the scalar field and it can be written as the gradient of the effective mass of the particle:

$$a^A = -\Pi^{AC}\tilde{\nabla}_C \ln(m^2 + h^2\varphi^2), \tag{29}$$

where $\Pi^{AC} = \tilde{g}^{AC} + \dot{x}^A\dot{x}^C$ is the projection tensor into the four-space orthogonal to the particles' proper velocity $\dot{x}^A$.

The equation of motion (28) cannot be applied to study the propagation of a massless particle in the thick brane, because, the proper acceleration a^A is not well defined in the center of brane for $m = 0$. Thus, we need to follow a new procedure to describe the motion of light rays in this context. A possible way is to start with the dispersion relation (27). Taking $m = 0$ and introducing a new function $G(\varphi)$, we can write

$$P^A P_A = -h^2 G^2(\varphi). \tag{30}$$

The function $G(\varphi)$ must satisfies the condition $G(0) = 0$, to guarantee that in the brane center the particle's mass is equal to zero. We could have $G(\varphi) = \varphi$, for instance. Equation (30) shows that the interaction with the scalar field generates mass for the particle if it moves in the extra dimension. As we shall see later, this mechanism is responsible for keeping the particle confined in the brane provided $G(\varphi)$ is a function with appropriate properties. On the other hand, if a particle is moving strictly in the center of the brane, then the usual relation for a massless particle $P_A P^A = 0$ is recovered. In this case, that particle would have no mass from the 4D-viewpoint.

Based on the wave mechanics, which establishes a relation between the linear momentum P^A of a free particle and the wave vector K^A of its corresponding plane wave, we are led to propose that

$$K^A K_A = -h^2 G^2(\varphi). \tag{31}$$

The equation of motion for a massless particle can be obtained by taking the covariant derivative of the above equation and admitting that $K_A = \nabla_A S$, i.e., the wave vector is the gradient of the wave phase S (a scalar function). Then, by using the condition $\nabla_B K_A = \nabla_A K_B$, it follows that

$$K^A \nabla_A K_B = -\frac{1}{2}h^2 \nabla_B G^2(\varphi). \tag{32}$$

Of course, the particle world-line is obtained by integrating the equation $\mathrm{d}x^A/\mathrm{d}\lambda = K^A$, where λ is some affine parameter.

With the help of Eq. (32), we now are able to study the motion of light rays. First, we want to discuss some general aspects of the motion when $M = 0$. In this case, the geometry of the thick brane is described by the metric (2). As that metric is warped, the transversal motion decouples from the movement in the other directions. After some manipulation, we can show that the evolution of the z-coordinate can be directly integrated and its first-integral can be put in the form

$$\mathrm{e}^{2a}\dot{z}^2 = \dot{z}_0^2 - V_0(z), \tag{33}$$

where $\dot{z}_0$ is a constant related to the initial condition and $V_0(z) \equiv h^2 \mathrm{e}^{2a} G^2(\varphi)$ works as an effective potential for the transversal motion. It is clear that, by an appropriate choice of G, the potential V_0 is capable of confining the particle around the brane center as we have already mentioned.

When the mass M is taken into account the spacetime geometry is no longer described by a warped metric and the decoupling between the motions does not exist anymore. Considering the perturbation on the geometry caused by the mass M, as described in the metric (23), the equation of the transversal motion now assumes the following form:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}[\mathrm{e}^{2a}(h^2 G^2(\varphi) + \dot{z}^2)] = \mathrm{e}^{2a}\dot{z}q_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{34}$$

where

$$q_{\beta\gamma}\mathrm{d}x^\beta\mathrm{d}x^\gamma = \mathrm{e}^{2a}\left[-\left(\frac{8GM\ell^2}{r^5}z\right)\mathrm{d}t^2+\left(\frac{4GM\ell^2}{r^5}z\right)\mathrm{d}r^2 -\left(\frac{6GM\ell^2}{r^3}z\right)\mathrm{d}\Omega^2\right]. \quad (35)$$

As we can see, the transverse motion is now coupled to the movement in the other directions of the brane through the term $q_{\mu\nu}$, which is of the order of GM. Thus, to proceed, we have to analyze the evolution of other coordinates. By exploring the axial and temporal symmetries of the spacetime we get the equations

$$-g_{tt}\dot{t} = E, \quad (36)$$
$$g_{\phi\phi}\dot{\phi} = L, \quad (37)$$

where, for a convenient choice of the affine parameter, E and L correspond to the particle's energy and angular momentum respectively, as measured by asymptotic observers that lie in the center of the brane. Examining Eq. (28) for $x^A=\theta$, we can also verify that $\theta=\pi/2$ is a possible solution. By using these equations and also (31), we can rewrite (34) in this way

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}[\mathrm{e}^{2a}(h^2G^2(\varphi)+\dot{z}^2)] = \left[\left(\frac{q_{tt}}{g_{tt}}-\frac{q_{rr}}{\hat{g}_{rr}}\right)\frac{E^2}{\tilde{g}_{tt}}+\left(\frac{q_{\phi\phi}}{g_{\phi\phi}}-\frac{q_{rr}}{\hat{g}_{rr}}\right)\frac{L^2}{\tilde{g}_{\phi\phi}} -\frac{q_{rr}}{\hat{g}_{rr}}(\dot{z}^2+h^2G^2)\right](\mathrm{e}^{2a}\dot{z}). \quad (38)$$

As the right hand side is of the order of GM, according to our approximation, we can use the zero order solution, Eq. (33), in the last term to obtain the following expression:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}[\mathrm{e}^{2a}(h^2G^2(\varphi)+\dot{z}^2)] = \left[-\left(\frac{q_{tt}}{g_{tt}}-\frac{q_{rr}}{\hat{g}_{rr}}\right)E^2+\left(\frac{q_{\phi\phi}}{g_{\phi\phi}}-\frac{q_{rr}}{\hat{g}_{rr}}\right)\frac{L^2}{r^2}-\frac{q_{rr}}{\hat{g}_{rr}}\dot{z}_0^2\right]\dot{z}. \quad (39)$$

To proceed let us make some considerations about the particle's motion. In the absence of M, as we have seen, the transversal motion is bounded and therefore the particle oscillates around the center of the brane with a certain characteristic frequency ω. It seems reasonable to admit that the frequency is very high in order to explain the absence of any phenomenological trace of extra dimensions in measurements so far. Therefore, we can assume that during a complete period of oscillation ($\sim 1/\omega$) the relative change of radial coordinate ($\delta r/r$) is negligible. Under this hypothesis, which is equivalent to the condition

$$\frac{\dot{r}}{r} \ll \omega, \quad (40)$$

we can integrate directly the above equation to find

$$\mathrm{e}^{2a}G^2\dot{z}^2 = \dot{z}_0^2-(V_0(z)+V_M(r,z)), \quad (41)$$

where V_M is a new term of the effective potential that arises due to presence of the mass M. It is given by

$$V_M(r,z) = -\int_0^z\left[-\left(\frac{q_{tt}}{g_{tt}}-\frac{q_{rr}}{\hat{g}_{rr}}\right)E^2 +\left(\frac{q_{\phi\phi}}{g_{\phi\phi}}-\frac{q_{rr}}{\hat{g}_{rr}}\right)\frac{L^2}{r^2}-\frac{q_{rr}}{\hat{g}_{rr}}\dot{z}_0^2\right]\mathrm{d}z'. \quad (42)$$

In the first order with respect to GM and for small amplitude oscillation we obtain

$$V_M(r,z) = \left(\frac{2GM\ell^2z^2}{r^5}\right)\left[E^2+\frac{5}{2}\frac{L^2}{r^2}+\dot{z}_0^2\right]. \quad (43)$$

Therefore, the transversal motion is still bounded for small initial perturbations $\dot{z}_0$. Furthermore, based on (41), we can say that the frequency of the oscillation and also the amplitude now depend on the radial position of the particle. This is the major effect of the mass M on the movement in the extra-dimension direction.

3.1 Deflection of light

Let us investigate the influence of the transverse motion over the particle path in the four-dimensional spacetime. Starting from Eq. (31) and parameterizing it in terms of the angular coordinate ϕ, we obtain in the first order of GM:

$$-E^2\left[1+\frac{2GM}{r}\left(1+\frac{2\ell^2}{3r^2}-\frac{2\ell^2}{r^4}z^2\right)\right] +L^2r^2\left(1+\frac{3GM\ell^2}{r^5}z^2\right) +\left[1+\frac{2GM}{r}\left(1+\frac{\ell^2}{r^2}+\frac{\ell^2}{r^4}z^2\right)\right]\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2\frac{L^2}{r^4} = -\mathrm{e}^{2a}(h^2G^2+\dot{z}^2). \quad (44)$$

This equation can be written only in terms of r and z if we eliminate $\dot{z}^2$ by using Eq. (41). Despite this simplification, it is not easy to solve that equation, since z is evolving in time. However, as the time scale of the oscillation is much smaller than the time scale corresponding to the variations of the 4D-motion (see condition (40)), we can replace z^2 by its average, $\sigma^2=\langle z^2\rangle$, taken over several oscillation periods. Proceeding in this way, we can write Eq. (44) as a function of the radial coordinate only. Finally, introducing the coordinate $u=1/r$ and rewriting $\dot{z}_0^2$ as μ^2, the equation assumes, up to σ^2-order, the following form:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = F(u), \tag{45}$$

where

$$F(u) = \left(\frac{E^2-\mu^2}{L^2}\right) - u^2 + \frac{\mu^2}{L^2}2GMu + 2GMu^3\left[1-\left(\frac{E^2-3\mu^2}{L^2}\right)\frac{\ell^2}{3}\right] + 2GMu^5\ell^2\left[1+5\left(\frac{\mu^2-E^2}{L^2}\right)\sigma^2\right] + 10GMu^7\ell^2\sigma^2. \tag{46}$$

A direct integration of Eq. (45) gives us the dependence of the angular coordinate on the radial position:

$$\phi(u) - \phi(0) = \int_0^u \frac{1}{\sqrt{F(x)}}\mathrm{d}x. \tag{47}$$

To find an explicit solution, it is convenient to note that, if $M = 0$, the particle's trajectory would be a straight line ($u = 1/b\sin\phi$), where $b = \sqrt{L^2/(E^2-\mu^2)}$ is the impact parameter. The mass M, in the weak field regime ($GMu \ll 1$), causes a small deviation in the particle's original path. Due to the gravitational attraction of M, the minimum separation distance (r_0) between the particle and the mass M will be lesser than b by an amount of the order of GM/b in the first approximation order. For the calculation of the above integral it is important to determine r_0 (or equivalently u_0). As we can see from (45), u_0 is a root of $F(u)$. If we write $u_0 = \frac{1}{b}(1+\delta)$, then, investigating the roots of $F(u)$, we find, in the first order of GM, that

$$\delta = \frac{GM}{b}\left(1-\frac{\mu^2}{E^2}\right)^{-1}\left[1+\frac{2}{3}\frac{\ell^2}{b^2}\right]. \tag{48}$$

Therefore the minimum distance to the mass M depends on the photon's energy. This is a consequence of the fact the particle has a non-null effective mass μ due to the oscillation in the transverse direction.

After some algebraic manipulations, we can rewrite $F(u)$ as

$$F(u) = (u_0^2-u^2)\left(1-\frac{2}{(u+u_0)b}\delta - a_5u^5 - a_3u^3 - a_1u\right), \tag{49}$$

where

$$a_5 = 10\,GM\ell^2\sigma^2 \tag{50}$$

$$a_3 = 2\,GM\ell^2 \tag{51}$$

$$a_1 = 2\,GM\left[1+\frac{2}{3}\frac{\ell^2}{b^2}\left(\frac{E^2}{E^2-\mu^2}\right)\right]. \tag{52}$$

With the help of this expression, the integral (47) can be obtained directly and, therefore, the deflection angle, $\Delta\phi \equiv 2|\phi(u_0)-\phi(0)| - \pi$, can be calculated. In the first order of GM, we find that the deflection of the light by the mass M in the thick brane is given by

$$\Delta\phi = \frac{4GM}{b}\left[\left(\frac{1-\mu^2/2E^2}{1-\mu^2/E^2}\right)+\left(\frac{1-\mu^2/3E^2}{1-\mu^2/E^2}\right)\frac{\ell^2}{b^2}+\frac{2}{3}\frac{\ell^2\sigma^2}{b^4}\right]. \tag{53}$$

First we have to emphasize that, if there is no transversal perturbation, i.e., if $\dot{z}_0 = 0$, then the particle moves strictly in the center of the brane and the expression above recovers the result found in Refs. [33,34] for the light deflection in the Garriga–Tanaka thin-brane geometry. On the other hand, when the particle oscillates along the extra-dimension direction the deflection angle is modified by two contributions. The first one is related to the effective mass of the particle. As the particle becomes massive, the deflection becomes dependent on the photon's energy. Therefore, photons of distinct frequencies will suffer different deviations. Thus, the mass M, by means of its gravitational influence, could produce a rainbow if a beam of white light passes near it. The second effect of the transversal motion on the angle deflection comes from the last term. As we know σ^2 is related to the amplitude of the oscillation in the transversal direction and its presence can be explained as follows. Since the particle oscillates rapidly in the fifth direction, we may say that it feels not the geometry of the hypersurface $z = 0$ but an effective four-dimensional geometry, which is described by a kind of an average metric. Of course, this effective geometry should depend on σ^2, since it measures how far the particle penetrates into the extra-dimension direction.

3.2 Time delay

The time delay can be studied following a very similar procedure. From the equation below:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\dot{\phi}}{\dot{t}}\frac{\mathrm{d}r}{\mathrm{d}\phi},$$

we can determine the evolution of the radial coordinate with respect to t-coordinate. By using Eqs. (36), (37), and (44), we can show that the time delay of a light ray that travels from the point r_1 to the point r_2 and then is reflected back to the starting point r_1 is given by

$$\Delta t = \frac{4GM}{1-\mu^2/E^2} + \frac{6GM}{1-\mu^2/E^2}\left(1-\frac{1}{3\left(1-\mu^2/E^2\right)^{1/2}}\right) \times \ln\left(\frac{4r_1r_2}{r_0^2}\right)$$

$$
\begin{aligned}
&+\frac{28GM\ell^2}{3r_0^2}+\frac{8GM\ell^2}{3r_0^2}\frac{1}{\left(1-\mu^2/E^2\right)^{1/2}}+\frac{16GM\ell^2\sigma^2}{3r_0^4}\\
&+2\left(r_1+r_2\right)\left(\frac{\mu^2/E^2}{1-\mu^2/E^2}\right)\\
&-\frac{2GM}{r_0}\frac{\left(r_1+r_2\right)}{\left(1-\mu^2/E^2\right)}\left(\frac{\mu^2/E^2}{1-\mu^2/E^2}\right)\\
&-\frac{4GM\ell^2}{3r_0^2}\left(r_1+r_2\right)\left(\frac{\mu^2/E^2}{1-\mu^2/E^2}\right),
\end{aligned}
\tag{54}
$$

where r_0 is the distance of the closest approach to the mass M. As it happens in the deflection of the light ray, the thickness has two major effects on the flight time of the light ray. As we can see from (54), the time delay depends on the energy of the light ray and also on how far the particle penetrates in the fifth direction.

4 Semi-classical approach

We can approach the problem of the particle motion in the thick brane from a semi-classical point of view. The idea is to consider the quantization of the motion equation in the z-direction, while the movement in the 4D-world will be still described classically. Substituting $\dot{z}$ by the operator $i\partial_z$ (applying naively the usual quantization rules), the equation assumes a similar form of the Schroedinger equation:

$$
-\mathrm{e}^{2a}\frac{\partial^2\psi}{\partial z^2}+V_0\psi=\mu^2\psi. \tag{55}
$$

In this picture, the parameter μ is interpreted as the eigenvalue of the operator that appears in the left hand side of the equation. This operator can be viewed as a kind of mass operator since the eigenvalues μ plays the role of the effective mass of the particle when its motion in the 4D-world is considered. The mass operator is hermitian if the warping factor e^{-2a} is taken as a weight function for the inner product. Choosing $G(\varphi)$ appropriately, then we can find a bound state with eigenvalue $\mu=0$, the so-called zero-mode state.

When the body of mass M is taken into account, the situation changes a bit because of the presence of the potential V_M. However, we can treat V_M, which depends on r-coordinate, as a small perturbation of the main operator. Of course, r is evolving in time, but according to the condition (40), this variation can be considered as an adiabatic process. Therefore, we can find corrections to the eigenfunctions and eigenvalues by using the standard procedure of perturbation method in quantum mechanics. The new ground state ψ_0^M, for instance, which originally was the zero-mode ψ_0, is now associated to the non-zero eigenvalue $\langle\psi_0|V_M|\psi_0\rangle$, i.e., the expected value of the potential V_M evaluated in the unperturbed zero-mode state ψ_0. Considering Eq. (43), we find

$$
\langle\psi_0|V_M\,|\psi_0\rangle=\left(\frac{2GM\ell^2\sigma^2}{r^5}\right)\left[E^2+\frac{5}{2}\frac{L^2}{r^2}\right], \tag{56}
$$

where $\sigma^2=\langle\psi_0|z^2|\psi_0\rangle$ measures now the width in the transverse direction of the wave function corresponding to the zero-mode.

On the other hand, the equations of motion in the 4D-world, as Eq. (44), depend on z and $\dot{z}$, which should be considered as operators in our scheme. Therefore, in order to describe the motion classically, we have to take the average of the equation in a certain quantum state that describes the particle's state with respect to the fifth direction. In particular, the equation of the light deflection for the light ray in the fundamental state ψ_0^M is given by

$$
\begin{aligned}
&-E^2\left(1+\frac{2GM}{r}\left(1+\frac{2\ell^2}{3r^2}-\frac{2\ell^2}{r^4}\sigma^2\right)\right)\\
&\quad+L^2r^2\left(1+\frac{3GM\ell^2}{r^5}\sigma^2\right)\\
&\quad+\left(1+\frac{2GM}{r}\left(1+\frac{\ell^2}{r^2}+\frac{\ell^2}{r^4}\sigma^2\right)\right)\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2\frac{L^2}{r^4}\\
&=\left(\frac{2GM\ell^2\sigma^2}{r^5}\right)\left[E^2+\frac{5}{2}\frac{L^2}{r^2}\right].
\end{aligned}
\tag{57}
$$

As the brane has a non-null thickness, σ is not zero. Following the same steps previously described, we can obtain the deviation angle for the light ray in the zero-mode:

$$
\Delta\phi=\frac{4GM}{b}\left[1+\frac{\ell^2}{b^2}+\frac{2}{3}\frac{\ell^2\sigma^2}{b^4}\right]. \tag{58}
$$

In comparison to the case of the deflection in a thin brane [33,34], the difference comes from the third term which depends on the width of the wave function with respect to the extra-dimensional direction. This term encodes the main effect of the brane thickness on the light bending. Following a similar procedure, we can show that the time delay of light rays in the zero-mode is given by Eq. (54) with $\mu=0$.

The data about the deflection of light in the solar system [51,52] impose an upper bound on ℓ. Roughly, we have $\ell<10^4$ Km. On its turn, the value of σ depends crucially on the model of interaction between the particle and the brane. However, we can make some estimates considering that $\sigma<\varepsilon$ for bound states. If we take as reference values for ε and b the thickness of a TeV-brane and the Solar radius $R_\odot$, respectively, then we can write

$$
\frac{\sigma}{b}\lesssim 10^{-25}\left(\frac{\sigma}{\varepsilon}\right)\left(\frac{R_\odot}{b}\right). \tag{59}
$$

Therefore, the influence of the brane thickness over the bending of the light is very tiny in the solar system. Nevertheless, it is reasonable to expected that the effects become more significant for microscopic black holes in the strong field regime. However, this situation cannot be considered within our scheme.

5 Final remarks

Previous works have investigated classical tests of General Relativity in the braneworld scenario, trying to find empirical constraints on the parameters of infinitely thin-brane models. Here, we have analyzed the deflection of light and the time delay in the context of a thick brane scenario in order to determine the effects of the brane thickness over the motion in the 4D-world. The thick brane here is treated as a self-gravitating domain wall that corresponds to a regularized version of an infinitely thin brane in the RSII model. Considering a confined mass M in the thick brane, we find an approximate solution of this configuration, propagating initial data from the center of the brane to the fifth direction by using the Einstein equations coupled to the scalar field. The solution is built from the Garriga–Tanaka metric, taken as part of the initial data. Hence we may say that it represents approximately the gravitational field of a black hole in the weak field limit and in the vicinity of the brane center.

Based on a mechanism that describes the confinement of massive test particles in a domain wall, by means of a direct interaction between the particle and the scalar field, we developed a formalism to deal with the motion of massless particles in a thick brane. According to this prescription the particle gains an effective mass when the motion has a transversal component. The variation of the mass with respect to the fifth direction is dictated by a certain function $G(\varphi)$ of the scalar field. Bound motions are found if $G(\varphi)$ is conveniently chosen.

With the help of this formalism, we have studied the motion of massless particles in the spacetime produced by a mass M confined in a self-gravitating thick brane. The metric in the bulk is not warped and, therefore, the transverse motion is not decoupled from the movement in the radial direction. However, if the motion in the transversal direction is a high frequency oscillation, then we can find an approximate solution for the light deflection. We have shown that the transverse motion influences the bending of the light rays in two different ways. The first one is related to the effective mass the particle acquires due to its motion in the fifth dimension. As a consequence, the deviation angle becomes dependent on the light ray energy and, because of this feature, the mass M may produce the interesting phenomenon of a gravitational rainbow. The second effect is related to the fact that, as the particle oscillates rapidly in the fifth direction, it sees a kind of effective four-dimensional geometry. It follows, therefore, that the deviation angle shows a dependence on the amplitude of the transversal motion.

We have also considered a semi-classical approach to the problem. Analyzing the quantization of the motion in the fifth direction, we have seen that the system may admit a zero-mode solution, if $G(\varphi)$ has appropriate features. Based on this solution, we have studied the motion of the zero-mode in the 4D-world at the classical level. Specifically, we have determined the effects of the thickness of the brane over the deflection of the light rays in the zero-mode by a mass M. Compared to the thin brane result, the deviation angle has an additional contribution, which depends on the width, or more precisely, the root-mean-square deviation (σ), of the zero-mode in the extra-dimension direction. This additional term is a consequence of the fact that in thick branes the confinement of particles is not a delta-like confinement in a hypersurface, once the wave function has non-null width in the extra-dimensional direction. Thus, we may conclude that our results suggest that, regarding its apparent motion in the 4D-world, the particle in the zero-mode does not feel the geometry of the hypersurface $z = 0$, instead it feels an effective four-dimensional geometry that depends on the profile of the zero-mode wave function in the fifth direction.

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