Eur. Phys. J. C (2015) 75:38
DOI 10.1140/epjc/s10052-014-3241-4

THE EUROPEAN PHYSICAL JOURNAL C

Letter

On the relation between boundary proposals and hidden symmetries of the extended pre-big bang quantum cosmology

S. Jalalzadeh[1,a], **T. Rostami**[1,b], **P. V. Moniz**[2,3,c]

[1] Department of Physics, Shahid Beheshti University, G. C., Evin, 19839 Tehran, Iran
[2] Centro de Matemática e Aplicações-UBI, Covilhã, Portugal
[3] Departmento de Física, Universidade da Beira Interior, 6200 Covilhã, Portugal

Received: 11 July 2014 / Accepted: 17 December 2014 / Published online: 29 January 2015

Abstract A framework associating quantum cosmological boundary conditions to minisuperspace hidden symmetries has been introduced in Jalalzadeh and Moniz (Phys Rev D 89:083504, 2014). The scope of the application was, notwithstanding the novelty, restrictive because it lacked a discussion involving realistic matter fields. Therefore, in the present letter, we extend the framework scope to encompass elements from a scalar–tensor theory in the presence of a cosmological constant. More precisely, it is shown that hidden minisuperspace symmetries present in a pre-big bang model suggest a process from which boundary conditions can be selected.

1 Introduction

Quantum geometrodynamics is the oldest and still active approach to quantum gravity and quantum cosmology [1]. Since it bears a canonical setup in its foundation, it contains constraints as central equations. In the case of a metric representation perspective, these are the Hamiltonian and the diffeomorphism constraints and in the case of the connection (Ashtekar) approach, the Gauss constraints are also added.

Quantum geometrodynamics has, nevertheless, many technical and conceptual challenges: the problem of time, the problem of observables, factor ordering issues, the global structure of spacetime manifold and the problem of boundary conditions (for more details, see [1]). The issue of boundary conditions for the wave function of the Universe has been one of the most active areas of quantum cosmology. Two leading lines have been the no-boundary [2] and the tunneling proposals [3,4]. Two other proposals (of a Dirichlet or a Neumann nature) have also been used, although less often in the literature, to deal with the presence of classical singularities. More precisely, the wave function should vanish at the classical singularity (De Witt boundary condition) [5], or its derivative with respect to the scale factor vanishes at the classical singularity [6,7].

All those boundary conditions are chosen ad hoc, with some particular physical intuition in mind [8,9], possibly with some characteristic symmetry element being imported to assist, but they are not part of a dynamical law. Due to the fact that the algebra intrinsic to any given minisuperspace is specified by the symmetries of the model, including the type of matter content, it may be of interest to investigate whether there is a relation between any set of allowed boundary conditions and the algebra associated to the Dirac observables of the cosmological model.

Let us be more concrete. In [10], a simple closed Friedmann–Lemaître–Robertson–Walker (FLRW) model in the presence of either dust or radiation was studied. It was shown that by means of the presence of a hidden symmetry, namely $su(1,1)$, the model admits a particular Dirac observable, subsequently allowing to establish boundary proposals admissible for the model. More precisely, the Casimir operator $J^2 = j(j-1)$ of the $su(1,1)$ algebra leads (for the value $-\frac{3}{16}$) to the discrete set $j = \{\frac{1}{4}, \frac{3}{4}\}$ for the Bargmann index, which is a gauge invariant observable. Then it was shown that those two gauge invariant quantities split the underlying Hilbert space into two disjoint invariant subspaces, each corresponding to a different choice of boundary conditions (namely, of a Dirichlet or Neumann type). Notwithstanding the interest of this case study, it is a fact that it was a model of a very restrictive range.

In what follows, we will extend the scope of that discussion and employ a scalar–tensor gravity theory. Scalar–tensor gravity theories seem to be relevant in explaining the very early universe as shown in [11–14]. These theories are defined through a non-minimal coupling of a scalar field

[a] e-mail: s-jalalzadeh@sbu.ac.ir
[b] e-mail: t_rostami@sbu.ac.ir
[c] e-mail: pmoniz@ubi.pt

to the spacetime curvature, which originates from the low energy limit of unified field theories such as superstring theory [12–15]. With a suitable conformal transformation, theories with higher-order terms in the Ricci scalar may lead to scalar–tensor form [16]. Moreover, higher-dimensional gravity leads to a scalar–tensor theory from a dimensional reduction [17]. We make our analysis more concrete by employing a spatially flat isotropic cosmology, in which the dilaton scalar field is non-minimally coupled to the spacetime curvature, in the presence of a cosmological constant. There is a hidden symmetry, which, leading to a particular Dirac observable, provides a setting where concrete boundary conditions can be subsequently extracted [18]. This paper is then organized as follows. In Sect. 2 we present our model from a classical canonical perspective. In Sect. 3 we formulate its quantization, briefly elaborating on the standard canonical procedure as well as from a reduced phase space and corresponding observables point of view. Furthermore, we describe how from a hidden minisuperspace symmetry, taking into account the observables' algebra, concrete boundary conditions can be identified within this framework. Section 4 contains a summary, a discussion as well as an outlook on our results.

2 The model

One of the simplest extension of Einstein gravity is the scalar–tensor theory. The reduced string action in spatially flat and homogeneous cosmologies has a scale factor duality (SFD) [19–22]. The symmetry group is ${Z_2}^{D-1}$, which relates the expanding dimensions to contracting ones. SFD is a special case of a more general $O(d,d)$ symmetry. It has been shown [23] that the SFD at the classical level is associated with a $N=2$ supersymmetry at quantum level [18]. In Ref. [24], it is described that the concept of pre-big bang cosmology can be extended beyond the truncated string effective action to include more general dilaton–graviton systems. It is interesting to study these types of theories in their own right and they also place the results and predictions of string cosmology in a wider scenery.

Let us start from a D-dimensional scalar–tensor theory, which is non-minimally coupled to the spacetime curvature as

$$\mathcal{S}=\int d^Dx\sqrt{-g}\,e^{-\phi}(R-\omega(\nabla\phi)^2-2\Lambda), \tag{1}$$

where $\mathcal{R}$ is Ricci scalar, ϕ is the dilaton field, which plays the role of varying gravitational constant, g is the determinant of spacetime metric, ω is a spacetime constant and Λ is the cosmological constant. When $\Lambda=0$, this theory is equivalent to the standard Brans–Dicke theory. The genus-zero effective action of the bosonic string reduces to the above action when the antisymmetric tensor field $B_{\mu\nu}$ vanishes, $\omega=-1$ and $\Lambda=(D-26)/3\alpha'$ (α' is the inverse string tension) [15]. Also, when Λ is proportional to $(D-10)$, the action (1) represents the effective action for the bosonic sector of the closed superstring. Before proceeding, let us mention that a vast part of the content of this section is a summarized extract of [23].

We employ a spatially flat, isotropic, and homogeneous FLRW model, parameterizing the metric by means of the line element

$$ds^2=-N(t)^2dt^2+e^{2\alpha(t)}dx_i^2,\quad i=1,2,\ldots,(D-1), \tag{2}$$

where $N(t)$ is the lapse function and $e^{\alpha(t)}$ is the scale factor of the universe. Then the action reduces to

$$\mathcal{S}=\int dte^{(D-1)\alpha-\phi}\left[\frac{1}{N}(-(D-1)(D-2)\dot{\alpha}^2 +(D-1)\dot{\alpha}\dot{\phi}+\omega\dot{\phi}^2)-2N\Lambda\right]. \tag{3}$$

SFD is a characteristic of our setting, allowing us to discuss string theory features within cosmology. In fact, the above action has SFD properties which are allowed by means of associating our analysis to a spatially flat FLRW model [23]. The action is symmetric under the (SFD) simultaneous transformation

$$\begin{aligned}\alpha&=\left[\frac{(D-2)+(D-1)\omega}{D+(D-1)\omega}\right]\tilde{\alpha}-\left[\frac{2(1+\omega)}{D+(D-1)\omega}\right]\tilde{\phi},\\ \phi&=-\left[\frac{2(D-1)}{D+(D-1)\omega}\right]\tilde{\alpha}-\left[\frac{(D-2)+(D-1)\omega}{D+(D-1)\omega}\right]\tilde{\phi}.\end{aligned} \tag{4}$$

A conserved quantity can be identified, $F=e^{((D-1)\dot{\alpha}-\phi)}\left[\dot{\alpha}+(1+\omega)\dot{\phi}\right]$, as introduced in [23]. The time reversal invariance of the action under $t=-\tilde{t}$, in addition to the above transformation (4), leaves F unchanged. If we use the following transformations:

$$\begin{cases} u=2\epsilon^{\frac{1}{2}}\left[\frac{D-1+(D-2)\omega}{D+(D-1)\omega}\right]^{\frac{1}{2}}e^{\frac{1}{2}((D-1)\alpha-\phi)}\\ \quad\times\sinh\left(\frac{\gamma}{2}\alpha+\frac{1}{2(D-2)}(\frac{D-1}{\gamma}-\gamma)\phi\right),\\ v=2\epsilon^{\frac{1}{2}}\left[\frac{D-1+(D-2)\omega}{D+(D-1)\omega}\right]^{\frac{1}{2}}e^{\frac{1}{2}((D-1)\alpha-\phi)}\\ \quad\times\cosh\left(\frac{\gamma}{2}\alpha+\frac{1}{2(D-2)}(\frac{D-1}{\gamma}-\gamma)\phi\right),\end{cases} \tag{5}$$

where

$$\begin{cases}\gamma=\left[\frac{D-1}{D-1+(D-2)\omega}\right]^{\frac{1}{2}},\\ \lambda=-2\Lambda\left[\frac{D+(D-1)\omega}{D-1+(D-2)\omega}\right],\\ \epsilon=\pm1.\end{cases} \tag{6}$$

then the action changes to the oscillator–ghost–oscillator form:

$$\mathcal{S} = \frac{1}{\epsilon}\int dt\left[\frac{1}{N}\left(\dot{u}^2 - \dot{v}^2\right) - \frac{\lambda}{4}\left(u^2 - v^2\right)N\right]. \tag{7}$$

It is obvious that if $\epsilon = +1$, $\omega > \frac{-D}{D-1}$ and if $\epsilon = -1$, $\omega < \frac{-D}{D-1}$. The minisuperspace signature is $(+, -)$, therefore in the reduced action u acts as the "spacelike" component and v as the "timelike" component. In the $\{u, v\}$ coordinate system, the duality symmetry is more apparent. In fact, the invariance of the action under time reversal and parity symmetry, which is introduced as

$$t \to -t, u \to -u, v \to v, \tag{8}$$

leaves the conserved quantity $F = \frac{\gamma}{2}(\dot{u}v - \dot{v}u)$ invariant. It should be noted that the transformation (4) can be seen emerging from the above parity symmetry. From the action (7) let us write its Lagrangian:

$$\mathcal{L} = \frac{1}{\epsilon}\left[\frac{1}{N}(\dot{u}^2 - \dot{v}^2) - \frac{\lambda N}{4}(u^2 - v^2)\right]. \tag{9}$$

In order to construct the Hamiltonian of the model, the momenta conjugate to u, v, and N are

$$\Pi_u = \frac{2}{N\epsilon}\dot{u}, \quad \Pi_v = \frac{-2}{N\epsilon}\dot{v}, \quad \Pi_N = 0, \tag{10}$$

which subsequently lead to

$$\mathcal{H} = \frac{N\epsilon}{4}\left[(\Pi_u^2 - \Pi_v^2) + \lambda(u^2 - v^2)\right]. \tag{11}$$

The existence of the constraint $\Pi_N = 0$ indicates that the Lagrangian of the system is singular and the Hamiltonian can be generalized by adding to it the primary constraints multiplied by arbitrary functions of time, ζ. The total Hamiltonian will then be

$$\mathcal{H}_T = \frac{N\epsilon}{4}(\Pi_u^2 - \Pi_v^2) + \frac{N\lambda\epsilon}{4}(u^2 - v^2) + \zeta\Pi_N. \tag{12}$$

The constraint must be satisfied at all times and therefore,

$$\dot{\Pi}_N = \{\Pi_N, \mathcal{H}_T\} \approx 0, \tag{13}$$

which leads to the secondary (Hamiltonian) constraint

$$H = \frac{N\epsilon}{4}(\Pi_u^2 - \Pi_v^2) + \frac{N\lambda\epsilon}{4}(u^2 - v^2) \approx 0. \tag{14}$$

The existence of the constraint (14) means that there are some degrees of freedom which are not physically relevant. Hence we can fix the gauge as $N =$ constant. Note that, by means of the coordinate transformation $v = R\cosh\theta$ and $u = R\sinh\theta$, the Hamiltonian (14) becomes

$$H = \frac{N\epsilon}{4}\left(-\Pi_R^2 + \frac{1}{R^2}\Pi_\theta^2 - \lambda R^2\right), \tag{15}$$

where $\Pi_R = -\frac{2}{N\epsilon}\dot{R}$ is the "radial" momentum and $\Pi_\theta = \frac{2R^2}{N\epsilon}\dot{\theta}$ denotes a conserved "angular momentum". It is easy to show that $\Pi_\theta = \frac{2}{N\epsilon}(\dot{u}v - \dot{v}u) = \frac{4}{N\epsilon\gamma}F$. Hence, the duality symmetry in these new coordinates is equivalent to the anticlockwise pseudo-rotation in time reversal; $\theta \to -\theta$, $t \to -t$.

3 Reduced phase space quantization and Dirac observables

3.1 Standard quantization

In this (most usual and straightforward procedure) the quantization of the system is made by replacing the canonical conjugate variables (u, Π_u), (v, Π_v) by operators satisfying the commutation relations $[x_i, \Pi_j] = -i\delta_{ij}$. Thus, if we neglect any ambiguities that may arise due to factor ordering, the Wheeler–De Witt (WDW) equation can be written, from the Hamiltonian constraint (14), as

$$\left[-\partial_u^2 + \partial_v^2 + \Omega^2(u^2 - v^2)\right]\Psi(u, v) = 0, \tag{16}$$

where $\Omega = \sqrt{\lambda}$. The wave function of the universe is easily obtained as

$$\Psi_{n_u,n_v}(u, v) = \mathcal{N} H_{n_u}(\sqrt{\Omega}u) H_{n_v}(\sqrt{\Omega}v) e^{-\Omega(u^2+v^2)/2}, \tag{17}$$

where H_n is the Hermite polynomial of order n and $\mathcal{N}$ is a normalization constant. These solutions form a discrete basis for any bounded wave function $\Psi = \sum c_n \Psi_n$, where c_n are complex coefficients. For the ground state, $n = 0$, and $n > 0$ correspond to the excited states. These states represents Euclidean geometries, as they do not oscillate. Lorentzian geometries may be obtained if the appropriate values for c_n are taken [18,23].

It should be noted that the classical solution has a singularity at $u = 0$ and $v = 0$. In order to avoid this singularity, we can adopt that the wave function vanishes at the classical singularity i.e., (De Witt boundary proposal)

$$\Psi(u, v)|_{u=0,v=0} = 0, \tag{18}$$

or, as proposed by Tipler in [25],

$$\frac{d\Psi}{dx_i}|_{u=0,v=0} = 0. \tag{19}$$

As is mentioned in [26], upon choosing any of the above boundary conditions, we obtain, for the oscillator–ghost–oscillator system,

$$n_u - n_v = 0, \tag{20}$$

which, for the De Witt (18) boundary condition, states that both n_u and n_v must be odd, whereas if the boundary condition (19) is taken, then both n_u and n_v must be even.

3.2 Reduced phase space and observables

In the reduced phase space quantization, we first identify the physical degrees of freedom of a given model at the classical level by means of the factorization of the constraint surface with respect to the action of the gauge group, generated by the constraints. Then the resulting Hamiltonian system is quantized as a usual unconstrained system [27]. The constraint surface is obtained by means of gauge transformations, generated by all the first class constraints. A gauge invariant function on this surface is an observable. A well known setting is general relativity, which is invariant under the group of spacetime diffeomorphisms and, consequently, the corresponding Hamiltonian can be expressed as a sum of constraints [1]. The point to take into consideration is that from the associated Poisson bracket algebra we can describe the classical dynamics of a system and any observable must commute with these constraints.

In more detail, in order to find the gauge invariant observables associated to the Lagrangian (9), we consider the unconstrained phase space Γ in $\mathbb{R}^4$ with global canonical coordinates (x_i, Π_i). Subsequently, let us define complex valued holomorphic functions $S = \{C_i, C_i^*, 1\}$ on Γ:

$$\begin{cases} C_i = (\frac{1}{2\Omega})^{\frac{1}{2}}[\Omega x_i + i\Pi_i], \\ C_i^* = (\frac{1}{2\Omega})^{\frac{1}{2}}[\Omega x_i - i\Pi_i]. \end{cases} \tag{21}$$

These functions satisfy the Poisson brackets $\{C_i, C_j^*\} = -i\delta_{ij}$. The Hamiltonian constraint can then be readily written as

$$H = \Omega\left[C_u^* C_u - C_v^* C_v\right]. \tag{22}$$

Moreover, let us consider on Γ the following two sets of functions:

$$\tilde{J}_0 = C_u^* C_u - C_v^* C_v \tag{23}$$

and

$$\begin{cases} J_0 = \frac{1}{2}\left(C_u^* C_u + C_v^* C_v\right), \\ J_+ = C_u^* C_v^*, \\ J_- = C_v C_u. \end{cases} \tag{24}$$

The second set of functions satisfy the following closed Poisson algebra:

$$\{J_0, J_\pm\} = \mp i J_\pm, \{J_+, J_-\} = 2i J_0. \tag{25}$$

Since the Poisson brackets of the above variables and the Hamiltonian vanish, $\{\tilde{J}_0, H\} = \{H, J_0\} = \{H, J_\pm\} = 0$, their values on the constraint surface are constants of motion. In addition, the phase space of the model is four-dimensional, which implies that there will be at most four independent constraints. The Hamiltonian constraint (22) implies $\tilde{J}_0 = 0$. Furthermore, if we define

$$J^2 := J_0^2 - \frac{1}{2}\left(J_+ J_- + J_- J_+\right), \tag{26}$$

then, by inserting definitions (24) into (26), we can easily show that on the constraint surface $H = 0$, the J's are not algebraically independent but satisfy the identity

$$J^2 = \tilde{J}_0^2 - \frac{1}{4} = \left(\frac{H}{\Omega}\right)^2 - \frac{1}{4} = -\frac{1}{4}. \tag{27}$$

Rewriting then the conserved quantity F in terms of these new set of variables, as

$$F = \frac{i\epsilon\gamma}{4}\left(J_+ - J_-\right), \tag{28}$$

the vanishing of its Poisson bracket with the Hamiltonian is obvious.

3.3 Reduced phase space quantization, hidden symmetries and boundary conditions

Regarding a quantum mechanical description, in order to determine the wave function we must specify certain conditions at the boundary of the system under consideration. However, in quantum cosmology there is nothing external to the universe. It is assumed that an independent physical law would define appropriate boundary conditions [28]. Or, as we discuss herein, the symmetries of the cosmological model under investigation may suggest arguments for such a selection. Indeed, using hidden symmetries associated to the dynamics of the model will lead us to extract specific boundary conditions—more concretely, by means of considering the Dirac observables of the cosmological model.

Let us start by introducing the quantum counterparts of the set S as $\hat{S} = \{C_i, C_i^\dagger, 1\}$, with the following commutator algebra:

$$[C_i, C_j^\dagger] = \delta_{ij}, \quad [C_i, 1] = [C_i^\dagger, 1] = 0. \tag{29}$$

The action of operators $C_i, C_i^\dagger$ on the physical Hilbert space is

$$C_i|n_i\rangle = \sqrt{n_i}|n_i - 1\rangle,$$
$$C_i^\dagger|n_i\rangle = \sqrt{n_i + 1}|n_i + 1\rangle. \tag{30}$$

To represent the Dirac observables of our model quantum mechanically, we define operators on the phase space. More concretely, we notice that the (classical) Poisson bracket algebra of the $u(1,1)$ for J_0 and $J_\pm$ and $\tilde{J}_0$ (operator of the $u(1)$ algebra) can be promoted into a commutator algebra by setting the $u(1)$ generator as [26]

$$\tilde{J}_0 = -C_u^\dagger C_u + C_v^\dagger C_v, \tag{31}$$

and the generators of $SU(1,1)$ in two-mode realization as [29,30]

$$\begin{cases} J_+ = C_u^\dagger C_v^\dagger, \\ J_- = C_v C_u, \\ J_0 = \frac{1}{2}[C_u^\dagger C_u + C_v^\dagger C_v + 1], \end{cases} \tag{32}$$

which satisfy the following commutation relations:

$$\left[J_+, J_-\right] = -2J_0, \left[J_0, J_\pm\right] = \pm J_\pm. \tag{33}$$

The above commutation relations represent the Lie algebra of $su(1,1)$. The action of the above generators on a set of basis eigenvectors $|j,m\rangle$, which are simultaneous eigenvectors of J_0 and J^2, is given by

$$\begin{cases} J_0|j,m\rangle = (j+m)|j,m\rangle, \\ J_+|j,m\rangle = \sqrt{(2j+m)(m+1)}|j,m+1\rangle, \\ J_-|j,m\rangle = \sqrt{m(2j+m-1)}|j,m-1\rangle. \end{cases} \tag{34}$$

The positive discrete series of this Lie algebra are labeled by the Bargmann index j, which is a positive real number, $j > 0$, and m is any nonnegative integer [31]. According to (30) and the constraint $\tilde{J}_0 = 0$, we have for the $u(1)$ generator

$$\tilde{J}_0|j,m\rangle = 0. \tag{35}$$

Moreover, the Casimir operator is defined as in [32]

$$\begin{cases} J^2 := J_0^2 - \frac{1}{2}(J_+J_- + J_-J_+), \\ J^2|j,m\rangle = j(j-1)|j,m\rangle, \end{cases} \tag{36}$$

with the following commutation relations:

$$\left[J^2, J_0\right] = 0, \left[J^2, J_\pm\right] = 0. \tag{37}$$

Thus, the irreducible representation of $u(1,1)$ is determined by the number j and the eigenstates of J^2, J_0, and $\tilde{J}_0$. Furthermore, the Hamiltonian can be written as

$$H = \Omega\left(C_u^\dagger C_u - C_v^\dagger C_v\right) = -\Omega\tilde{J}_0, \tag{38}$$

which allow that the Casimir operator commutes with the Hamiltonian

$$[J^2, H] = 0, \tag{39}$$

which shows that the Bargmann index is a Dirac observable. In the number operator representation, the basis states are $|n_v\rangle$ and $|n_u\rangle$, so that the Hilbert space of the "two-mode" field is the direct product

$$|n_v, n_u\rangle = |n_v\rangle \otimes |n_u\rangle. \tag{40}$$

However, it is desirable to map these direct product states on the $\mathcal{D}^+(j)$ unitary irreducible representation of $SU(1,1)$. To do this we first need to find the Bargmann index j. From the realization of definition (32), the Casimir operator can be shown to be

$$J^2 = \frac{1}{4}(\tilde{J}_0^2 - 1). \tag{41}$$

In general, according to the definition of $\tilde{J}_0$ in (31), the eigenvalues of this operator are just the difference between n_v and n_u. For fixed eigenvalues of $\tilde{J}_0$, with the condition $|n_v - n_u| \neq 0$, using (31) and (36), the Bargmann index will be

$$j = \frac{1}{2} + \frac{1}{2}|n_v - n_u| = 1, \frac{3}{2}, 2, \ldots, |n_v - n_u| \neq 0. \tag{42}$$

However, in our model, the Hamiltonian constraint and (38) indicate that $n_v - n_u = 0$. This means that the Hamiltonian constraint forces a degenerate case $n_v = n_u$, and, consequently, we get the unitary representation with a degenerate Bargmann index,

$$j = \frac{1}{2}. \tag{43}$$

Hence, the Bargmann index $\{\frac{1}{2}\}$ is a gauge invariant observable of the quantum cosmological model. As J^2, J_0, and $J_\pm$ commute with the Hamiltonian, they leave the physical Hilbert space $V_{H=0}$ invariant and consequently we choose $\{J_0, J^2, \tilde{J}_0, 1\}$ as physical operators of the model. In addition, from (32) we can easily examine how J_0 and $J_\pm$ act on $|n_u, n_v\rangle$:

$$\begin{cases} J_0|n_u, n_v\rangle = \frac{1}{2}(n_u + n_v + 1)|n_u, n_v\rangle, \\ J_+|n_u, n_v\rangle = \sqrt{(n_u+1)(n_v+1)}|n_u + 1, n_v + 1\rangle, \\ J_-|n_u, n_v\rangle = \sqrt{n_u n_v}|n_u - 1, n_v - 1\rangle. \end{cases} \tag{44}$$

Comparing Eq. (44) with Eq. (34) and using $j = 1/2$, we obtain

$$n_u = n_v = m. \tag{45}$$

According to (33), J_+ and J_- do not commute with each other, so they are not mutually compatible observables. It is therefore impossible to simultaneously measure J_+ and J_-. This means that odd m's are separated from even m's. For example, consider that m is an odd number, then $m+1$ is even, so, if we measure an odd m, then the information about $m+1$ (which is even) is not accessible.

Let us now investigate in more depth how the presence of the duality symmetry in our quantum cosmological model can point to concrete boundary conditions to consider. We recall from Subsect. 3.1 that a wave function of the universe can be constructed admitting the De Witt's boundary proposal (18) or the boundary proposal (19). We now aim to establish, for the found Dirac observables and algebraic features, how symmetry considerations will assist us in identifying a set of such conditions to choose one from. We therefore represent the time reversal mentioned in Sect. 2 concerning the action (3), by means of introducing the time reversal operator Θ, $\Theta \Pi_i \Theta^{-1} = -\Pi_i$. The behavior of the $u(1,1)$ operators under this time reversal operator is

$$\begin{aligned} &\Theta \tilde{J}_0 \Theta^{-1} = \tilde{J}_0, \Theta J_0 \Theta^{-1} = J_0, \\ &\Theta J^2 \Theta^{-1} = J^2, \Theta J_\pm \Theta^{-1} = J_\pm. \end{aligned} \tag{46}$$

In order to further illustrate the transformation (4), or equivalently the simultaneous change of $u \to -u$ and $v \to v$, we introduce the parity operator in minisuperspace as $\pi^\dagger u \pi = -u$. The parity operator acts then on the $u(1,1)$ operators as

$$\begin{aligned} &\pi^\dagger \tilde{J}_0 \pi = \tilde{J}_0, \pi^\dagger J_0 \pi = J_0, \\ &\pi^\dagger J^2 \pi = J^2, \pi^\dagger J_\pm \pi = -J_\pm. \end{aligned} \tag{47}$$

Hence, F respects the symmetry (cf. Eq. (8))

$$\pi^\dagger \Theta F \Theta^{-1} \pi = F. \tag{48}$$

The symmetry underlying the transformation (8) is equivalently represented into the action of the operator $\pi^\dagger\Theta$. Using (38), (46), and (47), this allows us to re-affirm that H is indeed invariant under the SFD and time reversal, here expressed in terms of $\pi^\dagger\Theta$, in correspondence to the classical description in Sect. 2. The relevant feature to stress is that as F commutes with the Hamiltonian, F is therefore also a gauge invariant observable.

At this point, having presented the necessary framework as well as essential elements for the argument, let us explain how from the hidden minisuperspace symmetries we can suggest a process from which boundary conditions can be chosen. For that purpose, to make concrete a physical realization of the SFD together with the time reversal, we consider the pre-big bang cosmological scenario within the context of the scalar–tensor theory of gravity [24]. The aim in any version of that scenario, which for convenience we are adopting to apply our framework, is to describe how the universe starts out in a contracting pre-big bang phase, then goes through a bounce, and finally emerges as an expanding post-big bang universe. Hence, the bounce is represented by the self-dual point in the SFD. However, as the literature and research work has shown, this has not been fully achieved in terms of an effective description or workable SFD cosmology: these two phases are separated by a curvature singularity. Assuming, nevertheless, that the singularity may disappear by including quantum gravitational or higher-order corrections from string theory, we can identify, from the duality invariance of the equations of cosmological model, a clear suggestion about a possible temporal completion, based on a "self-duality principle" [33,34].

An approach and methodology to achieve an exit from the pre-big bang phase, $(t < 0)$, as described in the above paragraph, could be provided by quantum cosmology [35,36]. If we indeed assume that quantum effects would eventually remove the curvature singularity, we can expect that there exists a smooth wave function for the whole of the universe, including pre- and post-big bang phases properly matched together.

Being therefore more concrete, by applying this context to the simple setting studied here, introduced and described in detail in the previous sections, let us take that the wave function of the universe $\Psi_n(u, v)$ in (17) encompasses the phases of the pre-big bang with $(t' = -t, u' = -u, v' = v)$ and post-big bang with (t, u, v). We now concentrate on the discrete Dirac observables of model. According to Eq. (47), $[H, \pi] = 0$. Consequently, $\Psi(u, v)$ is also an eigenfunction of the parity operator, with eigenvalues ± 1 which are Dirac observables. Similarly, the Hamiltonian commutes with the time reversal, $[H, \Theta] = 0$, which, together with the Hamiltonian constraint $H = 0$ [or equivalently $n_v = n_u$ (the eigenfunctions are nondegenerate)] implies that the wave function is real, $\Psi_n(u, v) = \Psi_n^*(u, v)$. Hence, a general wave function is given by $\sum_{n\ \text{even}} c_n \Psi_n$ (even parity) or $\sum_{n\ \text{odd}} c_n \Psi_n$ (odd parity), with real coefficients. Therefore, the states of the Hilbert space can be classified in terms of these two values of the parity operator, as

$$V_{H=0} = V_{H=0,\pi=+1} \oplus V_{H=0,\pi=-1}. \tag{49}$$

Thus, the gauge invariance of the parity implies a partition of the Hilbert space into two disjointed invariant subspaces, which are equivalent to the result of imposing boundary conditions (18) or (19), respectively.

4 Conclusions, discussion and outlook

In this letter, we extended the scope of [10] to obtain relevant boundary conditions for a quantum cosmological model, by means of identifying the corresponding necessary Dirac observables.

In more detail, we considered a non-minimally coupled scalar field in a FLRW universe with a cosmological constant in the context of scalar–tensor cosmology. More specifically, in order to include the SFD of the pre-big bang setting extracted from string theory features, we considered a spatially flat universe.

The reduced phase space quantization was then investigated. The irreducible operators of the Lie algebra $u(1, 1)$ were shown to have a vanishing commutation relation with the Hamiltonian. From the vanishing of the commutator of the $su(1, 1)$ generator with the Hamiltonian, together with the gauge invariance of the Bargmann index, this fixes the allowed states for the wave function of the universe. Let us be more clear and elaborate more broadly. In Ref. [10] a closed FLRW universe filled with either dust or radiation was considered, which was discussed by means of a Hamiltonian and pointed out to be equivalent to a one-dimensional simple harmonic oscillator. The hidden symmetry of that model was $su(1, 1)$, with the set of gauge invariant Bargmann values $\{\frac{1}{4}, \frac{3}{4}\}$, which is related to the so called Barut–Girardello (even–odd coherent) states [37]. This led to a split in the underlying Hilbert space into two disjoint invariant subspaces, each then subsequently having been shown to be corresponding to a different choice of boundary conditions, as (18) and (19), to be more precise.

In a similar procedure (but bearing intrinsic differences with respect to some elements in [10]), we extracted the hidden symmetries of the cosmological scenario under study, which led us to the set of Dirac observables of model. The presence of a non-minimally scalar field as the matter component in a spatially flat universe led us to extended symmetries, namely $u(1, 1)$ together with time reversal (with respect to the comoving time) and parity in minisuperspace. The Hamiltonian of the model studied in this paper is (tobe compared with that in [10]) instead is equivalent to the oscillator–ghost–oscillator system, which led here to a two-mode realization of the $su(1, 1)$ algebra. This specificity of our setup induced from the Hamiltonian constraint a degenerate Bargmann index for the model. Hence, unlike Ref. [10], the continuous symmetries of our model are not responsible for pointing out the boundary conditions. More concretely, there are instead the SFD of the cosmological model plus time reversal, which are equivalent to the operator $\pi^{\dagger}\Theta$ action, that allow one here to select specifically the boundary conditions, associated to the partition (49).

Notwithstanding the contribution we think this approach and framework brings to quantum cosmology, there are issues where additional more work is needed and indeed these constitute new lines to explore.

On the one hand, by employing a homogeneous (and isotropic) model we are neglecting an infinity of degrees of freedom, namely the inhomogeneous modes that a wider metric or matter fields would provide. Only the presence of the latter ones would bring about a more substantial realistic sense to this methodology. The scope of application of the framework is therefore still very restrictive, in spite of this paper being a development with respect to the scarce content in [10]. Extending the present framework to either FLRW models, where homogeneity and isotropy would be the background, with fluctuations or even some inhomogeneous simple models (e.g., the Gowdy model) is needed to test it and establish if the scope can reach a wider domain of cosmological models, taking into consideration the intrinsic symmetries of the corresponding minisuperpaces. In addition, the extension to

(i) anisotropic homogeneous cosmologies, as in [38], or

(ii) wider string settings as in [24,39], taking the symmetries therein as necessary ingredients, constitute tentative routes to consider.

On the other hand, the relation between SFD and other symmetries has been pointed out in the past (see, e.g., [23], [40]; it was widely elaborated in [18]), namely supersymmetry, therefore including anti-commuting variables in the corresponding minisuperspace configuration or phase space, by means of taking at the start an action based, e.g., on a (albeit simplified) supergravity setting. We can say that our guiding target is to establish a robust correspondence involving minisuperspace symmetries (for bosonic and fermionic degrees of freedom) and subsequently allowing one to specify how and which boundary conditions can be suggested to select. This has not yet been done. Enlarging the scope so that boundary conditions could be taken from fundamental symmetries, such as a supersymmetric type, would be interesting. In addition, we can consider including settings here; besides the usual (commuting) spacetime variables, non-commuting variables and deformed Poisson algebra [41] are present, in order to investigate the limits of applicability of this framework, so far discussed in the context of simple minisuperspace models [42–44]. Finally, factor ordering issues that emerge in the traditional direct canonical quantization should also be considered and acknowledged in the discussion, as a means of both enlarging and testing the range of use of this framework [45].

Acknowledgments This work was in part supported by the grant PEst-OE/MAT/UI0212/2014.

References

1. C. Kiefer, *Quantum Gravity*, 3rd edn. (Oxford University Press, Oxford, 2012)
2. A. Vilenkin, Phys. Rev. Lett **74**, 846 (1995)
3. J.B. Hartle, S.W. Hawking, Phys. Rev. D **28**, 2960 (1983)
4. J.J. Halliwell, S.W. Hawking, Phys. Rev. D **31**, 1777 (1985)
5. B.S. De Witt, Phys. Rev. **160**, 1113 (1967)
6. V.G. Lapchinskii, V.A. Rubakov, Theor. Math. Phys. **33**, 1076 (1977)
7. N.A. Lemos, J. Math. Phys. (N.Y.) **37**, 1449 (1996)
8. M. Bojowald, K. Vandersloot, Invited parallel talk at Xth Marcel Grossmann meeting, July 20–26, Rio de Jaeiro, 2003. arXiv:gr-qc/0312103
9. C.G. Callan, D. Friedan, E. Martinec, M.J. Perry, Nucl. Phys. B **262**, 593 (1985)
10. S. Jalalzadeh, P.V. Moniz, Phys. Rev. D **89**, 083504 (2014)
11. E.S. Fradkin, A.A. Tseytlin, Nucl. Phys. B **261**, 1 (1985)
12. J.A. Casas, J. García-Bellido, M. Quirós, Nucl. Phys. B **361**, 713 (1991)
13. D. La, P.J. Steinhardt, Phys. Rev. Lett. **62**, 376 (1989)
14. P.J. Steinhardt, F.S. Accetta, Phys. Rev. Lett. **64**, 2740 (1990)
15. M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, 1988)
16. D. Wands, Class. Quantum Grav. **11**, 269 (1994)
17. R. Holman, E.W. Kolb, S.L. Vadas, Y. Wang, Phys. Rev. D **43**, 995 (1991)
18. P.V. Moniz, in *Quantum Cosmology-The Supersymmetric Perspective*, vol. 1. Lecture Notes in Physics (Springer, Berlin, 2010), p. 803
19. G. Veneziano, Phys. Lett. B **265**, 287 (1991)
20. M. Gasperini, G. Veneziano, Phys. Lett. B **277**, 256 (1992)
21. K.A. Meissner, G. Venezziano, Phys. Lett. B **267**, 33 (1991)
22. M. Cadoni, M. Cavaglia, Phys. Rev. D **50**, 6435 (1994)
23. J.E. Lidsey, Phys. Rev. D **52**, 5407 (1995)
24. J.E. Lidsey, Phys. Rev. D **55**, 3303 (1997)
25. F.J. Tipler, Phys. Rep. **137**, 231 (1986)
26. I. Bars, Phys. Rev. D **79**, 045009 (2009)
27. L.D. Faddeev, Theor. Math. Phys. **1**, 1 (1970)
28. J.B. Hartle, in *Proceedings of the 11th Nishinomiya Yukawa Symposium*, ed by K. Kikkawa, H. Kunitomo, H. Ohtsubo (World Scientific, Singapore, 1997)
29. R.F. Bishop, A. Vourdas, J. Phys. A Math. Gem. **20**, 37279 (1987)
30. C.C. Gerry, R. Grobe, Phys. Rev. A **51**, 4123 (1995)
31. S-H. Dong, *Factorization Methods in Quantum Mechanics*, vol. 150 (Springer, Dordrecht, 2007)
32. J. Louko, Phys. Rev. D **48**, 2708 (1993)
33. M. Gasperini, G. Veneziano, Astropart. Phys. **1**, 317 (1993)
34. M. Gasperini, G. Veneziano, Phys. Rep. **373**, 1 (2003)
35. M.C. Bento, O. Bertolami, Class. Quantum Grav. **12**, 1919 (1995)
36. M.P. Dabrowski, C. Kiefer, Phys. Lett. B **397**, 185 (1997)
37. C. Brif, A. Vourdas, A. Mann, J. Phys. A **29**, 5873 (1996)
38. D. Clancy, J.E. Lidsey, R.K. Tavakol, Class. Quant. Grav. **15**, 257 (1998)
39. M. Cavaglia, P.V. Moniz, Class. Quant. Grav. **18**, 95 (2001)
40. P.V. Moniz, Nucl. Phys. Proc. Suppl. **88**, 57 (2000)
41. S. Jalalzadeh, S.M.M. Rasouli, P.V. Moniz, Phys. Rev. D **90**, 023541 (2014)
42. M. Gasperini, G. Veneziano, Gen. Rel. Grav. **28**, 1301 (1996)
43. M. Gasperini, J. Maharana, G. Veneziano, Nucl. Phys. **472**, 349 (1996)
44. M. Gasperini, Int. J. Mod. Phys. A **13**, 4779 (1998)
45. T. Rostami, S. Jalalzadeh, P.V. Moniz, A quantum cosmological conjecture: intertwining factor ordering and boundary conditions from hidden symmetries (in preparation)