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Transport coefficients of black MQGP $M3$-branes

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Abstract The Strominger–Yau–Zaslow (SYZ) mirror, in the 'delocalised limit' of Becker et al. (Nucl Phys B 702:207, 2004), of N $D3$-branes, M fractional $D3$-branes and N_f flavour $D7$-branes wrapping a non-compact four-cycle in the presence of a black hole (BH) resulting in a non-Kähler resolved warped deformed conifold (NKRWDC) in Mia et al. (Nucl Phys B 839:187, 2010), was carried out in Dhuria and Misra (JHEP 1311:001, 2013) and resulted in black $M3$-branes. There are two parts in our paper. In the first we show that in the 'MQGP' limit discussed in Dhuria and Misra (JHEP 1311:001, 2013) a finite g_s (and hence expected to be more relevant to QGP), finite g_sM, N_f, $g_s^2MN_f$ and very large g_sN, and very small $\frac{g_sM^2}{N}$, we have the following. (i) The uplift, if valid globally (like Dasgupta et al., Nucl Phys B 755:21, 2006) for fractional $D3$ branes in conifolds), asymptotically goes to $M5$-branes wrapping a two-cycle (homologously a (large) integer sum of two-spheres) in $\mathrm{AdS}_5 \times M_6$. (ii) Assuming the deformation parameter to be larger than the resolution parameter, by estimating the five $SU(3)$ structure torsion (τ) classes $W_{1,2,3,4,5}$ we verify that $\tau \in W_5$ in the large-r limit, implying the NKRWDC reduces to a warped Kähler deformed conifold. (iii) The local T^3 of Dhuria and Misra (JHEP 1311:001, 2013) in the large-r limit satisfies the same conditions as the maximal T^2-invariant special Lagrangian three-cycle of T^*S^3 of Ionel and Min-OO (J Math 52(3), 2008), partly justifying use of SYZ-mirror symmetry in the 'delocalised limit' of Becker et al. (Nucl Phys B 702:207, 2004) in Dhuria and Misra (JHEP 1311:001, 2013). In the second part of the paper, by either integrating out the angular coordinates of the non-compact four-cycle which a $D7$-brane wraps around, using the Ouyang embedding, in the DBI action of a $D7$-brane evaluated at infinite radial boundary, or by dimensionally reducing the 11-dimensional EH action to five $(\mathbb{R}^{1,3}, r)$ dimensions and at the infinite radial boundary, we then calculate in particular the $g_s \lesssim 1$ (part of the 'MQGP') limit, a variety of gauge and metric-perturbation-modes' two-point functions using the prescription of Son and Starinets (JHEP 0209:042, 2002). We hence study the following. (i) The diffusion constant $D \sim \frac{1}{T}$, (ii) the electrical conductivity $\sigma \sim T$, (iii) the charge susceptibility $\chi \sim T^2$, (iv) [using (i)–(iii)] the Einstein relation $\frac{\sigma}{\chi} = D$, (v) the R-charge diffusion constant $D_R \sim \frac{1}{T}$, and (vi) (using Dhuria and Misra, JHEP 1311:001, 2013) and Kubo's formula related to shear viscosity η) the possibility of generating $\frac{\eta}{s} = \frac{1}{4\pi}$ from solutions to the vector and tensor mode metric perturbations' EOMs, separately. The results are also valid for the limits of Mia et al. (Nucl Phys B 839:187, 2010): $g_s, \frac{g_sM^2}{N}, g_s^2MN_f \to 0, g_sN, g_sM >> 1$.

1 Introduction

Since the formulation of the AdS/CFT correspondence [1–3], it has been of profound interest to gain theoretical insight into the physics of a strongly coupled Quark Gluon Plasma (sQGP) produced in heavy-ion collisions at RHIC by using holographic techniques. Within this context, the holographic spectral functions play an important role because of their ability to provide a framework to compute certain transport properties of QGP, such as the conductivity and viscosity of the plasma which otherwise are not computationally tractable. The calculation scheme generally involves Kubo's formulae, which involve the long distance and low frequency limit of two-point Green functions. The methodology to calculate two-point Green functions by applying gauge–gravity duality has been developed in the series of papers [4–6] by considering $D3$-brane background. However, this approach is useful for transport coefficients that are universal within models with gravity duals. One of the biggest upshots of this is the viscosity-over-entropy ratio, which takes on the universal value $\frac{1}{4\pi}$, known as the KSS bound [7–9]. Though the bound was initially obtained by assuming a zero charge density, the same was shown to hold

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to satisfy for a wide class of theories including non-zero chemical potential, finite spatial momentum and other different backgrounds (see [10–13]). In addition to this, it is interesting to analyse the photon production in the context of non-perturbative theories as this reveals one of the most fascinating signatures of a QGP. The study of photoemission in strongly coupled theories by using the AdS/CFT correspondence was initiated in [14] in the presence of a vanishing chemical potential, and then continued in [15,16] in the presence of a non-zero chemical potential by considering $D3/D7$ and $D4/D6$ backgrounds where baryonic $U(1)$ symmetry is exhibited from the $U(N_f) = SU(N_f) \times U(1)$ global symmetry present on the world volume of a stack of coincident flavoured branes. In addition to the finite chemical potential, spectral functions were obtained in [13,17,18] by considering a finite spatial momentum, a finite electric field and an anisotropic plasma, respectively. The validity of the results of different transport coefficients obtained in the context of different gravitational backgrounds is manifested on the basis of certain universal bounds. For example, it is well known that the results of some transport coefficients should satisfy the famous Einstein relation according to which one can express the diffusion constant as the ratio of conductivity and susceptibility [17]. Based on [4,5], the diffusion coefficient can be extracted from the poles in the retarded Green functions corresponding to the density of conserved charges. Also, it was suggested in [19] that, similar to the viscosity-to-entropy ratio, there can exist a universal bound on the ratio of conductivity-to-viscosity in theories saturated by models with gravity duals. Further, based on the Minkowski space prescription of Son and Starinets, two-point functions were obtained in [20,21] for conserved R-symmetry current and different components of the stress–energy tensor for $M2$- and $M5$-branes in $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ backgrounds, respectively, using which a variety of transport coefficients, including diffusion coefficient, shear viscosity and the second speed of sound, were calculated.

The knowledge of UV-completion of gauge theories is important to handle issues related to the finiteness of the solution at short distances as well as to capture certain aspects of large-N thermal QCD. One of the earliest successful attempts to explain the RG flow in the dual background was made by Klebanov and Strassler in [22] by embedding $D5$-branes wrapped over S^2 in a conifold background, which was further extended to an Ouyang–Klebanov–Strassler (OKS) background in [23] in the presence of fundamental quarks (by including N_f $D7$-branes wrapped over non-compact four-cycle(s)). This was followed by the modified OKS-BH background [24] in the presence of a black hole. It would be very interesting to determine the transport coefficients by considering a non-conformal modified OKS-BH background, specially in the finite-string-coupling limit, which motivates an uplift to M-theory in the 'delocalised limit' of [25], and calculation of the transport coefficients in it.

In this paper, after explicitly showing that the M-theory uplift in the delocalised limit of [25], of the modified OKS-BH background of [24] to black $M3$-branes, is a solution to $D = 11$ supergravity in particular in the 'MQGP limit' described in [26], we intend to study the behaviour of transport coefficients of these black $M3$-branes.[1] One should keep in mind that the MQGP limit could be more suited to demonstrate the characteristics of sQGP [27] by exploiting gauge–gravity duality, and due to the finiteness of the string coupling can meaningfully only be addressed within an M-theoretic framework. We are not aware of previous attempts at calculation of transport coefficients of the string theory duals of large-N thermal QCD-like theories at finite string coupling – this work is hence expected to shed some light on what to expect in terms of the values of a variety of transport coefficients for sQGP. Using the background of [24] or its type IIA mirror in the delocalised limit of [25] or its M-theory uplift, we have already computed some of the transport coefficients such as the viscosity-to-entropy ratio and diffusion constant in [26] by adopting the KSS prescription [7]. It would be interesting to study important transport coefficients such as electrical conductivity, shear viscosity, etc. in the finite string/gauge coupling limit (part of the MQGP limit of [26]) by using the M-theory uplift of a non-conformal resolved warped deformed conifold background of [24], which asymptotically, we show, corresponds to black $M5$-branes wrapping a sum of two-spheres in $\mathrm{AdS}_5 \times \mathcal{M}_6$.

An overview of the paper is as follows: in Sect. 2.1, we review the framework of our M-theory uplift in the 'delocalised' limit of [25] of type IIB background of [24] as constructed in [26] using S(trominger) Y(au) Z(aslow) mirror symmetry (in the 'delocalised' limit of [25]). In this section, we also highlight results of various hydro/thermodynamical quantities obtained in the MQGP limit provided in [26], and then review the thermodynamical stability of the M-theory uplift. In Sect. 2.2, we explicitly demonstrate that the uplift described in Sect. 2.1 solves the $D = 11$ supergravity equations of motion, in the large-r limit and near the $\theta_{1,2} = 0, \pi$-branches in the delocalised limit of [25], and in the MQGP limit of [26]. In Sect. 2.3, we obtain magnetic charge of the black $M3$-branes due to four-form fluxes G_4 through all (non)compact four-cycles and point out that the black $M3$-branes asymptotically can be thought of as $M5$-branes wrapped around two-cycles given homologously by an integer sum of two-spheres. In Sect. 2.4, given that supersymmetry via the existence of a special Lagrangian (sLag) is necessary for implementing mirror symmetry as three T-dualities

[1] The near-horizon geometry of these branes, near the $\theta_{1,2} = 0, \pi$ branches, preserves $\frac{1}{8}$ supersymmetry [26].

à la SYZ, we show that in particular the MQGP limit of (4) and, assuming that the deformation parameter is larger than the resolution parameter, the local T^3 of [26], in the large-r limit, is the maximal T^2-invariant sLag of a deformed conifold as defined in [28]. In Sect. 3, we turn toward the discussion of various transport coefficients in the MQGP limit. A variety of transport coefficients can be obtained by considering fluctuations of the $U(1)_{N_f}$ gauge field corresponding to N_f $D7$-branes wrapping a non-compact four-cycle of (resolved) warped deformed conifold as discussed in [26], the fluctuations around $U(1)_R$ gauge field, and the stress–energy tensor. In this spirit, in Sect. 3.1, we first consider gauge field fluctuations around a non-zero temporal component of the gauge field background (non-zero chemical potential as worked out in [26]) in the presence of N_f $D7$-branes wrapping a non-compact four-cycle in the resolved warped deformed conifold background and work out the equations of motion (EOM). Further, we move on to the computation of two-point retarded Green functions by using the prescription first set out in [4] using which one can calculate the electrical conductivity by utilising Kubo's formula. In Sect. 3.2, we work out the charge susceptibility due to the non-zero baryon density, which turns out to be such that the ratio of the electrical conductivity and charge susceptibility satisfies the Einstein relation. Following the same strategy as described in Sect. 3.2, we compute the correlation function corresponding to the $U(1)_R$ current in the context of 11-dimensional M-theory background in Sect. 3.3, from the pole of which one simply reads off the "diffusion coefficient". In Sect. 3.3, we determine the correlation functions of the stress–energy tensor modes. After classifying different stress–energy tensor modes, we compute two-point correlation functions of the vector and tensor modes of a black $M3$-brane. Using Kubo's formula, we therefore calculate the shear viscosity and show that is possible to obtain the shear viscosity-to-entropy ratio (η/s) to be $1/4\pi$. We finally summarise our results, also giving a flow-chart of the flow of logic and (sub-)section-wise summary of results in the process in Fig. 1, and remark on the possible extensions of the current work in Sect. 4. There are four appendices. Appendix A includes simplified expressions of possible non-zero $G_{\mu\nu\lambda\rho}$'s obtained. In Appendix B we explicitly evaluate the five $SU(3)$ structure torsion classes for the resolved warped deformed conifold background of [24]. Appendix C gives the explicit embedding of the maximal T^2-invariant special Lagrangian three-cycle of a deformed conifold. In Appendix D, we collect various intermediate steps relevant to the evaluation of R-charge correlators of Sect. 3.2.

2 The black $M3$-branes of [26]

This section has two subsections. In Sect. 2.1, we will review the uplift of the type IIB background of [24] to M-theory in the delocalised limit of [25], as carried out in [26] using SYZ-mirror symmetry (in the delocalised limit of [25]), as well as its hydrodynamical and thermodynamical properties. In Sect. 2.2, we discuss the charge(s) of the black $M3$-branes of Sect. 2.1 and show that asymptotically the 11D spacetime is a warped product of AdS_5 and a sixfold with $M5$-branes wrapping a two-cycle that is homologous to an integer sum of two-spheres.

2.1 The uplift in the delocalised limit of [25]

To include fundamental quarks at finite temperature relevant to, e.g., the study of the deconfined phase of strongly coupled QCD i.e. the "Quark Gluon Plasma", a black hole and N_f flavour $D7$-branes 'wrapping' a (non-compact) four-cycle were inserted apart from N $D3$-branes placed at the tip of six-dimensional conifold and M $D5$-branes that are wrapping a two-cycle S^2. This takes place in the context of type IIB string theory in [24] which causes, both resolution and deformation of the two- and three-cycles of the conifold, respectively, at $r = 0$ (apart from warping). This results in a warped resolved deformed conifold. In that background, back-reaction due to the presence of a black hole as well as $D7$-branes is included in the 10-D warp factor $h(r, \theta_1, \theta_2)$. The metric from [24] is given by

$$\mathrm{d}s^2 = \frac{1}{\sqrt{h}}\left(-g_1\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right) + \sqrt{h}\left[g_2^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\mathcal{M}_5^2\right];$$

g_i's demonstrate the presence of a black hole and are given as follows: $g_{1,2}(r, \theta_1, \theta_2) = 1 - \frac{r_h^4}{r^4} + \mathcal{O}\left(\frac{g_s^2 M}{N}\right)$, where r_h is the horizon, and the (θ_1, θ_2) dependence comes from the $\mathcal{O}\left(\frac{g_s M^2}{N}\right)$ corrections. In (1),

$$\begin{aligned}\mathrm{d}\mathcal{M}_5^2 = {} & h_1(\mathrm{d}\psi + \cos\theta_1\,\mathrm{d}\phi_1 + \cos\theta_2\,\mathrm{d}\phi_2)^2 \\ & + h_2(\mathrm{d}\theta_1^2 + \sin^2\theta_1\,\mathrm{d}\phi_1^2) + h_4(h_3\mathrm{d}\theta_2^2 + \sin^2\theta_2\,\mathrm{d}\phi_2^2) \\ & + h_5\,[\cos\psi\,(\mathrm{d}\theta_1\mathrm{d}\theta_2 - \sin\theta_1\sin\theta_2\mathrm{d}\phi_1\mathrm{d}\phi_2) \\ & + \sin\psi\,(\sin\theta_1\,\mathrm{d}\theta_2\mathrm{d}\phi_1 + \sin\theta_2\,\mathrm{d}\theta_1\mathrm{d}\phi_2)]\,, \qquad (1)\end{aligned}$$

$r >> a, h_5 \sim \frac{(\text{deformation parameter})^2}{r^3} << 1 \forall r >>$ (deformation/parameter)$^{\frac{2}{3}}$. Due to the presence of black hole, h_i appearing in the internal metric (1) as well as M, N_f are not constant, and up to linear order depend on g_s, M, N_f as given below:

$$h_1 = \frac{1}{9} + \mathcal{O}\left(\frac{g_s M^2}{N}\right),\ h_2 = \frac{1}{6} + \mathcal{O}\left(\frac{g_s M^2}{N}\right),$$

$$h_4 = h_2 + \frac{6a^2}{r^2},\ h_3 = 1 + \mathcal{O}\left(\frac{g_s M^2}{N}\right),$$

$$M_{\rm eff}/N_f^{\rm eff} = M/N_f + \sum_{m\geq n}(a/b)_{mn}(g_sN_f)^m(g_sM)^n,$$

$$L = (4\pi g_s N)^{\frac{1}{4}}.$$

The warp factor that includes the back-reaction due to fluxes as well as black hole, for finite, but not large r, is given as

$$h = \frac{L^4}{r^4}\left[1 + \frac{3g_sM_{\rm eff}^2}{2\pi N}{\rm log}r\left\{1 + \frac{3g_sN_f^{\rm eff}}{2\pi}\left({\rm log}r + \frac{1}{2}\right)\right.\right.$$
$$\left.\left. + \frac{g_sN_f^{\rm eff}}{4\pi}{\rm log}\left({\rm sin}\frac{\theta_1}{2}{\rm sin}\frac{\theta_2}{2}\right)\right\}\right]. \quad (2)$$

For large r [24],

$$h = L^4\left[\frac{1}{r^{4-\epsilon_1}} + \frac{1}{r^{4-2\epsilon_2}} - \frac{2}{r^{4-\epsilon_2}} + \frac{1}{r^{4-r^{\frac{\epsilon_2^2}{2}}}}\right]$$
$$\equiv \sum_{\alpha=1}^{4}\frac{L_{(\alpha)}^4}{r_{(\alpha)}^4}, \quad (3)$$

where $\epsilon_1 \equiv \frac{3g_sM^2}{2\pi N} + \frac{g_s^2M^2N_f}{8\pi^2 N} + \frac{3g_s^2M^2N_f}{8\pi N}\ln\left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)$, $\epsilon_2 \equiv \frac{g_sM}{\pi}\sqrt{\frac{2N_f}{N}}$, $r_{(\alpha)} \equiv r^{1-\epsilon_{(\alpha)}}$, $\epsilon_{(1)} = \frac{\epsilon_1}{2}$, $\epsilon_{(2)} \equiv \epsilon_{(3)} = \frac{\epsilon_2}{2}$; $L_{(1)} = L_{(2)} = L_{(4)} = L^4$, $L_{(3)} = -2L^4$. It is conjectured that as $r \to \infty$, $\alpha \in [1, \infty)$.

In [26], we considered the following two limits:

(i) weak (g_s) coupling – large 'tHooft coupling limit:

$$g_s << 1, g_sN_f << 1, \frac{g_sM^2}{N} << 1, g_sM >> 1,$$
$$g_sN >> 1$$
$$\text{effected by } g_s \sim \epsilon^d, M \sim \epsilon^{-\frac{3d}{2}}, N \sim \epsilon^{-19d},$$
$$\epsilon \leq \mathcal{O}(10^{-2}), d > 0$$

(ii) MQGP limit: $\frac{g_sM^2}{N} << 1, g_sN >> 1$, finite g_s, M

$$\text{effected by } g_s \sim \epsilon^d, M \sim \epsilon^{-\frac{3d}{2}}, N \sim \epsilon^{-39d},$$
$$\epsilon \lesssim 1, d > 0. \quad (4)$$

Apart from other calculational simplifications, one of the advantages of working with either of the limits in (4) is that the ten-dimensional warp factor h (3) for large r approaches the expression (2) for finite/small r. The reason for the unusual scalings appearing in (4) is, as explained towards the end of this section, that this ensures that the finite part of the $D = 11$ supergravity action—the Gibbons–Hawking–York boundary term (like [29])—is independent of the angular $(\theta_{1,2})$ cutoff/regulator, and yields an entropy density ($s \sim r_h^3$) from a thermodynamical calculation matching this calculated from the horizon area.

The three-form fluxes, including the 'asymmetry factors', are given by [24]

$$\widetilde{F}_3 = 2MA_1\left(1 + \frac{3g_sN_f}{2\pi}\log r\right)$$
$$\times e_\psi \wedge \frac{1}{2}(\sin\theta_1\, {\rm d}\theta_1 \wedge {\rm d}\phi_1 - B_1\sin\theta_2\, {\rm d}\theta_2 \wedge {\rm d}\phi_2)$$
$$-\frac{3g_sMN_f}{4\pi}A_2\frac{{\rm d}r}{r} \wedge e_\psi$$
$$\wedge\left(\cot\frac{\theta_2}{2}\sin\theta_2\, {\rm d}\phi_2 - B_2\cot\frac{\theta_1}{2}\sin\theta_1\, {\rm d}\phi_1\right)$$
$$-\frac{3g_sMN_f}{8\pi}A_3\sin\theta_1\sin\theta_2$$
$$\times\left(\cot\frac{\theta_2}{2}{\rm d}\theta_1 + B_3\cot\frac{\theta_1}{2}{\rm d}\theta_2\right)$$
$$\wedge\, {\rm d}\phi_1 \wedge {\rm d}\phi_2,$$
$$H_3 = 6g_sA_4M$$
$$\times\left(1 + \frac{9g_sN_f}{4\pi}\log r + \frac{g_sN_f}{2\pi}\log\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)\frac{{\rm d}r}{r}$$
$$\wedge\frac{1}{2}(\sin\theta_1\, {\rm d}\theta_1 \wedge {\rm d}\phi_1 - B_4\sin\theta_2\, {\rm d}\theta_2 \wedge {\rm d}\phi_2)$$
$$+\frac{3g_s^2MN_f}{8\pi}A_5\left(\frac{{\rm d}r}{r}\wedge e_\psi - \frac{1}{2}{\rm d}e_\psi\right)$$
$$\wedge\left(\cot\frac{\theta_2}{2}{\rm d}\theta_2 - B_5\cot\frac{\theta_1}{2}{\rm d}\theta_1\right). \quad (5)$$

The asymmetry factors A_i, B_i encode information of the black hole and of the background and are given in [24]. The $\mathcal{O}(a^2/r^2)$ corrections included in asymmetry factors correspond to modified Ouyang background in the presence of a black hole. The values for the axion C_0 and the five-form F_5 are given by [30]

$$C_0 = \frac{N_f}{4\pi}(\psi - \phi_1 - \phi_2)\left[\text{since } \int_{S^1}{\rm d}C_0 = N_f\right],$$
$$F_5 = \frac{1}{g_s}\left[{\rm d}^4x \wedge {\rm d}h^{-1} + *({\rm d}^4x \wedge {\rm d}h^{-1})\right]. \quad (6)$$

It can be shown that the deviation from supersymmetry, one way of measuring which is the deviation from the condition of $G_3 \equiv \tilde{F}_3 - ie^{-\phi}H_3$ being imaginary self dual: $|iG_3 - *_6G_3|^2$, is proportional to the square of the resolution parameter a, which (assuming a negligible bare resolution parameter) in turn is related to the horizon radius r_h via $a^2 = \frac{g_sM^2}{N}r_h^2 + \frac{g_sM^2}{N}(g_sN_f)r_h^4$ [31]. We hence see that despite the presence of a black hole, in either limits of (4), SUSY is approximately preserved.

For embedding parameter $\mu << 1$, $D7$-branes monodromy arguments for Ouyang embedding: $x = (9a^2r^4 + r^6)^{1/4}e^{\iota/2(\psi-\phi_1-\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} = \mu^2$ (relabelling ρ as r) "$j(\tau(x)) = \frac{4.(24f)^3}{27g^2+f^3}$" $\sim \frac{1}{x-\mu^2}$ or $\tau \sim \frac{1}{2\pi\iota}\ln x$, implying [24]

$$e^{-\Phi} = \frac{1}{g_s} - \frac{N_f}{8\pi}\log(r^6 + 9a^2r^4) - \frac{N_f}{2\pi}\log\left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right).$$

Working around: $r \approx \langle r\rangle, \theta_{1,2} \approx \langle\theta_{1,2}\rangle, \psi \approx \langle\psi\rangle$ define the T-duality coordinates, $(\phi_1, \phi_2, \psi) \rightarrow (x, y, z)$ as

$$x = \sqrt{h_2} h^{\frac{1}{4}} \sin\langle\theta_1\rangle\langle r\rangle\phi_1, \;\; y = \sqrt{h_4} h^{\frac{1}{4}} \sin\langle\theta_2\rangle\langle r\rangle\phi_2,$$
$$z = \sqrt{h_1}\langle r\rangle h^{\frac{1}{4}}\psi. \tag{7}$$

Interestingly, around $\psi = \langle\psi\rangle$ (the 'delocalised' limit of [25], i.e., instead of a dependence on a local coordinate, an isometry arising by freezing a coordinate and redefining it away as also summarised below), under the coordinate transformation [25]

$$\begin{pmatrix} \sin\theta_2 d\phi_2 \\ d\theta_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\langle\psi\rangle & \sin\langle\psi\rangle \\ -\sin\langle\psi\rangle & \cos\langle\psi\rangle \end{pmatrix} \begin{pmatrix} \sin\theta_2 d\phi_2 \\ d\theta_2 \end{pmatrix}, \tag{8}$$

the h_5 term becomes $h_5\,[d\theta_1 d\theta_2 - \sin\theta_1 \sin\theta_2 d\phi_1 d\phi_2]$. Following closely [25], and utilising (8) and (7) one sees that

$$\theta_2 \rightarrow -\frac{\sqrt{6}\sin\langle\psi\rangle}{(4\pi g_s N)^{\frac{1}{4}}} y + \cos\langle\psi\rangle\theta_2 \quad \text{implying } \cos\theta_2 d\,\phi_2$$
$$\rightarrow \cot\left[-\frac{\sqrt{6}\sin\langle\psi\rangle}{(4\pi g_s N)^{\frac{1}{4}}} y + \cos\langle\psi\rangle\theta_2\right]$$
$$\times\left\{\frac{\sqrt{6}\cos\langle\psi\rangle dy}{(4\pi g_s N)^{\frac{1}{4}}} + \sin\langle\psi\rangle d\theta_2\right\}. \tag{9}$$

Defining $\bar{\theta}_2 \equiv -\frac{\sqrt{6}\sin\langle\psi\rangle}{(4\pi g_s N)^{\frac{1}{4}}} y + \cos\langle\psi\rangle\theta_2 = -\sin\langle\theta_2\rangle\sin\langle\psi\rangle$ $\phi_2 + \cos\langle\psi\rangle\theta_2 \stackrel{\theta_2\sim 0,\pi}{\longrightarrow} \cos\langle\psi\rangle\theta_2$, $e_\psi \rightarrow e_\psi + \cot\bar{\theta}_2\cos\langle\psi\rangle$ $\sin\langle\theta_2\rangle d\phi_2 + \cot\bar{\theta}_2\sin\langle\psi\rangle d\theta_2$, one hence sees that under $\psi \rightarrow \psi - \cos\langle\bar{\theta}_2\rangle\phi_2 + \cos\langle\theta_2\rangle\phi_2 - \tan\langle\psi\rangle \ln\sin\bar{\theta}_2$, $e_\psi \rightarrow e_\psi$. In the delocalised limit of [25], one thus introduces an isometry along ψ in addition to the isometries along $\phi_{1,2}$. This clearly is not valid globally—the deformed conifold does not possess a third global isometry.

To enable use of SYZ-mirror duality via three T dualities, one needs to ensure that the abovementioned local T^3 is a special Lagrangian (sLag) three-cycle. This will be explicitly shown in Sect. 3. For implementing mirror symmetry via SYZ prescription, one also needs to ensure a large base (implying large complex structures of the aforementioned two two-tori) of the $T^3(x, y, z)$ fibration. This is effected via [32]

$$d\psi \rightarrow d\psi + f_1(\theta_1)\cos\theta_1 d\theta_1 + f_2(\theta_2)\cos\theta_2 d\theta_2,$$
$$d\phi_{1,2} \rightarrow d\phi_{1,2} - f_{1,2}(\theta_{1,2})d\theta_{1,2}, \tag{10}$$

for appropriately chosen large values of $f_{1,2}(\theta_{1,2})$. The three-form fluxes remain invariant. The fact that one can choose such large values of $f_{1,2}(\theta_{1,2})$, was justified in [26]. The guiding principle was that one requires that the metric obtained after SYZ-mirror transformation applied to the resolved warped deformed conifold is like a warped resolved conifold at least locally, then $G^{IIA}_{\theta_1\theta_2}$ needs to vanish [26].

We can get a one-form type IIA potential from the triple T-dual (along x, y, z) of the type IIB $F_{1,3,5}$. We therefore can construct the following type IIA gauge field one-form [26] in the delocalised limit of [25]

- $$A^{F_3} = \Big[\bar{\tilde{\tilde{F}}}_{xr} x dr + \bar{\tilde{\tilde{F}}}_{x\theta_1} x d\theta_1 + \bar{\tilde{\tilde{F}}}_{x\theta_2} x d\theta_2 + \bar{\tilde{\tilde{F}}}_{y\theta_1} y d\theta_1$$
$$+ \bar{\tilde{\tilde{F}}}_{y\theta_2} y d\theta_2 + \bar{\tilde{\tilde{F}}}_{z\theta_2} z d\theta_2 + \bar{\tilde{\tilde{F}}}_{z\theta_1} z d\theta_1 + \bar{\tilde{\tilde{F}}}_{zr} z dr + \bar{\tilde{\tilde{F}}}_{yr} y dr\Big]$$
$$\times\left(\theta_{1,2} \rightarrow \langle\theta_{1,2}\rangle, \phi_{1,2} \rightarrow \langle\phi_{1,2}\rangle, \psi \rightarrow \langle\psi\rangle, r \rightarrow \langle r\rangle\right);$$
- $$A^{F_1} = \left(\bar{\tilde{\tilde{F}}}_{1yx} y dx + \bar{\tilde{\tilde{F}}}_{1zx} z dx\right)$$
$$\times\left(\theta_{1,2} \rightarrow \langle\theta_{1,2}\rangle, \phi_{1,2} \rightarrow \langle\phi_{1,2}\rangle, \psi \rightarrow \langle\psi\rangle, r \rightarrow \langle r\rangle\right);$$
- $$A^{F_5} = \left(\bar{\tilde{\tilde{F}}}_{5r\theta_1} r d\theta_1 + \bar{\tilde{\tilde{F}}}_{5\theta_1\theta_2}\theta_1 d\theta_2 + \bar{\tilde{\tilde{F}}}_{5r\theta_2} r d\theta_2\right)$$
$$\times\left(\theta_{1,2} \rightarrow \langle\theta_{1,2}\rangle, \phi_{1,2} \rightarrow \langle\phi_{1,2}\rangle, \psi \rightarrow \langle\psi\rangle, r \rightarrow \langle r\rangle\right),$$

resulting in the following local uplift:

$$ds^2_{11} = e^{-\frac{2\phi^{IIA}}{3}}\left[\frac{1}{\sqrt{h(r,\theta_1,\theta_2)}}\left(-g_1 dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right)\right.$$
$$\left. + \sqrt{h(r,\theta_1,\theta_2)}\left(g_2^{-1} dr^2\right) + ds^2_{IIA}(\theta_{1,2}, \phi_{1,2}, \psi)\right]$$
$$+ e^{\frac{4\phi^{IIA}}{3}}\left(dx_{11} + A^{F_1} + A^{F_3} + A^{F_5}\right)^2,$$

which are black $M3$-branes.

In the MQGP limit: $G^{\mathcal{M}}_{00} \sim \epsilon^{\frac{55d}{3}} r^2\left(1 - \frac{r_h^4}{r^4}\right)$, $G^{\mathcal{M}}_{rr} \sim \frac{\epsilon^{-\frac{59d}{3}}}{r^2\left(1 - \frac{r_h^4}{r^4}\right)}$. One can rewrite $G_{rr} dr^2 = \xi' \frac{d\omega^2}{\omega}$ where $\xi' \sim \frac{r_h \epsilon'}{\epsilon^{\frac{59d}{3}}}$ and $\xi' \frac{d\omega^2}{\omega} = dv^2$ or $\omega = \frac{v^2}{4\xi'}$. One sees that near $r = r_h \sim 1$, $G^{\mathcal{M}}_{tt} dt^2 \sim \epsilon^{38d} u^2 dt^2$, implying again $T^2 \sim \frac{1}{(\sqrt{g_s N})^2}$, in conformity with [24]. Similar result are obtained for the limits of [24]. Now, $G^{\mathcal{M}}_{00}, G^{\mathcal{M}}_{rr}$ have no angular dependence and hence the temperature $T = \frac{\partial_r G_{00}}{4\pi\sqrt{G_{00} G_{rr}}}$ [7,8] of the black $M3$-brane turns out to be given by

$$T = \frac{\sqrt{2}}{r_h\sqrt{\pi}\sqrt{\frac{g_s\left(18{g_s}^2 N_f \ln^2(r_h) {M_{\text{eff}}}^2 + 3 g_s(4\pi - g_s N_f(-3 + \ln(2)))\ln(r_h){M_{\text{eff}}}^2 + 8N\pi^2\right)}{{r_h}^4}}} \xrightarrow{\text{Both limits}} \frac{r_h}{\pi L^2}.$$

To get a numerical estimate for r_h, we see that equating T to $\frac{r_h}{\pi L^2}$, in both limits one then obtains $r_h = 1 + \epsilon$, where $0 < \epsilon < 1$.

The amount of near-horizon supersymmetry was determined in [26] by solving for the Killing spinor ϵ by the vanishing supersymmetric variation of the gravitino in $D = 11$ supergravity. In the weak (g_s)-coupling-large-'t Hooft-couplings limit [26]

$$\begin{aligned}&\varepsilon(\theta_{1,2}, \Phi_{1,2} \equiv \phi_1 \pm \phi_2, \psi, x_{10})\\&\quad = e^{\mp\beta\Phi_1\epsilon^{-\alpha_\Phi} G^{5678}(\epsilon)\Gamma_{56}} e^{\mp\beta\Phi_2\epsilon^{-\alpha_\Phi} G^{5678}(\epsilon)\Gamma_{56}}\\&\qquad\times e^{-\beta\psi\epsilon^{-\alpha_\psi}(\mp G^{5679}(\epsilon)\Gamma_{56}\mp G^{5689}(\epsilon)\Gamma_{56})}\\&\qquad\times e^{x_{10}\epsilon^{\alpha_{10}}(G^{567\overline{10}}(\epsilon)\Gamma_{56}\pm G^{568\overline{10}}(\epsilon)\Gamma_{56})}\varepsilon_0,\end{aligned}$$

with the constraints

$$\left(G^{5678}(\epsilon) \pm G^{567\overline{10}}(\epsilon)\Gamma_{\overline{10}} \pm G^{568\overline{10}}(\epsilon)\Gamma_{\overline{10}}\right)\varepsilon_0 = 0, \quad (11)$$

and

$$\Gamma_7\varepsilon_0 = \pm\varepsilon_0;\ \Gamma_8\varepsilon_0 = \pm\varepsilon_0, \quad (12)$$

for a constant spinor ε_0; $g_{\Phi_{1,2}\Phi_{1,2}} = \epsilon^{-\alpha_\Phi}$, $g_{\psi\psi} = \epsilon^{-\alpha_\psi}$, $g_{x_{10}x_{10}} = \epsilon^{\alpha_{10}}$ near $\theta_{1,2} = 0, \pi$. The near-horizon black $M3$-branes solution possesses 1/8 supersymmetry near $\theta_{1,2} = 0, \pi$. Similar arguments also work in the MQGP limit of [26].

Freezing the angular dependence on $\theta_{1,2}$ (there being no dependence on $\phi_{1,2}, \psi, x_{10}$ in both weak (g_s)-coupling-large-'t Hooft-couplings and MQGP limits), noting that $G^{IIA/\mathcal{M}}_{00,rr,\mathbb{R}^3}$ are independent of the angular coordinates (additionally possible to tune the chemical potential μ_C to a small value [26], using the result of [7]

$$\begin{aligned}\frac{\eta}{s} &= T\frac{\sqrt{|G^{IIA/\mathcal{M}}|}}{\sqrt{|G^{IIA/\mathcal{M}}_{tt}G^{IIA/\mathcal{M}}_{rr}|}}\Bigg|_{r=r_h}\\&\quad\times\int_{r_h}^{\infty}\mathrm{d}r\frac{|G^{IIA/\mathcal{M}}_{00}G^{IIA}_{rr}|}{G^{IIA/\mathcal{M}}_{\mathbb{R}^3}\sqrt{|G^{IIA/\mathcal{M}}|}} = \frac{1}{4\pi}. \end{aligned}\quad (13)$$

In the notations of [7] one can pull out a common $Z(r)$ in the angular part of the metrics as $Z(r)K_{mn}(y)\mathrm{d}y^i\mathrm{d}y^j$, (which for the type IIB/IIA backgrounds is $\sqrt{h}r^2$) in terms of which we have

$$\begin{aligned}D &= \frac{\sqrt{|G^{IIB/IIA}|}Z^{IIB/IIA}(r)}{G^{IIb/IIA}_{\mathbb{R}^3}\sqrt{|G^{IIB/IIA}_{00}G^{IIB/IIA}_{rr}|}}\Bigg|_{r=r_h}\\&\quad\times\int_{r_h}^{\infty}\mathrm{d}r\frac{|G^{IIB/IIA}_{00}G^{IIB/IIA}_{rr}|}{\sqrt{|G^{IIB/IIA}|}Z^{IIB/IIA}(r)} = \frac{1}{2\pi T}.\end{aligned}$$

Let us review the thermodynamical stability of a type IIB background in the Ouyang limit as discussed in [26]. Assuming $\mu(\neq 0) \in \mathbb{R}$ in Ouyang's embedding [33] $r^{\frac{3}{2}}e^{\frac{i}{2}(\psi-\phi_1-\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} = \mu$, which could be satisfied for $\psi = \phi_1 + \phi_2$ and $r^{\frac{3}{2}}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} = \mu$. Including a $U(1)$(of $U(N_f) = U(1) \times SU(N_f)$) field strength $F = \partial_r A_t \mathrm{d}r \wedge \mathrm{d}t$ in addition to B_2, in the $D7$-brane DBI action, taking the embedding parameter μ to be less than, but close to 1, the chemical potential μ_C was calculated up to $\mathcal{O}(\mu)$ in [26]:

$$\mu_C \approx g_s\left[\frac{{}_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{4}{3}; -\frac{C^2g_s^2}{{r_h}^6}\right)}{2{r_h}^2}\right] + g_sN_f\left[\frac{Cg_s\left({}_2F_1\left(\frac{2}{3}, \frac{3}{2}; \frac{5}{3}; -\frac{C^2g_s^2}{{r_h}^6}\right)a^2 + 8{r_h}^2\,{}_2F_1\left(\frac{1}{3}, \frac{3}{2}; \frac{4}{3}; -\frac{C^2g_s^2}{r_h^6}\right)\ln(\mu)\right)}{32\pi r_h^4}\right].$$

One thus sees

$$\begin{aligned}&\frac{\partial\mu_C}{\partial T}\Bigg|_{N_f, a=fr_h: f<<<1; C=\frac{\pi^3(4\pi g_sN)^{\frac{3}{2}}}{g_s}} \sim -\frac{1}{8\pi}\\&\quad\times\frac{T^6(f^2+4\ln\mu)}{\left(1+T^6\right)^{\frac{3}{2}}} + f^2g_sN_f\frac{{}_2F_1\left(\frac{2}{3}, \frac{3}{2}; \frac{5}{3}; -\frac{1}{T^6}\right)}{16\pi T^3} < 0.\end{aligned}$$

So, it is clear that $\frac{\partial\mu_C}{\partial T}\Big|_{N_f} = -\frac{\partial S}{\partial N_f}\Big|_T < 0$. Apart from $C_v > 0$, thermodynamic stability requires $\frac{\partial\mu_C}{\partial N_f}\,|_T > 0$, which for $C > 0, \mu = \lim_{\epsilon\to 0^+} 1 - \epsilon$ is satisfied.

Let us, towards the end of this section, review the arguments as regards the demonstration of thermodynamical stability from a $D = 11$ point of view by demonstrating the positivity of the specific heat, as shown in [26]. Keeping in mind that, as $r_h \sim l_s$, higher order α' corrections become important, the action is

$$\begin{aligned}\mathcal{S}_E &= \frac{1}{16\pi}\int_{\mathcal{M}}\mathrm{d}^{11}x\sqrt{G^{\mathcal{M}}}R^{\mathcal{M}} + \frac{1}{8\pi}\int_{\partial M}\mathrm{d}^{10}xK^{\mathcal{M}}\sqrt{\hat{h}}\\&\quad-\frac{1}{4}\int_M\left(|G_4|^2 - C_3 \wedge G_4 \wedge G_4\right)\\&\quad+\frac{T_2}{2\pi^4.3^2.2^{13}}\int_{\mathcal{M}}\mathrm{d}^{11}x\sqrt{G^{\mathcal{M}}}\left(J - \frac{1}{2}E_8\right)\\&\quad+T_2\int C_3 \wedge X_8 - \mathcal{S}^{\mathrm{ct}},\end{aligned}$$

where $T_2 \equiv M2$-brane tension, and

$$J = 3.2^8\left(R^{mijn}R_{pijq}R_m^{\ rsp}R^q_{\ rsn} + \frac{1}{2}R^{mnij}R_{pqij}R_m^{\ rsp}R^q_{\ rsn}\right),$$
$$E_8 = \epsilon^{abcm_1n_1\ldots m_4n_4}\epsilon_{abcm_1'n_1'\ldots m_4'n_4'}R^{m_1'n_1'}_{\ \ \ m_1n_1}\ldots R^{m_4'n_4'}_{\ \ \ m_4n_4},$$
$$X_8 = \frac{1}{192\cdot(2\pi^2)^4}\left[\mathrm{tr}(R^4) - (\mathrm{tr}R^2)^2\right],$$

for Euclideanised spacetime where $\mathcal{M}$ is a volume of spacetime defined by $r < r_\Lambda$, where the counterterm $\mathcal{S}^{\mathrm{ct}}$ is added such that the Euclidean action $\mathcal{S}_E$ is finite [34,35]. The reason why there is no $\frac{1}{\kappa_{11}^2}$ in the action is because of the tacit understanding that as one goes from the local T^3 coordinates (x, y, z) to global coordinates (ϕ_1, ϕ_2, ψ) via Eq. (7), the $(g_sN)^{\frac{3}{4}}$ cancels off κ_{11}^2. This also helps in determining how the cutoff r_Λ scales with ϵ, which appears in the scalings of g_s, M and N. Given that $\tilde{F}_3 \wedge H_3$ is localised, i.e., receives the dominant contributions near $\theta_{1,2} = 0$, the total D-brane charge in the very large-$N(>> M, N_f)$ limit can be estimated by [23] $\frac{1}{2\kappa_{10}^2T_3}\int\tilde{F}_3\wedge H_3 \sim N$. In the MQGP limit [26], defined by considering $g_s, g_sM, {g_s}^2MN_f \equiv$ finite, $g_sN >> 1$, $\frac{g_sM^2}{N} << 1$. We have

$$\int\tilde{F}_3\wedge H_3 \sim \int\Bigg\{\frac{3^4(g_sM)^2(g_sN_f)N_fA_1A_4(B_1+B_4)}{8\pi^2}\times\frac{(\ln r)^2}{r}\sin\theta_2\sin\theta_1 - \frac{3^2(g_sM)^2(g_sN_f)N_fA_3A_5}{2^6\pi^2}\frac{1}{r}\times\sin\theta_1\sin\theta_2\left(\cot^2\frac{\theta_2}{2} + B_3B_5\cot^2\frac{\theta_1}{2}\right)\Bigg\}\times d\theta_1 d\theta_2 dr d\phi_1 d\phi_2 d\psi.$$

Since the integrand receives the most dominant contribution near $\theta_{1,2} = 0, \pi$, we have introduced a cutoff $\epsilon_{\theta_{1,2}} \sim \epsilon^{\frac{3}{2}}$ in the MQGP limit of [26]. Using this,

$$\int\tilde{F}_3\wedge H_3 \sim 3^3.\,2^4\pi(g_sM)^2(g_sN_f)N_f(\ln r_\Lambda)^3.$$

We have $\frac{1}{\kappa_{11}^2} \sim \frac{1}{3^3.\,2^5\pi\frac{(g_sM)^2}{N}(g_sN_f)^2}\frac{1}{(\ln r_\Lambda)^3}$. Assuming $\frac{1}{3^3.\,2^5\pi\frac{(g_sM)^2}{N}(g_sN_f)^2}\frac{1}{(\ln r_\Lambda)^3} \sim (g_sN)^{-\frac{3}{4}}$, we get $\ln r_\Lambda \sim \left\{\frac{N^{\frac{7}{4}}g_s^{-\frac{13}{4}}}{3^3.\,2^5\pi M^2N_f^2}\right\}^{\frac{1}{3}}$.

The action, apart from being divergent (as $r\to\infty$) also possesses pole singularities near $\theta_{1,2} = 0, \pi$. We regulate the second divergence by taking a small $\theta_{1,2}$-cutoff ϵ_θ, $\theta_{1,2} \in [\epsilon_\theta, \pi - \epsilon_\theta]$, and demanding $\epsilon_\theta \sim \epsilon^\gamma$, for an appropriate γ. We then explicitly checked that the finite part of the action turns out to be independent of this cutoff ϵ/ϵ_θ. It was shown in [26] that in [24]'s and the MQGP limit, $S^{\mathrm{finite}}_{\mathrm{EH+GHY}+|G_4|^2+\mathcal{O}(R^4)} \sim -r_h^3$ and the counterterms are given by

$$\int\left(\epsilon^{-\kappa_{\mathrm{EH}}\text{-surface}^{(i)}}\sqrt{h^{\mathcal{M}}}R^{\mathcal{M}},\ \epsilon^{-\kappa^{(i)}_{\mathrm{cosmo}}}\sqrt{h},\ \epsilon^{-\kappa^{(i)}_{\mathrm{flux}}}\sqrt{h}|G_4|^2\right)\Bigg|_{r=r_\Lambda}\times\left(\frac{a_{\mathrm{EH}}}{\epsilon} + a_{\mathrm{GHY}}\text{-boundary} - \epsilon^9a_{G_4} + a_{R^4}\epsilon^{23}\right),$$

and

$$\int\left(\epsilon^{-\kappa_{\mathrm{EH}}\text{-surface}^{(ii)}}\sqrt{h^{\mathcal{M}}}R^{\mathcal{M}},\ \epsilon^{-\kappa^{(ii)}_{\mathrm{cosmo}}}\sqrt{h},\ \epsilon^{-\kappa^{(ii)}_{\mathrm{flux}}}\sqrt{h}|G_4|^2\right)\Bigg|_{r=r_\Lambda}\times\left(\frac{a_{\mathrm{EH}}}{\epsilon^3} + a_{\mathrm{GHY}}\text{-boundary} - \epsilon^{19}a_{G_4} + a_{R^4}\epsilon^{45}\right),$$

respectively, in the two limits—in both limits the counterterms could be given an ALD-gravity-counterterms [35] interpretation. It was also shown in [26] that the entropy (density s) is positive and one can approximate it by a $s \sim r_h^3$ result, which is what one also obtains (as shown in [26]) by calculation of the horizon area. Using this, therefore, $C \sim \frac{T(r_h)}{\frac{\partial T}{\partial r_h}}\frac{\partial(r_h^3)}{\partial r_h} > 0$, implying a stable uplift!

2.2 Satisfying $D = 11$ SUGRA EOMs locally in the MQGP limit

In this subsection we explicitly verify that the uplift to M-theory in the delocalised limit of [25] in the MQGP limit of [26] (involving $g_s \lesssim 1$) constitutes a bona-fide solution of $D = 11$ supergravity EOMs in the presence of (wrapped) $M5$-brane sources. These for a single $M5$-brane source are given by [36]

$$R^{\mathcal{M}}_{MN} - \frac{1}{2}G^{\mathcal{M}}_{MN}R = \frac{1}{12}\left(G_{MPQR}G_N^{\ PQR} - \frac{1}{8}G^{\mathcal{M}}_{MN}G_{PQRS}G^{PQRS}\right) + \kappa_{11}^2T^{M5}_{MN}, \tag{14}$$

and in the absence of a Dirac 6-brane:

$$d*_{11}G_4 + G_4\wedge G_4 = -2\kappa_{11}^2T_5(H_3 - A_3)\wedge *_{11}J_6, \tag{15}$$

where the $M5$-brane current $J_6 \sim \frac{dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge d\theta_1\wedge d\phi_1}{\sqrt{-G^{\mathcal{M}}}}$; the Bianchi identity is

$$dG_4 = 2\kappa_{11}^2T_5 *_{11} J_6. \tag{16}$$

Taking the trace of (14),

$$-\frac{9}{2}R = -\frac{1}{32}G_{PQRS}G^{PQRS} + \kappa_{11}^2T^Q_Q. \tag{17}$$

Incorporating the above in Eq. (14), one gets

$$R^{\mathcal{M}}_{MN} = \frac{1}{12}\left(G_{MPQR}G_N^{PQR} - \frac{1}{12}G^{\mathcal{M}}_{MN}G_{PQRS}G^{PQRS}\right) +\kappa_{11}^2\left(T_{MN} - \frac{1}{9}G^{\mathcal{M}}_{MN}T^Q_Q\right) \tag{18}$$

and the spacetime energy–momentum tensor T_{MN} for a single M5-brane wrapped around $S^2(\theta_1, \phi_1)$ is given by

$$T^{MN}(x) = \int_{\mathcal{M}_6} d^6\xi \sqrt{-\det G^{M5}_{\mu\nu}}G^{(M5)\mu\nu}\partial_\mu X^M \partial_\nu X^N \times \frac{\delta^{11}(x - X(\xi))}{\sqrt{-\det G^{\mathcal{M}}_{MN}}} \tag{19}$$

where $X = 0, 1, ..., 11$ and $\mu, \nu = 0, 1, 2, 3, \theta_1, \phi_1$.

Now, we have

$$\det G^{\mathcal{M}}_{MN} \sim \frac{2\pi r^6 f_2(\theta_2)^2 g_s N \cos^2(\theta_1)\cot^6(\theta_1)\sin^2(\theta_2)\cos^2(\theta_2)}{15943233^{2/3} g_s{}^{16/3}},$$

$$\det G^{M5}_{\mu\nu} \sim \frac{9\left(1 - \frac{r_h^4}{r^4}\right) r^8 \sin^4(\theta_1)(\cos(2\theta_2) - 5)}{64\pi g_s{}^{15/3} N\left(\sin^2(\theta_1)\cos^2(\theta_2) + \cos^2(\theta_1)\sin^2(\theta_2)\right)}$$

$$G^{\mathcal{M}}_{\theta_1\theta_1} \sim \frac{\sqrt{\pi}N}{\sqrt[3]{3}\sqrt[3]{\frac{1}{g_s}}\sqrt{g_s N}}. \tag{24}$$

Utilising the above, we have

$$T_{\theta_1\theta_1} - \frac{1}{9}G^{\mathcal{M}}_{\theta_1\theta_1}T^Q_Q \sim \frac{8}{9}G^{\mathcal{M}}_{\theta_1\theta_1}\frac{\sqrt{-\det G^{M5}_{\mu\nu}}}{\sqrt{-\det G^{\mathcal{M}}_{MN}}} \sim -\frac{\left(\sqrt{1 - \frac{r_h^4}{r^4}}\right) r \tan^4(\theta_1)\sqrt{\frac{5-\cos(2\theta_2)}{2-\cos[2(\theta_1-\theta_2)]-\cos[2(\theta_1+\theta_2)]}}\csc(\theta_2)\sec(\theta_2)\sin(\theta_1)}{\sqrt{g_s N} f_2(\theta_2)}. \tag{25}$$

Now, in the MQGP limit, the most dominant contribution is governed by $R_{\theta_1\theta_1}$. So

$$R^{\mathcal{M}}_{\theta_1\theta_1} = \frac{1}{12}\left(G_{\theta_1 PQR}G_{\theta_1}^{PQR} - \frac{1}{8}G^{\mathcal{M}}_{\theta_1\theta_1}G_{PQRS}G^{PQRS}\right) +\kappa_{11}^2\left(T_{\theta_1\theta_1} - \frac{1}{9}G^{\mathcal{M}}_{\theta_1\theta_1}T^Q_Q\right). \tag{20}$$

Using Eq. (19), one sees that

$$T^Q_Q \sim \frac{\sqrt{-\det G^{M5}_{\mu\nu}}}{\sqrt{-\det G^{\mathcal{M}}_{MN}}} \tag{21}$$

Similarly, in both limits,

$$T_{\theta_1\theta_1} = G^{\mathcal{M}}_{\theta_1 M_1}G^{\mathcal{M}}_{\theta_1 M_2}T^{M_1M_2} \sim G^{\mathcal{M}}_{\theta_1\theta_1}\frac{\sqrt{-\det G^{M5}_{\mu\nu}}}{\sqrt{-\det G^{\mathcal{M}}_{MN}}}. \tag{22}$$

So, we get

$$T_{\theta_1\theta_1} - \frac{1}{9}G^{\mathcal{M}}_{\theta_1\theta_1}T^Q_Q \approx \frac{8}{9}\times G^{\mathcal{M}}_{\theta_1\theta_1}\frac{\sqrt{-\det G^{M5}_{\mu\nu}}}{\sqrt{-\det G^{\mathcal{M}}_{MN}}}. \tag{23}$$

We will estimate (25) at an $r_\Lambda \sim e^{\left(\frac{N^{7/4}}{g_s^{13/4}M^2N_f^2}\right)^{\frac{1}{3}}}$ (see Sect. 2.2) resolution/deformation parameter and also larger than $r_h = 1 + \epsilon(\epsilon \to 0^+)$, near $\theta_{1,2} = 0$ (effected as $\theta_{1,2} \sim \alpha_\theta\epsilon^{\frac{3}{2}}$), for $\epsilon \sim 0.83$, $\alpha_\theta << 1$ yields $r_\Lambda \sim 5$ in the MQGP limit. In this limit, from (25), in the same spirit as [37] one estimates

$$T_{\theta_1\theta_1} - \frac{1}{9}G^{\mathcal{M}}_{\theta_1\theta_1}T^Q_Q\Big|_{\sim\alpha_\theta\epsilon^{\frac{3}{2}}, \alpha_\theta<<1} \sim N_{wM_5}\mathcal{O}(1)\alpha_\theta^4 << 1, \tag{26}$$

where N_{wM_5} is the number of wrapped $M5$-branes approximately given by N, which in the MQGP limit and with $\epsilon = 0.83$ is around 10^3; $\alpha_\theta << 10^{-\frac{3}{4}}$. In [26], it was shown that, in the MQGP limit, the EH action $\int\sqrt{-G^{\mathcal{M}}}R$ yields a divergent contribution: $\frac{r_\Lambda^4}{\alpha_\theta^3\epsilon^3}$, which can be cancelled by a boundary counterterm: $\epsilon^{-155/6}\left(\frac{a(\alpha_{1,2,3})}{\alpha_\theta^3\epsilon^3}\right)\int\sqrt{-h}R$ evaluated at $r = r_\Lambda$, wherein $g_s\alpha_1\epsilon$, $N \sim \alpha_2\epsilon^{3/2}$, $N \sim \alpha_3\epsilon^{-39}$. So, at $r = r_\Lambda$, $\int\sqrt{-G^{\mathcal{M}}}R - \epsilon^{155/6}\left(\frac{a(\alpha_{1,2,3})}{\alpha_\theta^3\epsilon^3}\right)\int\sqrt{-h}R$ will not yield any finite contribution to R_{MN}. Further,

$$F_{MPQR}G_N^{PQR} - \frac{1}{12}G^{\mathcal{M}}_{MN}G_{PQRS}G^{PQRS}$$
$$\sim \frac{\sin^9(\theta_1)\cos(\theta_1)\cot^3(\theta_2)}{\left(\sin^2(\theta_1)\cos^2(\theta_2)+\cos^2(\theta_1)\sin^2(\theta_2)\right)^2}\Bigg|_{\theta_{1,2}\sim\alpha_\theta\epsilon^{\frac{3}{2}},\alpha_\theta<<1}$$
$$\sim \alpha_\theta^2\epsilon^2 << 1. \tag{27}$$

Further:

$$*_{11}J_6 \sim \epsilon^{x^0x^1x^2x^3\theta_1\phi_1}_{r\theta_2\phi_2\psi x_{10}}\mathrm{d}r\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\phi_2\wedge\mathrm{d}\psi\wedge\mathrm{d}x_{10}$$
$$\sim -\prod_{\mu\in\mathbb{R}^{1,3}}G^{\mu\mu}\prod_{m\in S^2(\theta_1,\phi_1)}G^{mm}\mathrm{d}r\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\phi_2\wedge\mathrm{d}\psi$$
$$\wedge\,\mathrm{d}x_{10}\sim\frac{g_s^5N\sin^6\theta_1}{r^8}\mathrm{d}r\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\phi_2\wedge\mathrm{d}\psi\wedge\mathrm{d}x_{10}$$
$$\overset{r=r_\Lambda,MQGP,\epsilon=0.83,\theta_{1,2}\to0}{\longrightarrow} << 1. \tag{28}$$

From [26], $G_4\wedge G_4=0$ and

$$\begin{aligned}A_3 &= B_2\wedge\mathrm{d}x_{10},\\ H_3 &= H_{\theta\theta_2\phi_1}\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\phi_1 + H_{\theta_1\theta_2\phi_2}\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\\ &\quad\wedge\mathrm{d}\phi_2 + H_{r\theta_1\theta_2}\mathrm{d}r\wedge\mathrm{d}\theta_1\mathrm{d}\theta_2\\ &\quad+H_{\theta_1\phi_1\psi}\mathrm{d}\theta_1\mathrm{d}\phi_1\mathrm{d}\psi + H_{\theta_1\theta_2\phi_2}\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\phi_2\\ &\quad+H_{\theta_1\theta_2\phi_1}\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\phi_1\\ &\quad+H_{\theta_1\phi_1\psi}\mathrm{d}\theta_1\wedge\mathrm{d}\phi_1\wedge\mathrm{d}\psi.\end{aligned} \tag{29}$$

So from (29) and (28), $(H_3-A_3)\wedge *_{11}J_6=0$. Using this in (15), G_{MNPQ} satisfies $\partial_M\left(\sqrt{-G}G^{MNPQ}\right)=0$, which in the MQGP limit is given by

$$\partial_{\theta_1}\left(\sqrt{-G^{\mathcal{M}}}G^{\theta_1NPQ}\right)+\partial_{\theta_2}\left(\sqrt{-G^{\mathcal{M}}}G^{\theta_2NPQ}\right)$$
$$+\partial_{\phi_1}\left(\sqrt{-G^{\mathcal{M}}}G^{\phi_1NPQ}\right)=0. \tag{30}$$

Using the non-zero expressions for the various four-form flux components in the MQGP limit requires one to consider:

- $N=\theta_2,\ P=\phi_1,\ Q=x_{10}$ and one hence obtains

$$\partial_{\theta_1}\left(\sqrt{-G^{\mathcal{M}}}G^{\theta_1\theta_1}G^{\theta_2\theta_2}G^{\phi_1\phi_2}G^{x_{10}x_{10}}G_{\theta_1\theta_2\phi_1x_{10}}\right)$$
$$\sim\frac{1}{g_s^{\frac{29}{12}}N^{\frac{7}{4}}}$$
$$\times\left[\frac{r\theta_2\tan(\theta_1)\sin(\theta_2)\tan(\theta_2)\left(\left(3\tan^2(\theta_1)+1\right)\sin^2(\theta_2)+2\tan^4(\theta_1)\right)}{\left(\sin^2(\theta_1)+\sin^2(\theta_2)\right)^2}\right]$$
$$\downarrow r=r_\Lambda,\ MQGP\text{ limit},\ \theta_{1,2}\sim\alpha_\theta\epsilon^{\frac{3}{2}},\alpha_\theta<<1,\epsilon=0.83$$
$$\sim\alpha_\theta^2 10^{-5}<<1. \tag{31}$$

- $N=\phi_1,\ P=\psi,\ Q=\phi_1$;

$$\partial_{\theta_1}\left(\sqrt{-G}G^{\theta_1\phi_2\psi\phi_1}\right)+\partial_{\theta_2}\left(\sqrt{-G}G^{\theta_2\phi_2\psi\phi_1}\right)$$
$$\downarrow r=r_\Lambda,\ MQGP\text{ limit},\ \theta_{1,2}\sim\alpha_\theta\epsilon^{\frac{3}{2}},$$
$$\alpha_\theta<<1,\ \epsilon=0.83$$
$$\sim N_f\sin\phi_2\left(\frac{r_\Lambda\sin^5\theta_1}{g_s^{\frac{3}{4}}N^{\frac{3}{4}}}+\frac{r_\Lambda\sin^{11}\theta_1}{g_s^{\frac{3}{4}}N^{\frac{3}{4}}}\right)$$
$$\sim\alpha_\theta^5 10^{-3}N_f\sin\phi_2<<1. \tag{32}$$

- $N=\theta_2,\ P=\phi_1,\ Q=\phi_2$:

$$\partial_{\theta_1}\left(\sqrt{-G}G^{\theta_1\theta_2\phi_1\phi_2}\right)\sim N_f\sin\phi_2\frac{r_\Lambda^2\sin^{10}\theta_2}{g_s^{\frac{41}{12}}N^{\frac{3}{4}}}$$
$$\sim\alpha_\theta^{10}10^{-2}N_f\sin\phi_2<<1. \tag{33}$$

Utilising the expressions for the non-zero G_{MNPQ} flux components and (28) (which tells us that $*_{11}J_6$'s only non-zero component is $(*_{11}J_6)_{r\theta_2\phi_2\psi x_{10}}$), at $\theta_{1,2}\sim\alpha_\theta\epsilon^{\frac{3}{2}}$, $\alpha_\theta<<1$:

$$\partial_{[r}G_{\theta_2\phi_2\psi x_{10}]}=0\approx(*_{11}J_6)_{r\theta_2\phi_2\psi x_{10}}\,. \tag{34}$$

Also:

$$\partial_{[r]}G_{\theta_1\phi_2\psi\phi_1]}=0;\ \partial_{[r}G_{\theta_2\phi_2\psi x_{10}]}=0;$$
$$\partial_{[r]}G_{\theta_1\theta_2\phi_1\phi_2]}\big|_{r=r_\Lambda,\theta_{1,2}\to0,\ \text{MQGP limit}}$$
$$\sim\frac{g_s^{\frac{7}{4}}M^2}{N^{\frac{1}{4}}r_\Lambda}\cos\phi_2+\frac{g_s^{\frac{7}{4}}MN^{\frac{1}{4}}}{r_\Lambda}\cos\phi_2$$
$$\downarrow g_s\sim\alpha_{g_s}\epsilon,\ M\sim\alpha_M\epsilon^{-\frac{3}{2}},\ N\sim\alpha_N\epsilon^{-39},$$
$$\epsilon=0.83:\alpha_{M,N}\sim\frac{1}{\mathcal{O}(1)}\sim10^{-2}<<1;$$
$$\partial_{[\psi}G_{\theta_1\theta_2\phi_1\phi_2]}$$
$$\downarrow\theta_1\sim\alpha_\theta\epsilon^{\frac{3}{2}},\theta_2\sim\mathcal{O}(1)\alpha_\theta\epsilon^{\frac{3}{2}}:\alpha_\theta<<1;$$
$$\text{MQGP limit: }\epsilon=0.83$$
$$\frac{N_f\,(g_sN)^{\frac{3}{4}}}{(\mathcal{O}(1))^4}\sin\phi_2\sim10^{-2}\sin\phi_2\quad\text{for }\mathcal{O}(1)\sim7. \tag{35}$$

We therefore conclude that the D = 11 SUGRA EOMs are nearly satisfied near $r=r_\Lambda$ and the $\theta_1=\theta_2=0$ branch.

2.3 Black $M3$-branes as wrapped $M5$-branes around two-cycle

Let us turn our attention towards figuring out the charge of the black $M3$-brane. We utilise the non-zero expressions of the components of the four-form flux G_4 as given in (152), and the following features. (i) We assume the expressions obtained in the 'delocalised' limit of [25] and hence in conformity with the non-locality of T duality transformations, are valid globally ($\langle\theta_{1,2}\rangle$ was shown in [38] to be replaced by $\theta_{1,2}$ for $D5$ branes wrapping two-cycles in conifolds). (ii) We use the principal values of the integrals over $\theta_{1,2}$, i.e., assuming: $\int_0^\pi \mathcal{F}(\theta_i)\mathrm{d}\theta_i = \lim_{\epsilon_{\theta_i}\to 0}\int_{\epsilon_{\theta_i}}^{\pi-\epsilon_{\theta_i}} \mathcal{F}(\theta_i)\mathrm{d}\theta_i = \lim_{\epsilon_{\theta_i}\to 0}\int_{\epsilon_{\theta_i}}^{\pi-\epsilon_{\theta_i}} \mathcal{F}(\pi-\theta_i)\mathrm{d}\theta_i$ (iii) We assume the $f_i(\theta_i)$ introduced in (10) to make the base of the local T^3-fibered resolved warped deformed conifold to be large, it will globally be $\cot\theta_i$. (iv) We believe that the distinction between results with respect to ϕ_1 and ϕ_2 arising due to the asymmetric treatment of this, while constructing A^{IIA} from triple T-duals of $F^{IIB}_{1,3,5}$, being artificial and hence ignoring this. Then using (152) one sees that in the limit of [24]

$$\int_{C_4^{(1)}(\theta_1,\theta_2\phi_1,\phi_2)} G_4\Big|_{\langle\psi\rangle,\langle r\rangle,\langle x_{10}\rangle} \sim \epsilon^{-14}N_f\sin\frac{\langle\psi\rangle}{2};$$
$$\int_{C_4^{(2)}(\theta_1,\theta_2,\phi_{1/2},x_{10})} G_4\Big|_{\langle\psi\rangle,\langle\phi_{2/1}\rangle,\langle r\rangle} \sim \epsilon^{-9};$$
$$\int_{C_4^{(3)}(r,\theta_1,\theta_2,x_{10})} G_4\Big|_{\langle\psi\rangle,\langle\phi_{1,2}\rangle} = 0;$$
$$\int_{C_4^{(4)}(r,\theta_1,\phi_{1/2},x_{10})} G_4\Big|_{\langle\theta_2\rangle,\langle\psi\rangle,\langle\phi_{2/1}\rangle} = 0;$$
$$\int_{C_4^{(5)}(r,\theta_2,\phi_{1/2},x_{10})} G_4\Big|_{\langle\psi\rangle,\langle\theta_1\rangle,\langle\phi_{2/1}\rangle} = 0;$$
$$\int_{C_4^{(6)}(r,\theta_2,\psi,x_{10})} G_4\Big|_{\langle\theta_1\rangle,\langle\phi_{1,2}\rangle} = 0.$$

Strictly $G_{r\theta_2\psi x_{10}} \sim \mathcal{O}\left(\frac{g_s^2MN_f}{N}\right)\mathcal{F}_1(\theta_{1,2}) + \mathcal{O}(g_s^2MN_f)$ $\mathcal{F}_2(\theta_{1,2})$.

We first take the weak (g_s) coupling-large 't Hooft coupling limit in $G_{r\theta_2\psi x_{10}}$ itself to annul this flux. Also, $G_{r\theta_2\psi x_{10}} \sim \frac{N_f}{r}$ in the weak (g_s) coupling-large 't Hooft coupling limit, and hence it is negligible for large r.

Similarly $\int_{C_4^{(7)}(r,\theta_1,\psi,x_{10})} G_4\Big|_{\langle\theta_2\rangle,\langle\phi_{1,2}\rangle} = 0;$

$$\int_{C_4^{(8)}(\theta_1,\phi_{1/2},\psi,x_{10})} G_4\Big|_{\langle\phi_{2/1}\rangle,\langle r\rangle,\langle\theta_2\rangle} = 0;$$
$$\int_{C_4^{(9)}(\theta_1,\theta_2,\phi_{1/2},x_{10})} G_4\Big|_{\langle\phi_{2/1}\rangle,\langle\psi\rangle,\langle\phi_{2/1}\rangle,\langle r\rangle} \sim \epsilon^{-8};$$
$$\int_{C_4^{(10)}(\theta_1,\theta_2,\psi,x_{10})} G_4\Big|_{\langle\phi_{1,2}\rangle,\langle r\rangle} = 0.$$

Strictly $G_{\theta_1\theta_2\psi x_{11}} \sim \mathcal{O}\left[(g_s^2MN_f)(g_sN_f)\left(\frac{g_sM^2}{N}\right)\right]$ $\mathcal{F}_3(\theta_{1,2})$, Again we first take the weak (g_s) coupling-large 't Hooft coupling limit in $G_{\theta_1\theta_2\psi x_{10}}$ itself to annul this flux,

$$\int_{C_4^{(11)}(r,\theta_2,\phi_1,\phi_2)} G_4\Big|_{\langle\theta_1\rangle,\langle\psi\rangle,\langle x_{10}\rangle} = 0;$$
$$\int_{C_4^{(12)}(r,\theta_2,\phi_1,\psi)} G_4\Big|_{\langle\theta_1\rangle,\langle\phi_2\rangle,\langle x_{10}\rangle} = 0;$$
$$\int_{C_4^{(13)}(r,\theta_1,\phi_{1/2},\psi)} G_4\Big|_{\langle\theta_2\rangle,\langle\phi_{2/1}\rangle,\langle x_{10}\rangle} = 0;$$
$$\int_{C_4^{(14)}(\theta_1,\theta_2,\phi_{1/2},\psi)} G_4\Big|_{\langle\phi_{2/1}\rangle,\langle r\rangle,\langle x_{10}\rangle} = 0$$
$$\int_{C_4^{(15)}(\theta_{1/2},\phi_1,\phi_2,\psi)} G_4\Big|_{\langle\phi_{2/1}\rangle,\langle x_{10}\rangle,\langle r\rangle} = 0, \tag{36}$$

where we calculate the flux of G_4 through various four-cycles C_4^I. We have dropped the contribution of $G_{r\theta_1\theta_2\psi}$, $G_{r\theta_2\phi_1\phi_2}$ to $G_{rmnp}\mathrm{d}r\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\wedge\mathrm{d}x^p$ as it is $\mathcal{O}\left(\frac{1}{r}\right)$-suppressed as compared to the ones retained, which is hence, dropped, at large r. From (36), one sees that the most dominant contribution to all possible fluxes arises (near $\langle\psi\rangle$), in the large-r limit, from $G_{\theta_1\theta_2\phi_1x_{10}}$ and $G_{\theta_1\theta_2\phi_2x_{10}}$. Using (37), the large-$r$ limit of the $D=11$ metric can be written as

$$\begin{aligned}\mathrm{d}s^2_{11} &= \mathrm{d}s^2_{\mathrm{AdS}_5} + g_sM^2(\ln r)^2\Sigma_1(\theta_{1,2})\mathrm{d}\theta_1^2\\ &+\frac{g_s^{\frac{11}{6}}M^2}{\sqrt{N}}a^4r^4(\ln r)^2\Sigma_2(\theta_{1,2})\mathrm{d}\theta_2^2 + g_s^{\frac{11}{6}}r\ln r\\ &\times\left[(g_sN)^{\frac{1}{4}}M\Sigma_3(\theta_{1,2})\mathrm{d}\theta_1 + g_s^2a^2r^2\Sigma_4(\theta_{1,2})\mathrm{d}\theta_2\right]\\ &\times\mathrm{d}x_{10} + g_s^{\frac{4}{3}}\mathrm{d}x_{10}^2 + g_s^{\frac{4}{3}}M[N_fr\ln r\Sigma_5(\theta_{1,2})\mathrm{d}\theta_1\\ &+a^2r^2\ln r\tilde{\Sigma}_5(\theta_{1,2})\mathrm{d}\theta_2]\mathrm{d}\phi_{1/2}\\ &+\mathrm{d}s^2_{\mathcal{M}_3(\phi_{1,2},\psi)}(\theta_{1,2}).\end{aligned} \tag{37}$$

Hence, asymptotically, the $D=11$ spacetime is a warped product of $\mathrm{AdS}_5(\mathbb{R}^{1,3}\times\mathbb{R}_{>0})$ and an $\mathcal{M}_6(\theta_{1,2},\phi_{1,2},\psi,x_{10})$ where $\mathcal{M}_6$ has the following fibration structure:

$$\begin{array}{lll} & \mathcal{M}_6(\theta_{1,2},\phi_{1,2},\psi,x_{10}) \longleftarrow S^1(x_{10}) & \\ & \downarrow & \\ \mathcal{M}_3(\phi_1,\phi_2,\psi) & \longrightarrow \mathcal{M}_5(\theta_{1,2},\phi_{1,2},\psi) & \\ & \downarrow & \\ & \mathcal{B}(\theta_1,\theta_2) \longleftarrow [0,1]_{\theta_1} & . \\ & \downarrow & \\ & [0,1]_{\theta_2} & \end{array} \tag{38}$$

Analogous to the $F_3^{IIB}(\theta_{1,2})$ (with non-zero components being $F_{\psi\phi_1\theta_1}$, $F_{\psi\phi_2\theta_2}$, $F_{\phi_1\phi_2\theta_1}$ and $F_{\phi_1\phi_2\theta_2}$) in a Klebanov–

Strassler background corresponding to $D5$-branes wrapped around a two-cycle which homologously is given by $S^2(\theta_1, \phi_1) - S^2(\theta_2, \phi_2)$ [22], the black $M3$-brane metric asymptotically can be thought of as black $M5$-branes wrapping a two-cycle homologously given by $n_1 S^2(\theta_1, x_{10}) + n_2 S^2(\theta_2, \phi_{1/2}) + m_1 S^2(\theta_1, \phi_{1/2}) + m_2 S^2(\theta_2, x_{10})$ for some large $n_{1,2}, m_{1,2} \in \mathbb{Z}$.

A similar interpretation is expected to hold in the MQGP limit of [26]—in the same limit, however, the analogs of (152) are very tedious and unmanageable to work out for arbitrary $\theta_{1,2}$. But we have verified that, in the MQGP limit, $\int_{C_4(\theta_{1,2},\phi_{1/2},x_{10})} G_4\Big|_{\langle\phi_{2/1}\rangle,\langle\psi\rangle,\langle r\rangle}$ will be very large. Warped products of AdS_5 and an M_6 corresponding to wrapped $M5$-branes have been considered in the past—see [39].

2.4 Kählerity from torsion classes and a warped deformed conifold sLag in the 'Delocalised' Limit of [25], and large-r and MQGP limits

In this subsection, first, by calculating the five $SU(3)$ structure torsion (τ) classes $W_{1,2,3,4,5}$, we show that in the MQGP limit of [26] and assuming the deformation parameter to be larger than the resolution parameter, the non-Kähler resolved warped deformed conifold of [24], in the large-r limit, reduces to a warped Kähler deformed conifold for which $\tau \in W_5$. Then, in the large-r limit, we show that the local T^3 of [26] satisfies the same constraints as the one satisfied by a maximal T^2-invariant sLag sub manifold of a T^*S^3 so that the application of mirror symmetry as three T-dualities à la SYZ, in the MQGP limit of [26], could be implemented on the type IIB background of [24].

The $SU(3)$ structure torsion classes [40,41] can be defined in terms of J, Ω, dJ, dΩ and the contraction operator $\lrcorner : \Lambda^k T^\star \otimes \Lambda^n T^\star \to \Lambda^{n-k} T^\star$, J being given by

$$J = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6,$$

and the (3, 0)-form Ω being given by

$$\Omega = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6).$$

The torsion classes are defined in the following way:

- $W_1 \leftrightarrow [dJ]^{(3,0)}$, given by real numbers $W_1 = W_1^+ + W_1^-$ with $d\Omega_+ \wedge J = \Omega_+ \wedge dJ = W_1^+ J \wedge J \wedge J$ and $d\Omega_- \wedge J = \Omega_- \wedge dJ = W_1^- J \wedge J \wedge J$;
- $W_2 \leftrightarrow [d\Omega]_0^{(2,2)}$: $(d\Omega_+)^{(2,2)} = W_1^+ J \wedge J + W_2^+ \wedge J$ and $(d\Omega_-)^{(2,2)} = W_1^- J \wedge J + W_2^- \wedge J$;
- $W_3 \leftrightarrow [dJ]_0^{(2,1)}$ is defined as $W_3 = dJ^{(2,1)} - [J \wedge W_4]^{(2,1)}$;
- $W_4 \leftrightarrow J \wedge dJ$: $W_4 = \frac{1}{2} J \lrcorner dJ$;
- $W_5 \leftrightarrow [d\Omega]_0^{(3,1)}$: $W_5 = \frac{1}{2}\Omega_+ \lrcorner d\Omega_+$ (the subscript 0 indicative of the primitivity of the respective forms).

Depending on the classes of torsion one can obtain different types of manifolds, some of which are:

1. (complex) special-hermitian manifolds with $W_1 = W_2 = W_4 = W_5 = 0$, which means that $\tau \in W_3$;
2. (complex) Kähler manifolds with $W_1 = W_2 = W_3 = W_4 = 0$ which means $\tau \in W_5$;
3. (complex) balanced Manifolds with $W_1 = W_2 = W_4 = 0$, which means $\tau \in W_3 \oplus W_5$;
4. (complex) Calabi–Yau manifolds with $W_1 = W_2 = W_3 = W_4 = W_5 = 0$, which means $\tau = 0$.

The resolved warped deformed conifold can be written in the form of the Papadopoulos–Tseytlin ansatz [42] in the string frame:

$$ds^2 = h^{-1/2} ds^2_{\mathbb{R}^{1,3}} + e^x ds^2_{\mathcal{M}} = h^{-1/2} dx^2_{1,3} + \sum_{i=1}^{6} G_i^2, \tag{39}$$

where [43,44]

$$\begin{aligned}
G_1 &\equiv e^{(x(\tau)+g(\tau)/2}\, e_1,\; G_2 \equiv \mathcal{A}\, e^{(x(\tau)+g(\tau))/2}\, e_2 \\
&\quad + \mathcal{B}(\tau)\, e^{(x(\tau)-g(\tau))/2}\, (\epsilon_2 - a e_2), \\
G_3 &\equiv e^{(x(\tau)-g(\tau))/2}\, (\epsilon_1 - a e_1), \\
G_4 &\equiv \mathcal{B}(\tau)\, e^{(x(\tau)+g(\tau))/2}\, e_2 - \mathcal{A}\, e^{(x(\tau)-g(\tau))/2}\, (\epsilon_2 - a e_2), \\
G_5 &\equiv e^{x(\tau)/2}\, v^{-1/2}(\tau) d\tau, \\
G_6 &\equiv e^{x(\tau)/2}\, v^{-1/2}(\tau)(d\psi + \cos\theta_2 d\phi_2 + \cos\theta_1 d\phi_1),
\end{aligned} \tag{40}$$

wherein $\mathcal{A} \equiv \frac{\cosh\tau + a(\tau)}{\sinh\tau}$, $\mathcal{B}(\tau) \equiv \frac{e^{g(\tau)}}{\sinh\tau}$. The e_is are one-forms on S^2,

$$e_1 \equiv d\theta_1, \qquad e_2 \equiv -\sin\theta_1 d\phi_1, \tag{41}$$

and the ϵ_is a set of one-forms on S^3

$$\begin{aligned}
\epsilon_1 &\equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2, \\
\epsilon_2 &\equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2,\; \epsilon_3 \equiv d\psi + \cos\theta_2 d\phi_2.
\end{aligned} \tag{42}$$

These one-forms are quite convenient to work with since they allow us to write down very simple expressions for the holomorphic (3, 0) form,

$$\Omega = (G_1 + iG_2) \wedge (G_3 + iG_4) \wedge (G_5 + iG_6), \tag{43}$$

and the fundamental (1, 1) form,

$$\begin{aligned}
J = \frac{i}{2}\big[&(G_1 + iG_2) \wedge (G_1 - iG_2) + (G_3 + iG_4) \\
&\wedge (G_3 - iG_4) + (G_5 + iG_6) \wedge (G_5 - iG_6)\big].
\end{aligned} \tag{44}$$

Substituting (40) into (39), one obtains

$$
\begin{aligned}
ds_6^2 = & \frac{1}{\sqrt{h}} ds^2_{\mathbb{R}^{1,3}} + \frac{e^{x(\tau)}}{v(\tau)} \\
& \times \left(d\tau^2 + [d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2]^2 \right) \\
& + d\theta_1^2 \left(e^{g(\tau)+x(\tau)} + a(\tau)^2 e^{-g(\tau)+x(\tau)} \right) \\
& + d\phi_1^2 \left(\frac{1}{2} \left(a^2(\tau) + e^{2g(\tau)} \right) \right. \\
& \times e^{x(\tau)-g(\tau)} \mathrm{csch}^2(\tau) \sin^2(\theta_1) \\
& \times \left(2a(\tau)^2(\tau) + 4a \cos h(\tau) + 2e^{2g(\tau)} \right. \\
& + \cos h(2\tau) + 1)) + d\theta_2^2 e^{x(\tau)-g(\tau)} \\
& \times \left(\left(a^2(\tau) + e^{2g(\tau)} \right) \sin^2(\psi) \mathrm{csch}^2(\tau) \right. \\
& + 2a(\tau) \sin^2(\psi) \coth(\tau) \mathrm{csch}(\tau) \\
& \left. + \sin^2(\psi) \coth^2(\tau) + \cos^2(\psi) \right) \\
& + d\phi_2^2 e^{x(\tau)-g(\tau)} \sin^2(\mathrm{theta2}) \\
& \times \left(\left(a^2(\tau) + e^{2g(\tau)} \right) \cos^2(\psi) \mathrm{csch}^2(\tau) \right. \\
& + 2a \cos^2(\psi) \coth(t) \mathrm{csch}(\tau) + \cos^2(\psi) \coth^2(t) \\
& \left. + \sin^2(\psi) \right) - 2a(\tau) e^{-x(\tau)+g(\tau)} \cos\psi \, d\theta_1 d\theta_2 \\
& + d\phi_1 d\phi_2 a e^{x(\tau)-g(\tau)} \cos(\psi) \mathrm{csch}^2(\tau) \sin(\theta_1) \sin(\theta_2) \\
& \times \left(2a^2 + 4a \cos h(t) + 2e^{2g(\tau)} + \cos h(2\tau) + 1 \right) \\
& - a(\tau) e^{x(\tau)-g(\tau)} \sin(\psi) \mathrm{csch}^2(\tau) \sin(\theta_1) \\
& \times \left(2a^2(\tau) + 4a(\tau) \cos h(\tau) + 2e^{2g(\tau)} + \cos h(2\tau) + 1 \right) \\
& \times d\theta_2 d\phi_1 - 2a(\tau) e^{-x(\tau)+g(\tau)} \sin\psi \sin\theta_2 d\theta_1 d\phi_2.
\end{aligned} \tag{45}
$$

As $r \sim e^{\frac{\tau}{3}}$, the large-$r$ limit is equivalent to the large-τ limit, in which (45) approaches

$$
\begin{aligned}
ds_6^2 = & \frac{e^{x(\tau)}}{v(\tau)} \left(d\tau^2 + [d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2]^2 \right) \\
& + \left(e^{x(\tau)+g(\tau)} + a^2(\tau) e^{x(\tau)-g(\tau)} \right) d\theta_1^2 \\
& + e^{x(\tau)-g(\tau)} \sin^2\theta_1 d\phi_1^2 + e^{x(\tau)-g(\tau)} d\theta_2^2 \\
& + e^{x(\tau)-g(\tau)} \sin^2\theta_2^2 d\phi_2^2 - 2a e^{x(\tau)-g(\tau)} \\
& \times [\cos\psi \, (d\theta_1 d\theta_2 - \sin\theta_1 \sin\theta_2 d\phi_1 d\phi_2) \\
& + \sin\psi \, (d\theta_1 d\phi_2 \sin\theta_2 + d\theta_2 d\phi_1 \sin\theta_1)].
\end{aligned} \tag{46}
$$

In the MQGP limit of [26], the metric (1) matches the metric (1) and (1) with the identifications:

$$
\begin{aligned}
& \frac{e^x(\tau)}{v(\tau)} \sim \frac{\sqrt{4\pi g_s N}}{9} \left(1 + \mathcal{O}(r_h^2 e^{-\frac{2\tau}{3}}) \right); \\
& v(\tau) \sim \frac{3}{2} \left[1 + \mathcal{O}\left(\left\{ \frac{g_s M^2}{N} a^2_{\mathrm{res}}, r_h^2 \right\} e^{-\frac{2\tau}{3}} \right) \right]; \\
& e^{x(\tau)} \sim \frac{\sqrt{4\pi g_s N}}{6} \left[1 + \mathcal{O}\left(\frac{g_s M^2}{N} a^2_{\mathrm{res}} e^{-\frac{2\tau}{3}} \right) \right]; \\
& g(\tau) \sim 0; \; a(\tau) \sim -2e^{-\tau}.
\end{aligned} \tag{47}
$$

The reason for the choice of $a(\tau)$ is the following. The $\cos\psi, \sin\psi$ terms in [45] are given by $\gamma(\tau)\frac{\mu^2}{\rho^2}$, where $\rho^2 \sim e^\tau$ and, for large $\rho, \gamma(\tau) \sim \frac{(\sin h\tau \cos h\tau - \tau)^{\frac{1}{3}}}{\tanh\tau} \to \rho^{\frac{4}{3}}$. Redefining $r \sim \rho^{\frac{2}{3}}$, these terms will appear with a coefficient $r^2\left(\frac{\mu^2}{r^3}\right)$. Comparing with (1), we see: $h_5 \sim \frac{1}{r^3} \sim e^{-\tau}$. Note that (47) satisfies the identity $e^{2g(\tau)} + a^2(\tau) = 1$ [46]. Now, $v(\tau)$ of [44] is related to $p(\tau), x(\tau)$ of [43] via $e^{p(\tau)} = v(\tau)^{\frac{1}{6}} e^{-\frac{x(\tau)}{3}}$. Using this, the five $SU(3)$ structure torsion classes were worked out in [43] and are given in (48).

For the sixfold to be complex, $W_1 = W_2 = 0$. This was shown in [43] to be equivalent to the condition

$$
\frac{e^{2g(\tau)} + 1 + a^2(\tau)}{2a(\tau)} = -\cos h\tau. \tag{48}
$$

Using (47), we see that in the large-r limit subject to discussion below Sect. 2.1, (48) is approximately satisfied. Using that in the large-τ limit, $\mathcal{A} \sim 1$ and $\mathcal{B}(\tau) \sim e^{-\tau}$ (without worrying about numerical pre-factors in the various terms in the final expressions) the five torsion classes are evaluated in (153). Hence, looking at the most dominant terms in the MQGP limit of [26], from (153), we see that

$$
W_1 \to 0; \; \frac{(W_2)_{\phi_1\phi_2}, (W_2)_{\phi_1\theta_2}}{\left(W_5^{\bar{3}}\right)_{\psi_1/\phi_1/\phi_2}} \sim (4\pi g_s N)^{-\frac{1}{4}}; \tag{49}
$$

$$
\begin{aligned}
& \frac{(W_3)_{\theta_1\phi_1\tau}, (W_3)_{\phi_1\phi_2\tau}, (W_3)_{\phi_1\theta_2\tau}}{\left(W_5^{\bar{3}}\right)_{\psi_1/\phi_1/\phi_2}} \sim (4\pi g_s N)^{-\frac{1}{4}}; \\
& \frac{(W_4)_\tau}{\left(W_5^{\bar{3}}\right)_{\psi_1/\phi_1/\phi_2}} \sim (4\pi g_s N)^{-\frac{3}{4}},
\end{aligned} \tag{50}
$$

implying that in the large-τ/r limit, $\tau \in W_5$ predominantly, and hence the warped deformed conifold is Kähler. Obviously, in the strict $\tau \to \infty$ limit, one obtains a Calabi–Yau three-fold in which $W_{1,2,3,4,5} = 0$.

Switching gears, we will now show that in the large-r limit, the local T^3 of [26] satisfies the constraints of a special Lagrangian three-cycle of the deformed conifold. Using [28], the following gives the embedding equation of a $T^2(\phi_1, \phi_2)$-invariant sLag $C_3(\phi_1, \phi_2, \psi)$ in the deformed conifold T^*S^3:

$$
\begin{aligned}
& K'(r^2)\Im m(z_1\bar{z}_2) = c_1, \; K'(r^2)\Im m(z_3\bar{z}_4) = c_2, \\
& \Im m(z_1^2 + z_2^2) = c_3,
\end{aligned} \tag{51}
$$

which using the same complex structure as that for the singular conifold and $K'(r^2) \stackrel{r>>1}{\longrightarrow} r^{-\frac{2}{3}}$ and (51) yields

$$r^{\frac{7}{3}}\left(\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}+\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\right)\cos(\phi_1+\phi_2)=c_1,$$
$$r^{\frac{7}{3}}\left(\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}+\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}\right)\cos(\phi_1+\phi_2)=c_2,$$
$$r^3\sin\theta_1\sin\theta_2\cos\psi=c_3. \tag{52}$$

Equation (52) can be solved to yield (154)–(155). From (154) and (155), one sees that $dr = r_{\phi_1}d\phi_1 + r_{\phi_2}d\phi_2$, $d\theta_{1,2} = \theta^{(1),(2)}_{\phi_1}d\phi_1 + \theta^{(1),(2)}_{\phi_2}d\phi_2 + \theta^{(1),(2)}_{\psi}d\psi$. One can show from (154) and (155) that for $(\theta_1,\theta_2) = (0/\pi, 0/\pi)$, $(\psi,\phi_1,\phi_2) \approx (0/2\pi/4\pi, \frac{\pi}{5}, \frac{\pi}{4})$. Now, the Kähler form J and nowhere-vanishing holomorphic three-form Ω are, respectively, given by (156)–(157) [47]. For the $\mu \equiv$, deformation parameter $<< 1, r >> 1$ limit near $\theta_1 = \theta_2 = 0$, implying near $\psi = 0$ (and $\phi_1 = \frac{\pi}{5}, \phi_2 = \frac{\pi}{4}$), one sees that

$$J \sim r^{\frac{1}{3}}dr\wedge(d\phi_1+d\phi_2) = r^{\frac{1}{3}}(r_{\phi_1}-r_{\phi_2})d\phi_1\wedge d\phi_2,$$
$$\Omega\sim 2\imath r dr\wedge d\theta_1\wedge d\theta_2 + r^2 d\theta_1\wedge d\theta_2\wedge(d\psi+d\phi_1+d\phi_2). \tag{53}$$

One can show that near $\psi = 0/2\pi/3\pi, \phi_1 = \frac{\pi}{5}, \phi_2 = \frac{\pi}{4}$ corresponding to $\theta_{1,2} \approx 0, \pi$:

$$r_{\phi_{1,2}} \sim \mathcal{O}(10)r_\Lambda^< \text{ implying } r_{\phi_1} \sim r_{\phi_2},$$
$$\theta^{(1),(2)}_{\phi_{1,2}} \sim -\mathcal{O}(1),$$
$$\theta^{(1),(2)}_{\psi} = 0, \tag{54}$$

which, defining the embedding $i : C_3(\phi_1,\phi_2,\psi) \hookrightarrow T^*S^3$, implies

$$i^*J = 0,$$
$$\Im m\left(i^*\Omega\right) = 0, \tag{55}$$

and

$$\Re e\left(i^*\Omega\right) \sim r^2\mathcal{O}(1)d\phi_1\wedge d\phi_2\wedge d\psi. \tag{56}$$

Now $x \sim (g_sN)^{\frac{1}{4}} sin\langle\theta\rangle\phi_1, y \sim (g_sN)^{\frac{1}{4}} sin\langle\theta\rangle\phi_2, z \sim (g_sN)^{\frac{1}{4}}\psi$. In the MQGP limit of [26], we take $\theta_{1,2} \sim \epsilon^{\frac{3}{2}}$, $g_s \sim \epsilon$, $N \sim \epsilon^{-39}$. So, as $r \to r_\Lambda$, one obtains

$$\Re e\left(i^*\Omega\right) \sim \frac{r_\Lambda^2}{\epsilon^{-\frac{51}{2}}}dx\wedge dy\wedge dz, \tag{57}$$

implying that if $r_\Lambda \to \infty$ as $\epsilon^{-\frac{51}{4}}$, then

$$\Re e\left(i^*\Omega\right) \sim \text{volume form}(T^3(x,y,z)).$$

A similar argument can be given in the limits of [24].

We hence see that in the large-r limit and the limits of (4), we were justified in [26] to have used mirror symmetry à la SYZ prescription.

3 Two-point functions of D7-brane gauge field, R-charge and stress–energy tensor, and transport coefficients

In this section, our basic aim is to calculate two-point correlation functions and hence transport coefficients due to the gauge field present in type IIB background and $U(1)_R$ gauge field and stress–energy tensor modes induced in the M-theory background by using the gauge–gravity prescription originally formulated in [4]. Let us briefly discuss the strategy of calculations related to the evaluation of retarded two-point correlation functions to be used in further subsections and the basic formulae of different transport coefficients in terms of the retarded two-point Green function.

Step 1: Given that the 11-dimensional background geometrically can be represented as $AdS_5\times\mathcal{M}^6$ asymptotically, we evaluate the kinetic term corresponding to a particular field in the gravitational action by integrating out the angular direction, i.e.

$$S = \int d^6y\mathcal{P}(\theta_i,\phi_i,\psi,x_{11}) \times\int d^4x du\; Q(u)(\partial_u\phi)^2+\cdots. \tag{58}$$

Step 2: Solve the EOM corresponding to a particular field in that background by considering fluctuations around it. Further, the solution can be expressed in terms of boundary fields as

$$\phi(q,u) = f_q(u)\phi_0(q), \tag{59}$$

where $\phi(\mathbf{x},u) = \int\frac{d^4q}{(2\pi)^4}e^{-iwt+i\mathbf{q}.\mathbf{x}}\phi(q,u)$ in the momentum space and $u = \frac{r_h}{r}$. The solution can be evaluated by using boundary conditions $\phi(u,q) = \phi(0)$ at $u = 0$ and the incoming-wave boundary condition according to which $\phi(u,q) \sim e^{-iwt}$ at $u = 1$.

Step 3: Evaluate the two-point Green functions by using

$$G^R(k) = 2\,F(q,u)|^{u=1(\text{horizon})}_{u=0(\text{boundary})},$$
$$\text{where } F(u,q) = \mathcal{P}(\theta_i,\phi_i,\psi,x_{11})Q(u)f_{-q}(u)\partial_u f_q(u). \tag{60}$$

In general, the transport coefficients in hydrodynamics are defined as response to the system after applying small perturbations. The small perturbations in the presence of external source can be evaluated in terms of retarded two-point correlation functions. For example, the shear viscosity from a dual background is calculated from the stress–energy tensor involving spatial components at zero momentum, i.e.

$$\eta = \lim_{w \to 0} \frac{1}{2w} \int \mathrm{d}t \mathrm{d}\mathbf{x} e^{iwt} \theta(t) \left\langle \left[T_{xy/yz}(0), T_{xy/yz}(0) \right] \right\rangle . \tag{61}$$

By applying a small perturbation to the metric of curved space, and assuming homogeneity of space, one gets the Kubo relation to calculate the shear viscosity [8]:

$$\eta = -\lim_{w \to 0} \frac{i}{w} G^R(w, q), \tag{62}$$

where the retarded two-point Green function for the stress–energy tensor components is defined as

$$G^R_{\mu\rho,\nu\sigma}(w, \mathbf{q}) = -i \int \mathrm{d}^4 x e^{-iq.x} \theta(t) \left\langle \left[T_{\mu\rho}(x), T_{\nu\sigma}(0) \right] \right\rangle . \tag{63}$$

One can compute the above correlation function for different stress–energy tensor modes by using the steps defined above.

Similarly, when one perturbs the system by applying an external field, the response of the system to the external source coupled to current is governed by [5]

$$G^R_{\mu\nu}(w, \mathbf{q}) = -i \int \mathrm{d}^4 x e^{-i\mathbf{q}.\mathbf{x}} \theta(t) \left\langle \left[j^R_\mu(x), j^R_\nu(0) \right] \right\rangle . \tag{64}$$

In the low frequency and long distance limit, the j^0 evolves according to the overdamped diffusion equation given as

$$\partial_0 j^0 = D \nabla^2 j^0, \tag{65}$$

with dispersion relation $w = -iDq^2$, generally known as Fick's law. The diffusion coefficient 'D' can be evaluated from the pole located at $w = -iDq^2$ in the complex w-plane in the retarded two-point correlation function of j^0.

In $(d+1)$-dimensional SU(N) gauge theory, in thermal equilibrium, the differential photon emission rate per unit volume and time at leading order in the electromagnetic coupling constant e is given as [15]

$$\frac{\mathrm{d}\Gamma}{d^d q} = \frac{e^2}{(2\pi)^d 2|\mathbf{q}|} n_B(w) \sum_{s=1}^{d-1} \epsilon^\mu_{(s)}(\mathbf{q}) \epsilon^\nu_{(s)}(\mathbf{q}) \chi_{\mu\nu}(q) \Bigg|_{w=q}, \tag{66}$$

where $q = (w, \mathbf{q})$ is the photon momentum and $\chi_{\mu\nu}$ is the spectral density given as

$$\chi_{\mu\nu}(q) = -2\Im m G^R_{\mu\nu}(q), \tag{67}$$

and

$$G^R_{\mu\nu}(w, \mathbf{q}) = -i \int \mathrm{d}^4 x e^{-iq.x} \theta(t) \left\langle \left[j^{EM}_\mu(x), j^{EM}_\nu(0) \right] \right\rangle \tag{68}$$

is the retarded correlation function of two electromagnetic currents. The trace of the spectral function will be given as

$$\chi^\mu_\mu(q) = \eta^{\mu\nu} \chi_{\mu\nu}(q) = \sum_{s=1}^{d-1} \epsilon^\mu_{(s)}(\mathbf{q}) \epsilon^\nu_{(s)}(\mathbf{q}) \chi_{\mu\nu}(q) \tag{69}$$

The electrical conductivity in terms of trace of the spectral function is defined as

$$\sigma = \frac{e^2}{2(d-1)} \lim_{w \to 0} \frac{1}{w} \chi^\mu_\mu(q) \Bigg|_{w=q}. \tag{70}$$

We utilise the above expression to calculate the conductivity in the next subsection.

3.1 D7-brane gauge field fluctuations

The gauge field in the type IIB background as described in Sect. 2 appears due to the presence of coincident N_f D7-branes wrapped around a non-compact four-cycle in the resolved warped deformed conifold background. The $U(1)$ symmetry acting on the centre of the $U(N_f) =$ SU$(N_f) \times U(1)$ group living on the world volume of D7-branes wrapped around a non-compact four-cycle induces a non-zero chemical potential. The spectral functions in the presence of a non-zero chemical potential are obtained by computing two-point correlation functions of the gauge field fluctuations about the background field [48] which includes only a non-zero temporal component of the gauge field.

3.1.1 EOMs

Considering fluctuations of the gauge field around the non-zero temporal component of the gauge field, we have

$$\hat{A}_\mu(u, \vec{x}) = \delta^0_\mu A_t(u) + \tilde{\tilde{A}}_\mu(\vec{x}, u). \tag{71}$$

It is assumed that fluctuations are gauged to have non-zero components only along the Minkowski coordinates.

The action for the gauge field in the presence of flavour branes is given as

$$\mathcal{I}_{D7} = T_7 \int \mathrm{d}^8\sigma e^{-\phi(r)} \sqrt{\det(i^*g + F)}, \tag{72}$$

where 'g' corresponds to the determinant of type IIB metric as given in Eq. (1), and i^*g gives the pull back of a 10D metric onto the $D7$-brane world volume [26]. Introduce fluctuations around $A_t(u)$, $F_{\mu\nu} \to \hat{F}_{\mu\nu} = \partial_{[\mu}\hat{A}_{\nu]}$ and $\hat{A}_\mu = A_t + \tilde{\tilde{A}}_\mu$. Defining

$$G = i^*g + F,$$

the DBI action will be given by

$$\mathcal{I}_{D7} = T_7 \int \mathrm{d}^8\sigma e^{-\phi(r)} \sqrt{\det(G + \tilde{\tilde{F}})}. \tag{73}$$

Expanding the above in the fluctuations quadratic in the field strength, the DBI action as worked out in [48] can be written as

$$\mathcal{I}_{D7}^2 = T_7 \int \mathrm{d}^8\sigma\ e^{-\phi(r)} \sqrt{|\det G|} \times \left(G^{\mu\alpha}G^{\beta\gamma}\tilde{\tilde{F}}_{\alpha\beta}\tilde{\tilde{F}}_{\gamma\mu} - \frac{1}{2}G^{\mu\alpha}G^{\beta\gamma}\tilde{\tilde{F}}_{\mu\alpha}\tilde{\tilde{F}}_{\beta\gamma}\right), \tag{74}$$

and the EOM for $\tilde{A}_\mu$ is

$$\partial_\nu\left[\sqrt{|\det G|} \times \left(G^{\mu\nu}G^{\sigma\gamma} - G^{\mu\sigma}G^{\nu\gamma} - G^{[\nu\sigma]}G^{\gamma\nu}\right)\partial_{[\gamma}\tilde{\tilde{A}}_{\mu]}\right] = 0. \tag{75}$$

Using the above, the DBI action at the boundary will be given as

$$\mathcal{I}_{D7}^2 = T_7 \int \mathrm{d}^8\sigma\ e^{-\phi(r)} \sqrt{|\det G|} \times \left((G^{tu})^2 \tilde{\tilde{A}}_t \partial_u \tilde{\tilde{A}}_t - G^{rr}G^{ik}\tilde{\tilde{A}}_i\partial_u\tilde{\tilde{A}}_k - A_t G^{ut}\mathrm{tr}(G^{-1}F)\right)\Bigg|_{u=1}^{u=0}. \tag{76}$$

We have already obtained the expression of F_{rt} in [26] and this is given as

$$F_{rt} = \frac{Ce^{\phi(r)}}{\sqrt{C^2e^{2\phi(r)} + r^6}}. \tag{77}$$

Considering $u = \frac{r_h}{r}$, we have

$$F_{ut} = -\frac{Cu^2e^{\phi(u)}}{r_h\sqrt{C^2e^{2\phi(u)} + r^6}}. \tag{78}$$

Looking at the above expression, we see that at $u = 0$ (boundary), $F_{ut} \to 0$. The action in Eq. (76) will get simplified and is given as

$$\mathcal{I}_{D7}^{(2)} = T_7 \int \mathrm{d}^8\sigma\ e^{-\phi(u)} \sqrt{|detG|} \times \left(-G^{uu}G^{ii}\tilde{\tilde{A}}_i\partial_u\tilde{\tilde{A}}_i\right)\Big|_{u=1}^{u=0}, \tag{79}$$

where $i \in \mathbb{R}^{1,3}(t, x_1, x_2, x_3)$ and

$$G^{x_1x_1} = G^{x_2x_2} = G^{x_3x_3} = g^{x_1x_1} = g^{x_2x_2} = g^{x_3x_3} = \frac{u^2L^2}{r_h^2},$$
$$G^{tt} = \frac{u^2L^2}{g_1(r_h^2 - u^4F_{ut}^2)}$$
$$G^{uu} = \frac{g_1u^2r_h^2}{L^2(r_h^2 - F_{ut}^2u^4)}, \sqrt{-G} = \frac{r_h^3}{u^5}\sqrt{r_h^2 - u^4F_{ut}^2}. \tag{80}$$

Defining the gauge invariant field components

$$E_{x_1} = w\tilde{\tilde{A}}_{x_1} + q\tilde{\tilde{A}}_t, E_\alpha = w\tilde{\tilde{A}}_\alpha, \alpha = (x_2, x_3),$$

the DBI action in terms of these coordinates (using Eq. (100)) will be given as

$$\mathcal{I}_{D7}^{(2)} = T_7 \int \frac{\mathrm{d}w\mathrm{d}q}{(2\pi)^3}\frac{e^{-\phi(u)}r_h^2}{u} \times \left[\frac{E_{x_1}\partial_u E_{x_1}}{\left(q^2 - \frac{w^2}{g_1}\right)} - \frac{1}{w^2}E_{x_2}\partial_u E_{x_2})\right]\Bigg|_{u=1}^{u=0}. \tag{81}$$

Defining the longitudinal electric field as $E_{x_1}(q, u) = E_0(q)\frac{E_q(u)}{E_q(u=0)}$, the flux factor as defined in [4] in the zero-momentum limit will be given as

$$\mathcal{F}(q, u) = -\frac{e^{-\phi(u)}r_h^2}{w^2u}\frac{E_{-q}(u)\partial_u E_q(u)}{E_{-q}(u=0)E_q(u=0)}, \tag{82}$$

and the retarded Green function for E_{x_1} will be $\mathcal{G}(q, u) = -2\mathcal{F}(q, u)$. The retarded Green function for $\tilde{\tilde{A}}_{x_1}$ is w^2 times the above expression and, for $q = 0$, it gives

$$\mathcal{G}_{x_1x_!} = -2\mathcal{F}(q, u) = \frac{2e^{-\phi(u)}r_h^2}{u}\frac{\partial_u E_q(u)}{E_q(u)}\Bigg|_{u=0}. \tag{83}$$

The spectral functions in the zero-momentum limit will be given as

$$\mathcal{X}_{x_1x_1}(w, q = 0) = -2Im\mathcal{G}_{x_1x_1}(w, 0) = e^{-\phi(u)} r_h^2 Im \times \left[\frac{1}{u}\frac{\partial_u E_q(u)}{E_q(u)}\right]_{u=0}. \quad (84)$$

Since we are interested in obtaining the two-point Green function in the zero-momentum limit ($q = 0$), the transverse and longitudinal components of electric field will be simply given as $E_{x_1} = w\tilde{\tilde{A}}_{x_1}$, $E_{x_2} = w\tilde{\tilde{A}}_{x_2}$. In these coordinates, the above equations take the form

$$E''_{x_1} + \frac{\partial_u\left(\sqrt{-G}G^{uu}G^{x_1x_1}\right)}{\sqrt{-G}G^{uu}G^{x_1x_1}}E'_{x_1} - \frac{G^{tt}}{G^{uu}}w^2E_{x_1} = 0,$$
$$E''_{x_2} + \frac{\partial_u\left(\sqrt{-G}G^{uu}G^{x_2x_2}\right)}{\sqrt{-G}G^{uu}G^{x_2x_2}}E'_{x_2} - \frac{G^{tt}}{G^{uu}}w^2E_{x_2} = 0, \quad (85)$$

where the prime denotes the derivative w.r.t. u. The coordinates in the Minkowski directions are chosen such that the momentum four-vector exhibits only one spatial component i.e. $(-w, q, 0, 0)$. Writing $A_\mu(\vec{x}, u) = \int \frac{d^4q}{(2\pi)^4} e^{-iwt+iqx_1} \tilde{A}_\mu(q, u)$, incorporating the values of various metric components, setting $E(u) = E_{x_1}(u) = E_{x_2}(u)$, and $w_3 = \frac{w}{\pi T}$, and substituting the value of F_{ut} and g_1, we have

$$E''(u) + \left(-\frac{4u^3}{1-u^4} - \frac{1}{u} + \sqrt{\frac{3C^2u^5}{g_s^2r_h^6 + C^2u^6}}\right) \times E'(u) + \frac{w_3^2}{(1-u^4)^2}E(u) = 0. \quad (86)$$

From (86), one sees that $u = 1$ is a regular singular point and the roots of the indicial equation about this are given by $\pm\frac{iw_3}{4}$; choosing 'incoming-wave' solution, the solutions to (86) sought will be of the form

$$E(u) = (1-u)^{-\frac{iw_3}{4}}\mathcal{E}(u) \equiv (1-u)^{-\frac{iw_3}{4}}\mathcal{E}(u), \quad (87)$$

where $\mathcal{E}(u)$ is analytic in u. As one is interested in solving for $\mathcal{E}(u)$, analytic in u, near $u = 0$, (86) will be approximated by

$$(u-1)^2E''(u) - \frac{(u-1)^2E'(u)}{u} + \frac{w_3^2}{(1+u^2)^2(1+u)^2}E(u) \approx 0. \quad (88)$$

One converts (88) into a differential equation in $\mathcal{E}$. Performing a perturbation theory in powers of w_3, one looks for a solution of the form

$$\mathcal{E}(u) = \mathcal{E}^{(0)}(u) + w_3\mathcal{E}^{(1)}(u) + w_3^2\mathcal{E}^{(2)}(u) + \cdots \quad (89)$$

Near $u = 0$: the solutions are given by

$$\mathcal{E}^{(0)}(u \sim 0) = \frac{C_1^{(0)}}{2}u^2 + C_2^{(0)};$$
$$\mathcal{E}^{(1)}(u \sim 0) = \frac{C_1^{(0)}u^2}{2} + C_2^{(0)} + \frac{i\beta}{4}\left(\frac{C_1^{(1)}u^3}{2} - \frac{C_2^{(1)}u^2}{2}\right). \quad (90)$$

To get a non-zero conductivity, we will require $C_1^{(0)} \in \mathbb{C}$.

3.1.2 *Electrical conductivity, charge susceptibility and Einstein relation*

The conductivity, using (84), (90) and $T = \frac{r_h}{\pi\sqrt{4\pi g_s N}}$ [26], will be given as

$$\sigma = \lim_{w\to 0}\frac{\mathcal{X}_{x_1x_1}(w, q = 0)}{w} = \frac{r_h^2 C_1^{(0)}}{g_s\pi T} = \frac{\pi(4\pi g_s N)T}{g_s}. \quad (91)$$

Another physically relevant quantity is the charge susceptibility, which is thermodynamically defined as the response of the charge density to the change in the chemical potential [49]. We have

$$\chi = \left.\frac{\partial n_q}{\partial \mu}\right|_T, \quad (92)$$

where $n_q = \frac{\delta S_{\rm DBI}}{\delta F_{rt}}$, and the chemical potential is defined as $\mu = \int_{r_h}^{r_B} F_{rt} dr$. Using this, the charge susceptibility will be given as

$$\chi = \left(\int_{r_h}^{r_B}\frac{dF_{rt}}{dn_q}\right)^{-1}. \quad (93)$$

From [26],

$$n_q \sim \frac{r^3F_{rt}\left(\frac{1}{g_s} - \frac{N_f \ln\mu}{2\pi}\right)}{\sqrt{1-F_{rt}^2}}, \quad \mu \equiv \text{embedding parameter}, \quad (94)$$

which using (77) implies

$$\frac{1}{\chi} \sim \frac{1}{\left(\frac{1}{g_s} - \frac{N_f \ln\mu}{2\pi}\right)} \int_{r_h}^{\infty} \frac{\mathrm{d}r}{r^3} \left(1 - F_{rt}^2\right)^{\frac{3}{2}} = \frac{1}{6 {r_h}^{2\frac{(g_s N_f \log(\mu)-1)}{C g_s}} \left(C^2 {g_s}^2 + {g_s}^2 {N_f}^2 {r_h}^6 \log^2(\mu) - 2 g_s N_f {r_h}^6 \log(\mu) + {r_h}^6\right)}$$
$$\times \Bigg[\left(C^2 {g_s}^2 - 2{g_s}^2 {N_f}^2 {r_h}^6 \log^2(\mu) + 4 g_s N_f {r_h}^6 \log(\mu) - 2{r_h}^6\right) {}_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{4}{3}; -\frac{C^2 {g_s}^2}{{r_h}^6 (g_s N_f \log(\mu) - 1)^2}\right)$$
$$+5{r_h}^6 (g_s N_f \log(\mu) - 1)^2 {}_2F_1\left(-\frac{1}{2}, \frac{1}{3}; \frac{4}{3}; -\frac{C^2 {g_s}^2}{{r_h}^6 (g_s N_f \log(\mu) - 1)^2}\right)\Bigg]. \quad (95)$$

For $\mu = 1 - \epsilon, \epsilon \to 0^+, C << 1$ [26], in the MQGP limit [26] (and also the weak (g_s) coupling-strong 't Hooft coupling limit [24]), one sees $\chi \sim \frac{r_h^2}{g_s} \sim \pi(4\pi N)T^2$.

The diffusion coefficient corresponding to non-zero charge density appear by demanding the longitudinal component of electric field strength to be zero i.e. $E_{x_2} = 0$. As given in [49], the condition provides us with the expression of the diffusion coefficient,

$$D = e^{-\phi}\sqrt{G G^{00} G^{uu}}\, G^{ii}\Big|_{u=1} \int_{u=1}^{u=0} \frac{\mathrm{d}u}{e^{-\phi}\sqrt{G G^{00} G^{uu}}}. \quad (96)$$

Substituting the values for the above metric components as given in Eq. (80), after integration, we get $D = \frac{L^2}{r_h} + \mathcal{O}(C^2 g_s^2/r_h^8) \sim \frac{1}{T}$. Using Eqs. (92), (95) and (96), we get $\frac{\sigma}{\chi} \sim \frac{1}{T} \sim D$, hence verifying the Einstein relation.

3.2 R-charge correlators

The $U(1)_R$-charges are defined in the bulk gravitational background dual to the rank of the isometry group corresponding to the spherical directions transverse to the AdS space. Given that the 11-dimensional M-theory background corresponds to black $M3$-branes which asymptotically can be expressed as $M5$-branes wrapped around two-cycles defined homologously as an integer sum of two-spheres (as described in Sect. 2), there will be a rotational (R)-symmetry group dual to the isometry group $U(1) \times U(1)$ corresponding to $\phi_{1/2}$ and ψ in the directions transverse to $M5$-branes wrapped around $S^2(\theta_1, \phi_{(2/1)}) + S^2(\theta_2, x_{10})$. To determine the diffusion coefficient due to the R-charge, one needs to evaluate the two-point correlation function of $\tilde{A}_\mu$, which basically will be a metric perturbation of the form $h_{M\mu}$ where M is a spherical direction and μ is an asymptotically AdS direction. As a first step, the $\tilde{A}_\mu$ EOM is

$$\partial_\beta \left[g^{\mu\nu} g^{\alpha\beta} \sqrt{g} \tilde{F}_{\nu\alpha}\right] = 0. \quad (97)$$

By defining $u = \frac{r_h}{r}$ so that $g_1 = g_2 = 1 - u^4$, the black $M3$-brane metric of [26] reduces to the form

$$\mathrm{d}s^2 = -\frac{g_1 {g_s}^{-\frac{2}{3}} r_h^2}{u^2 L^2}\mathrm{d}t^2 + \frac{{g_s}^{-\frac{2}{3}} r_h^2}{u^2 L^2}\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right)$$
$$+\frac{{g_s}^{-\frac{2}{3}} L^2}{u^2 g_1}\mathrm{d}u^2 + d\mathcal{M}_6^2, \quad (98)$$

where $d\mathcal{M}_6^2 = d\mathcal{M}_5^2 + g_s^{\frac{4}{3}}\left(d_{x_{11}}^2 + A^{F_1} + A^{F_2} + A^{F_5}\right)^2$, and in the weak (g_s) coupling-strong 't Hooft coupling/ MQGP limit, the non-zero components include

$$\begin{aligned} \mathrm{d}\mathcal{M}_5^2 = {} & G^{\mathcal{M}}_{\theta_1\theta_1}\mathrm{d}\theta_1^2 + G^{\mathcal{M}}_{\theta_1\theta_2}\mathrm{d}\theta_1\mathrm{d}\theta_2 \\ & + G^{\mathcal{M}}_{\phi_1\theta_1}\mathrm{d}\phi_1\mathrm{d}\theta_1 + G^{\mathcal{M}}_{\phi_2\theta_1}\mathrm{d}\phi_2\mathrm{d}\theta_1 + G^{\mathcal{M}}_{z\theta_1}\mathrm{d}\psi\mathrm{d}\theta_1 \\ & + G^{\mathcal{M}}_{10\theta_1}\mathrm{d}x_{10}\mathrm{d}\theta_1 + G^{\mathcal{M}}_{\theta_2\theta_2}\mathrm{d}\theta_2^2 + G^{\mathcal{M}}_{\phi_1\theta_2}\mathrm{d}\phi_1\mathrm{d}\theta_2, \\ & G^{\mathcal{M}}_{\phi_2\theta_2}\mathrm{d}\phi_2\mathrm{d}\theta_2 + G^{\mathcal{M}}_{\psi\theta_2}\mathrm{d}yz\mathrm{d}\theta_2 + G^{\mathcal{M}}_{10\theta_2}\mathrm{d}x_{10}\mathrm{d}\theta_2 \\ & + G^{\mathcal{M}}_{\phi_1\psi_1}\mathrm{d}\phi_1^2 + G^{\mathcal{M}}_{\phi_1\phi_2}\mathrm{d}\phi_1\mathrm{d}\phi_2 + G^{\mathcal{M}}_{\phi_1\psi}\mathrm{d}\phi_1\mathrm{d}\psi \\ & + G^{\mathcal{M}}_{yy}\mathrm{d}\phi_2^2 + G^{\mathcal{M}}_{\phi_2\psi}\mathrm{d}\phi_2\mathrm{d}\psi \\ & + G^{\mathcal{M}}_{\psi\psi}\mathrm{d}\psi^2 + G^{\mathcal{M}}_{10\ 10}\mathrm{d}x_{10}^2. \end{aligned} \quad (99)$$

The simplified expressions of the aforementioned metric components are given in [26].

Writing $\tilde{A}_\mu(x_1) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} e^{-i\omega t + iqx_1}\tilde{A}_\mu(q, u)$ and working in the $A_u = 0$ gauge, by setting $\mu = u, x_1, t, \alpha(\in \mathbb{R}^3)$ in (97) one ends up with the following equations:

$$\begin{aligned} & w_3 \tilde{A}_t' + g_1 q_3 \tilde{A}_{x_1}' = 0, \\ & \tilde{A}_{x_1}'' - \frac{1}{u}\tilde{A}_{x_1}' + \frac{g_1'}{g_1}\tilde{A}_{x_1}' - \frac{1}{g_1^2}\left(w_3 q_3 \tilde{A}_t + w_3^2 \tilde{A}_{x_1}\right) = 0, \\ & \tilde{A}_t'' - \frac{1}{u}\tilde{A}_t' - \frac{1}{g_1}\left(w_3 q_3 \tilde{A}_t + q_3^2 \tilde{A}_{x_1}\right) = 0, \\ & \tilde{A}_\alpha'' - \frac{1}{u}\tilde{A}_\alpha' + \frac{g_1'}{g_1}\tilde{A}_\alpha' + \frac{1}{g_1^2}\left(w_3^2 - g_1 q_3^2\right)\tilde{A}_\alpha = 0, \end{aligned} \quad (100)$$

where $\alpha = (x_2, x_3)$. The simplest to tackle is the last equation as it is decoupled from the previous three equations. One notices that the horizon $u = 1$ is a regular singular point and the exponents of the indicial equation about this are given by $\pm\frac{i w_3}{4}$; one chooses the incoming-wave solution and hence writes $\tilde{A}_\alpha(u) = (1-u)^{-\frac{iw_3}{4}}\tilde{\mathcal{A}}_\alpha(u)$. The last equation in (100) then is rewritten as

$$(1-u)^2\tilde{\mathcal{A}}''_\alpha(u) + \left(\frac{iw_3}{2}(1-u) - (1-u)\right.$$
$$\left.\times\left[\frac{4u^3}{(u+1)(u^2+1)} - \frac{(u-1)}{u}\right]\right)\tilde{\mathcal{A}}'_\alpha(u)$$
$$+\left(\frac{iw_3}{4}\left[\frac{iw_3}{4}+1\right] + \frac{\left[w_3^2+(u^4-1)q_3^2\right]}{(u+1)^2(u^2+1)^2}\right)\tilde{\mathcal{A}}_\alpha(u) = 0. \tag{101}$$

We make the following double perturbative ansatz for the solution to $\tilde{\mathcal{A}}_\alpha(u)$'s EOM (101):

$$\tilde{\mathcal{A}}_\alpha(u) = \tilde{\mathcal{A}}^{(0,0)}_\alpha(u) + q_3\tilde{\mathcal{A}}^{(1,0)}_\alpha(u) + w_3\tilde{\mathcal{A}}^{(0,1)}_\alpha(u)$$
$$+q_3w_3\tilde{\mathcal{A}}_\alpha(u)^{(1,1)} + q_3^2\tilde{\mathcal{A}}^{(2,0)}_\alpha(u) + \cdots. \tag{102}$$

Plugging (102) into (101) yields as solutions, near $u=0$ up to $\mathcal{O}(u)$, (D1). In the $w\to 0, q\to 0$ limit, one can take $\tilde{\mathcal{A}}_\alpha(0)\sim c_2$. Also,

$$(1-u)^{-\frac{iw_3}{4}}\tilde{\mathcal{A}}^{(0,0)\,\prime}_\alpha(u) = -c_1u + \mathcal{O}(u^2);$$
$$(1-u)^{-\frac{iw_3}{4}}\tilde{\mathcal{A}}^{(0,1)\,\prime}_\alpha(u) = -\frac{ic_2}{4}$$
$$+\frac{1}{16}(5\pi c_1 + w3c_2 - 16c_3)\,u + O\left(u^2\right)$$
$$(1-u)^{-\frac{iw-3}{4}}\tilde{\mathcal{A}}^{(2,0)}_\alpha(u) = (-c_3 + c_2\ln u)\,u + \mathcal{O}(u^2). \tag{103}$$

The kinetic terms relevant to the evaluation of two-point correlators of $\tilde{A}_{u,t,\alpha}$ in the MQGP limit are

$$S = g_s^{-\frac{4}{3}}r_h^2L^2\frac{1}{\kappa_{11}^2\epsilon^{\frac{9}{2}}}\int d^4x du\frac{1}{u}$$
$$\times\left[(1-u^4)\left(\tilde{A}'_\alpha\right)^2 + (1-u^4)\left(\tilde{A}'_x\right)^2 - \left(\tilde{A}'_t\right)^2\right]. \tag{104}$$

Hence, the retarded Green function $G^R_{\alpha\alpha}$ will be given as

We now go to the $\tilde{A}_{x_1,t}$ EOMs and observe that one can decouple $\tilde{A}_{x_1}$ and $\tilde{A}_t$, and obtain, e.g., the following third order differential equation for $\tilde{A}_{x_1}$:

$$(1-u^4)^2\tilde{A}'''_{x_1}(u) - (1-u^4)\{(1-u^4)+12u^3\}\tilde{A}''_{x_1}(u)$$
$$-\left[4u^2(1-u^4) - 16u^6 - q_3^2(1-u^4) + w_3^2\right]\tilde{A}'_{x_1} = 0. \tag{106}$$

One notes that $u=1$ is a regular singular point of the second order differential equation (106) in $\tilde{A}'_{x_1}$. The exponents of the indicial equation are $-1\pm\frac{iw_3}{4}$; choosing the incoming-boundary-condition solution, we write $\tilde{A}'_{x_1} = (1-u)^{-1-\frac{iw_3}{4}}\tilde{\mathcal{A}}'_{x_1}$ and assume a double perturbative series for the solution to the second order differential equation satisfied by $\tilde{\mathcal{A}}'_{x_1}$ of the form

$$\tilde{\mathcal{A}}'_{x_1}(u) = \tilde{\mathcal{A}}^{(0,0)\,\prime}_{x_1}(u) + w_3\tilde{\mathcal{A}}^{(0,1)\,\prime}_{x_1}(u) + q_3^2\tilde{\mathcal{A}}^{(2,0)\,\prime}_{x_1}(u)$$
$$+\mathcal{O}(w_3^2, w_3q_3^2). \tag{107}$$

The equations that one obtains at various orders of (107), if to be solved exactly, are intractable. We will be content with their solutions near $u=0$. They are given in (D3) and are of the form

$$\tilde{\mathcal{A}}^{(0,0)\,\prime}_{x_1} \equiv a + bu + cu^2 + \mathcal{O}(u^2).$$

The solutions of $\tilde{\mathcal{A}}^{(2,0)\,\prime}_{x_1}$ and $\tilde{\mathcal{A}}^{(0,1)\,\prime}_{x_1}$ require more work. Going even up to $\mathcal{O}(u^2)$ for them is intractable. Given that we would need for the purpose of evaluation of the retarded Green function $G^R_{x_1x_1}$ solutions only up to $\mathcal{O}(u\to 0)$, we give below solutions up to $\mathcal{O}(u)$.

One can show that $\tilde{\mathcal{A}}^{(0,1)\,\prime}_{x_1}(u)$, up to $\mathcal{O}(u)$, will be given by the expansion of (D2) up to $\mathcal{O}(u)$. After some MATHEMATICAlgebra, one can show:

$$G^{R\ \text{finite}}_{\alpha\alpha} \sim \frac{1}{u}\frac{\tilde{A}_{\alpha,-q}(u)}{\tilde{A}_{\alpha,-q}(0)}\partial_u\left(\frac{\tilde{A}_{\alpha,q}(u)}{\tilde{A}_{\alpha,q}(0)}\right)\Bigg|_{u=0}$$
$$\sim T\left[iw + \frac{q^2}{\pi T}\left(\frac{\frac{1}{24}\left[-3\left\{8c_2c_3 + c_1\left[8c_4 + c_1(2Li_3(-i) + 2Li_3(i) + 3\zeta(3))\right]\right\} + i\pi^3c_1^2 - \pi^2c_2c_1 - 6i\pi c_3c_1\right]}{-\frac{1}{64}\pi^2c_1^2 + \frac{1}{8}\pi(-ic_2 + 2c_3)c_1 - ic_4c_1 - \frac{c_2^2}{4} + ic_2c_3}\right)\right]. \tag{105}$$

On comparison with $G^R_{\alpha\alpha}\sim iw + 2D_Rq^2$ [20], one sees that there is a three-parameter family (which in the $c_2 >> c_{1,3,4}$ limit are $c_{1,3,4}$) of solutions to the R-charge fluctuation $\tilde{A}_\alpha$, which would generate $D^\alpha_R\sim\frac{1}{\pi T}$.[2]

[2] In $c_2 >> c_{1,3,4}$ limit, $D^\alpha_R = \frac{24c_3+\pi^2c_1}{12\pi\tilde{A}^0_\alpha}$

$$\tilde{\mathcal{A}}^{01}_{x_1}(u) \approx (7.4-1.8i)c_1 + (25.5-45.4i)c_2$$
$$+[(-4.8+5i)c_1 - (55.7-108.3i)c_2]u + \mathcal{O}(u^2). \tag{108}$$

One can similarly show that $\tilde{\mathcal{A}}^{(2,0)\,\prime}_{x_1}(u)$, up to $\mathcal{O}(u)$, will be given by the expansion, up to $\mathcal{O}(u)$, of (D4). Again after some MATHEMATICAlgebra, one hence obtains

$$\tilde{\mathcal{A}}^{(2,0)}_{x_1}(u) \approx (7.4-1.8i)c_1 + (40.2-5.4i)c_2 + [(-4.8+5i)c_1 - (12.1-26.7i)c_2]u + \mathcal{O}(u^2). \tag{109}$$

Now, writing

$$\begin{pmatrix} \tilde{\mathcal{A}}^{(0,0)\,\prime}_{x_1}(u) \\ \tilde{\mathcal{A}}^{(0,1)\,\prime}_{x_1}(u) \\ \tilde{\mathcal{A}}^{(2,0)\,\prime}_{x_1}(u) \end{pmatrix} = \begin{pmatrix} a(c_2) & b(c_1,c_2) \\ \tilde{a}(c_1,c_2) & \tilde{B}(\tau)(c_1,c_2) \\ \tilde{\tilde{a}}(c_1,c_2) & \tilde{\tilde{B}}(\tau)(c_1,c_2) \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix}, \tag{110}$$

one obtains

$$\begin{aligned} \tilde{A}'_{x_1} &= (1-u)^{-1-\frac{iw_3}{4}}\Big[a(c_2) + w_3\tilde{a}(c_1,c_2) + q_3^2\tilde{\tilde{a}}(c_1,c_2) \\ &\quad + \Big(b(c_1,c_2) + w_3\tilde{B}(\tau)(c_1,c_2) \\ &\quad + q_3^2\tilde{\tilde{B}}(\tau)(c_1,c_2)\Big)u + \mathcal{O}(u^2)\Big], \\ \tilde{A}''_{x_1}(u) &= \left(1+\frac{iw_3}{4}\right)(1-u)^{-2-\frac{iw_3}{4}} \\ &\quad \times\Big[a(c_2) + w_3\tilde{a}(c_1,c_2) + q_3^2\tilde{\tilde{a}} + \Big(b(c_1,c_2) \\ &\quad + w_3\tilde{B}(\tau)(c_1,c_2) + q_3^2\tilde{\tilde{B}}(\tau)(c_1,c_2)\Big)u + \mathcal{O}(u^2)\Big] \\ &\quad + (1-u)^{-1-\frac{iw_3}{4}}\Big[\Big(b(c_1,c_2) + w_3\tilde{B}(\tau)(c_1,c_2) \\ &\quad + q_3^2\tilde{\tilde{B}}(\tau)(c_1,c_2)\Big) + \mathcal{O}(u)\Big]. \end{aligned} \tag{111}$$

As in the differential equation

$$\tilde{A}''_{x_1}(u) - \left(\frac{4u^3}{1-u^4} + \frac{1}{u}\right)\tilde{A}'_{x_1}(u) - \frac{w_3q_3\tilde{A}_t + w_3^2\tilde{A}_{x_1}}{(1-u^4)} = 0, \tag{112}$$

we require $\tilde{A}'_{x_1}(u)$ to be well defined at $u=0$, one has to impose the following constraint on c_1, c_2:

$$a(c_2) + w_3\tilde{a}(c_1,c_2) + q_3^2\tilde{\tilde{a}}(c_1,c_2) = 0. \tag{113}$$

We have

$$\begin{pmatrix} a \\ \tilde{a} \\ \tilde{\tilde{a}} \end{pmatrix} \approx \begin{pmatrix} 0 & \alpha \\ a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{114}$$

and one sees that

$$c_1 = c_2\left(\frac{-\alpha - w_3b_1 - q_3^2b_2}{w_3a_1 + q_3^2a_2}\right). \tag{115}$$

Substituting (111) into (112) evaluated at $u=0$, one obtains

$$w_3q_3\tilde{A}^0_t + w_3^2\tilde{A}^0_{x_1} = 0. \tag{116}$$

As

$$\begin{pmatrix} b \\ \tilde{B}(\tau) \\ \tilde{\tilde{B}}(\tau) \end{pmatrix} \approx \begin{pmatrix} 0 & \beta \\ p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{117}$$

one realises that

$$\tilde{A}'_{x_1}(u) = (1-u)^{-1-\frac{iw_3}{4}}c_2u \times \frac{\left[w_3(a_1\beta - p_1\alpha) + q_3^2(a_2\beta - p_2\alpha) + \mathcal{O}(w_3^2, w_3q_3^2, u^2)\right]}{(a_1w_3 + a_2q_3^2)}. \tag{118}$$

This yields

$$\tilde{A}_{x_1}(u) = (1-u)^{-\frac{iw_3}{4}}c_2 \times \frac{\left[w_3(a_1\beta - p_1\alpha) + q_3^2(a_2\beta - p_2\alpha) + \mathcal{O}(w_3^2, w_3q_3^2, u^2)\right]}{(a_1w_3 + a_2q_3^2)} \times \frac{4 - iw_3u}{w_3(4i + w_3)} + c_3. \tag{119}$$

Thus, the retarded Green function G^R_{xx} will be given by

$$G^R_{x_1x_1} \sim \lim_{u\to 0}\frac{1}{u}\tilde{A}_{x_1,q}\partial_u\tilde{A}_{x_1,-q}(u). \tag{120}$$

The constants of integration c_2, c_3 must satisfy

$$c_2\frac{\left[w_3(a_1\beta - p_1\alpha) + q_3^2(a_2\beta - p_2\alpha)\right]}{(a_1w_3 + a_2q_3^2)} \times \frac{4}{w_3(4i + w_3)} + c_3 = \tilde{A}^0_{x_1}. \tag{121}$$

Choose

$$c_2\frac{\left[w_3(a_1\beta - p_1\alpha) + q_3^2(a_2\beta - p_2\alpha)\right]}{w_3(4i + w_3)} \sim \mathcal{O}(w_3), \tag{122}$$

which can be fine tuned to ensure $c_3 \approx \tilde{A}^0_{x_1}$. Thus, the retarded Green function $G^R_{x_1x_1}$ will be given by

$$\begin{aligned} G^R_{x_1x_1} &\sim \lim_{u\to 0}\frac{1}{u}\left(\frac{\tilde{A}_{x_1,q}(u)}{\tilde{A}_{x_1,q}(u=0)}\right)\partial_u\left(\frac{\tilde{A}_{x_1,-q}(u)}{\tilde{A}_{x_1,-q}(u=0)}\right) \\ &\sim T\left(\frac{w^2}{iw - \frac{ia_2}{\pi Ta_1}q^2}\right). \end{aligned} \tag{123}$$

Now $a_1 = a_2$ and if one were to consider the $\mathcal{O}(w_3q_3^0)$ term to be $iw_3\tilde{A}^{(0,1)}_{x_1}(u)$, then upon comparison with $G^R_{x_1x_1} \sim \frac{w^2}{iw - D_Rq^2}$ [20], one sees that $D^{x_1}_R = \frac{1}{\pi T}$. By requiring

$D_R^{x_1} = D_R^\alpha$, one sees that in fact that there is a two-parameter family of solutions that would generate $D_R = \frac{1}{\pi T}$ as, in the $c_2 >> c_{1,3,4}$ limit as an example, one generates the constraint $\frac{24c_3+\pi^2 c_1}{12A_\alpha^0} = 1$.

3.3 Stress–energy tensor modes

The two-point function of stress–energy tensors are obtained by considering small perturbations of the five-dimensional metric, $g_{\mu\nu} = g_{\mu\nu(0)} + h_{\mu\nu}$. Up to first order in the metric perturbation (dropping G_4 flux contributions) the Einstein equation is given in [20],

$$\mathcal{R}^{(1)}_{\mu\nu} = \frac{2}{d-2}\Lambda h_{\mu\nu}, \tag{124}$$

where d is the dimension of AdS space. To linear order in $h_{\mu\nu}$ the Ricci scalar will be given by [50]

$$\mathcal{R}^{(1)}_{\mu\nu} = \Gamma^\alpha_{(1)\mu\nu,\alpha} - \Gamma^\alpha_{(1)\mu\alpha,\nu} + \Gamma^\beta_{(0)\mu\nu}\Gamma^\alpha_{(1)\alpha\beta} + \Gamma^\beta_{(1)\mu\nu}\Gamma^\alpha_{(0)\alpha\beta} - \Gamma^\beta_{(0)\mu\alpha}\Gamma^\alpha_{(1)\nu\beta} - \Gamma^\beta_{(1)\mu\alpha}\Gamma^\alpha_{(0)\nu\beta}, \tag{125}$$

where

$$\Gamma^{(1)\alpha}_{\mu\nu} = -g^{\alpha\beta}_{(0)} h_{\beta\gamma}\Gamma^\gamma_{\mu\nu} + \frac{1}{2} g^{\alpha\beta}_{(0)}\left(h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}\right). \tag{126}$$

The five-dimensional metric in M-theory background in the limits of (4) is as follows:

$$ds^2 = -\frac{g_1 g_s^{-\frac{2}{3}} r_h^2}{u^2 L^2} dt^2 + \frac{g_s^{-\frac{2}{3}} r_h^2}{u^2 L^2}\left(dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{g_s^{-\frac{2}{3}} L^2}{u^2 g_1} du^2. \tag{127}$$

We assume the perturbation of the metric of $M3$-branes to be dependent on x_1 and t only i.e. after Fourier decomposing this, we have $h_{\mu\nu}(\vec{x}, t) = \int \frac{d^4 q}{(2\pi)^4} e^{-iwt+iqx_1} h_{\mu\nu}(q, w)$ and choose the gauge where $h_{\mu u} = 0$. In the case of $M3$-branes, there will be rotation group $SO(2)$ acting on the directions transverse to u, t, and x_1. Based on the spin of different metric perturbations under this group, it can be classified into groups as follows:

(i) Vector modes: $h_{x_1x_2}, h_{tx_2} \neq 0$ or $h_{x_1x_3}, h_{tx_3} \neq 0$, with all other $h_{\mu\nu} = 0$.
(ii) Scalar modes: $h_{x_1x_1} = h_{x_2x_2} = h_{x_3x_3} = h_{tt} \neq 0$, $h_{x_1t} \neq 0$, with all other $h_{\mu\nu} = 0$.
(iii) Tensor modes: $h_{x_2x_3} \neq 0$, with all other $h_{\mu\nu} = 0$.

We are interested in calculating the shear viscosity in the context of the $M3$-brane by obtaining correlator functions corresponding to the vector and tensor modes.

3.3.1 Metric vector mode fluctuations

The vector mode fluctuations will be given by considering non-zero h_{tx_2} and $h_{x_1x_2}$ components with all other $h_{\mu\nu} = 0$ [20]. Since the aforementioned metric is conformally flat near $u = 0$, one can make a Fourier decomposition at large r such that

$$h_t^{x_2} = e^{-iwt+iqx_1} H_t(u), \quad h_{x_1}^{x_2} = e^{-iwt+iqx_1} H_{x_1}(u). \tag{128}$$

Using Eqs. (125) and (126), we get the following linearised Einstein equation for H_t and H_{x_1}:

$$\begin{aligned} & w_3 H_t' + g_1 q_3 H_{x_1}' = 0,\\ & H_{x_1}'' - \frac{(4-\frac{5}{2}g_1)}{u g_1} H_{x_1}' + \frac{1}{g_1^2}\left(w_3 q_3 H_t + w_3^2 H_{x_1}\right) + \left(\frac{8}{u^2 g_1} - \mathcal{O}(1)\frac{g_s^{\frac{2}{3}} L^2 \Lambda}{u^2 g_1}\right) H_{x_1} = 0,\\ & H_t'' - \frac{3}{u} H_t' - \frac{1}{g_1}\left(w_3 q_3 H_{x_1} + q_3^2 H_t\right) + \left(\frac{8}{u^2 g_1} - \mathcal{O}(1)\frac{g_s^{\frac{2}{3}} L^2 \Lambda}{u^2 g_1}\right) H_t = 0, \end{aligned} \tag{129}$$

where Λ is the cosmological constant arising from $|G_4|^2$ and higher order corrections ($\mathcal{O}(R^4)$). It is shown in [26] that the higher order corrections are very subdominant as compared to the flux term in both limits.

The dominant flux term as calculated in [26] is given by

$$\frac{|G_4^2|}{\sqrt{G^M}} \sim H^2_{\theta_1\theta_2\phi_2} G^{\mathcal{M}\ \theta_1\theta_1} G^{\mathcal{M}\ \theta_2\theta_2} G^{\mathcal{M}\ \phi_2\phi_2} G^{\mathcal{M}\ 10\ 10}, \tag{130}$$

and the simplified components are given as

$$\begin{aligned} G^{\mathcal{M}\ \theta_1\theta_1} &\sim \frac{\sqrt[3]{3}\sqrt[3]{\frac{1}{g_s}}\sqrt{g_s N}}{\sqrt{\pi} N},\\ G^{\mathcal{M}\ \theta_2\theta_2} &\sim \frac{27\sqrt[3]{3}\sqrt{g_s N}\tan^2(\theta_1)\csc^2(\theta_2)}{2\sqrt{\pi}\sqrt[3]{g_s} N f_2(\theta_2)^2},\\ G^{\mathcal{M}\ \phi_2\phi_2} &\sim \frac{6912\sqrt[3]{3}\sin^7(\theta_1)\cos(\theta_1)\cot^3(\theta_2)\csc^4(\theta_2)}{\sqrt{\pi}\left(\frac{1}{g_s}\right)^{2/3}\sqrt{g_s N}(\cos(2\theta_1)-5)^3},\\ G^{\mathcal{M}\ 10\ 10} &\sim 3\sqrt[3]{3}\left(\frac{1}{g_s}\right)^{4/3}, \end{aligned}$$

$$H_{\theta_1\theta_2\phi_2} \sim \frac{\sqrt{\pi} f_2(\theta_2)\sqrt{g_s N}\sin(\theta_1)\cos(\theta_1)\sin^3(\theta_2)\sin(2\theta_2)\cos(\theta_2)}{3\left(\sin^2(\theta_1)\cos^2(\theta_2)+\cos^2(\theta_1)\sin^2(\theta_2)\right)^2}. \tag{131}$$

Incorporating the aforementioned expressions in (130) and assuming that the integrand receives a dominant contribution along $\theta_{1,2} = 0, \pi$, we introduce a cutoff $\theta_{1,2} \sim \epsilon^{\frac{3}{2}}$ in the MQGP limit of [26]. Utilising this and integrating over the rest of the 11-dimensional components, we have

$$\Lambda = \int du d\mathcal{M}_6 \left|\frac{G_4^2}{\sqrt{G^M}}\right|^2 \sim c_1 \frac{2g_s^{\frac{2}{3}}\epsilon^3}{3L^2}, \tag{132}$$

where c_1 is constant. Analogous to a partial cancellation in [20] of the coefficient of H_x originating from $R^{(1)}_{\mu\nu}$ and the cosmological constant contribution from the $\Lambda h_{\mu\nu}$ term in (124), assuming that $\frac{2c_1\epsilon}{3} = 8$ in the weak (g_s) coupling-string 't Hooft limit and $\frac{2\tilde{c}_1\epsilon^3}{3} = 8$ in the MQGP limit after incorporating the value of Λ in (129), we assume that the $\frac{1}{u^2 g_1}$-term will be cancelled out with a term appearing in the coefficient of $H_{x_1,t}$ in $H_{x_1}(x_1,t)$'s EOM (129). Hence the linearised set of the EOM will be given as

$$\begin{aligned}
&w_3 H_t' + g_1 q_3 H_{x_1}' = 0,\\
&H_{x_1}'' - \frac{(4-\frac{5}{2}g_1)}{u g_1}H_{x_1}' + \frac{1}{g_1^2}\left(w_3 q_3 H_t + w_3^2 H_{x_1}\right) = 0,\\
&H_t'' - \frac{3}{u}H_t' - \frac{1}{g_1}\left(w_3 q_3 H_{x_1} + q_3^2 H_t\right) = 0.
\end{aligned} \tag{133}$$

To obtain a two-point function at the asymptotic boundary, one needs to determine the kinetic term for the vector modes H_{x_1} and H_t. They can be calculated using the Einstein–Hilbert action up to quadratic order in $h_{\mu\nu}$ given as [50]

$$\begin{aligned}
S = &\frac{1}{2K_{11}^2}\int \sqrt{g}g^{\mu\nu}\left(\Gamma^{(1)\alpha}_{\mu\beta}\Gamma^{(1)\beta}_{\mu\nu} - \Gamma^{(1)\alpha}_{\mu\nu}\Gamma^{(1)\beta}_{\alpha\beta}\right) + \frac{1}{2\kappa_{11}^2}\\
&\times\int\sqrt{g}g^{\mu\nu}\left(\Gamma^{(2)\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{(2)\beta}_{\nu\alpha} - \Gamma^{(2)\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}\right.\\
&\left. - \Gamma^{\alpha}_{\mu\nu}\Gamma^{(2)\beta}_{\alpha\beta}\right) + \frac{1}{16\kappa_{11}^2}\int\\
&\times\sqrt{g}\left[8h^{\mu}_{\sigma}h^{\sigma\nu} - 4hh^{\mu\nu} + \left(h^2 - 2h_{\sigma\tau}h^{\sigma\tau}\right)g^{\mu\nu}\right]\\
&\times\left(\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}\right) + \frac{1}{4\kappa_{11}^2}\int\sqrt{g}\left(g^{\mu\nu}h - 2h^{\mu\nu}\right)\\
&\times\left(\Gamma^{(1)\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{(1)\beta}_{\nu\alpha} - \Gamma^{(1)\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{(1)\beta}_{\alpha\beta}\right),
\end{aligned} \tag{134}$$

where

$$\begin{aligned}
\Gamma^{(1)\alpha}_{\mu\nu} &= \frac{1}{2}g^{\alpha\beta}\left(\nabla_\nu h_{\beta\mu} + \nabla_\mu h_{\beta\nu} - \nabla_\beta h_{\mu\nu}\right)\\
\Gamma^{(2)\alpha}_{\mu\nu} &= -g^{\alpha\beta}g^{\gamma\delta}h_{\beta\delta}\left(\nabla_\mu h_{\gamma\nu} + \nabla_\nu h_{\mu\gamma} - \nabla_\gamma h_{\mu\nu}\right).
\end{aligned} \tag{135}$$

Only the first two terms in the action will be relevant to get the kinetic term for the vector modes. Solving them, we get

$$\begin{aligned}
S = &\frac{1}{8\kappa_{11}^2}\int d^{11}x\sqrt{G^{\mathcal{M}}}G^{\mathcal{M}uu}G^{\mathcal{M}}_{x_2x_2}\\
&\times\left[-G^{\mathcal{M}x_1x_1}\left(H_{x_1}'\right)^2 - G^{\mathcal{M}tt}\left(H_t'\right)^2 + \cdots\right].
\end{aligned} \tag{136}$$

The very simplified form of the 11-dimensional metric in the $\theta_i \to 0$ limit will be given by

$$\sqrt{G^{\mathcal{M}}} \sim \frac{g_s^{-\frac{8}{3}}L^2 r_h^4}{u^5}\cot^3\theta_2\sin\theta_2 f_2(\theta_2). \tag{137}$$

Using the fact that the integrand possesses a maximum contribution along $\theta = 0, \pi$, we assume that the result of the integration along $\theta_{1,2}$ will be given by the sum of the contribution of the integrand at $\theta_{1,2} = 0, \pi$. In [26], we have introduced a cutoff $\theta_{1,2} \sim \alpha_\theta\epsilon^{\frac{3}{2}}$ in the 'MQGP' limit where $\epsilon \lesssim 1, \alpha_\theta << 1$ in the MQGP limit of [26]. Using this and Eq. (127), the simplified action, in the MQGP limit, will be given as

$$S \sim \epsilon^{-\frac{9}{2}}\frac{r_h^4}{K_{11}^2 g_s^2}\int du\, d^4x \frac{1}{u^3}\left[\left(H_t'\right)^2 - g_1\left(H_{x_1}'\right)^2\right]. \tag{138}$$

According to the Kubo formula as mentioned in the beginning of Sect. 4, the shear viscosity is defined as $\eta = -\lim_{w\to 0}\left[\frac{1}{w}\Im m G_{x_1x_2,x_1x_2}\right]$. In the $q_3 \to 0$ limit, the H_t and H_{x_1} decouple and the EOM for 'H_{x_1}' becomes

$$\begin{aligned}
&H_{x_1}''(u) - \frac{[4-\frac{5}{2}(1-u^4)]}{u(1-u^4)}H_{x_1}'(u)\\
&+\left[\frac{\omega_3^2}{(1-u^4)^2} + \frac{\alpha}{u^2(1-u^4)}\right]H_{x_1}(u) = 0,
\end{aligned} \tag{139}$$

where $u = 1$ is thus seen to be a regular singular point with exponents of the corresponding indicial equation given by $\pm\frac{iw_3}{4}$. Choosing the 'incoming-boundary-condition' exponent, we will look for the solutions of the form $H_{x_1}(u) = (1-u)^{-\frac{iw_3}{4}}\mathcal{H}_{x_1}(u)$, $\mathcal{H}_{x_1}(u)$ being analytic in u. Assuming a perturbative ansatz for $\mathcal{H}_{x_1}(u)$: $\mathcal{H}_{x_1}(u) = \mathcal{H}^{(0)}_{x_1}(u) + w_3\mathcal{H}^{(1)}_{x_1}(u) + w_3^2\mathcal{H}^{(2)}_{x_1} + \mathcal{O}(w_3^3)$, we obtain

$$\mathcal{O}(w_3^0) : (u-1)^2\mathcal{H}_{x_1}^{(0)\,\prime\prime}(u) + \frac{[4-\frac{5}{2}(1-u^4)](u-1)}{u(1+u)(1+u^2)}\mathcal{H}_{x_1}^{(0)\,\prime}(u) + \frac{\alpha(u-1)}{u^2(u+1)(u^2+1)}\mathcal{H}_{x_1}^{(0)}(u) = 0, \quad (140)$$

where $\alpha \equiv 8 - \mathcal{O}(1)g_s^{-\frac{2}{3}}L^2\Lambda$. The solution to (140) for arbitrary α is given as

$$\mathcal{H}_{x_1}^{(0)}(u,\alpha) = (-1)^{\frac{1}{16}(5-\sqrt{16\alpha+25})}u^{\frac{5}{4}-\frac{1}{4}\sqrt{16\alpha+25}} \times \left(\tilde{c}_1\, {}_2F_1\left[\frac{1}{16}\left(5-\sqrt{25+16\alpha}\right),\right.\right. \times \frac{1}{16}\left(11-\sqrt{25+16\alpha}\right), 1-\frac{1}{8}\sqrt{16\alpha+25}; u^4\Big] \times (-1)^{\frac{1}{8}\sqrt{16\alpha+25}}\tilde{c}_2 u^{\frac{1}{2}\sqrt{16\alpha+25}} \times {}_2F_1\left[\frac{1}{16}\left(5+\sqrt{25+16\alpha}\right), \frac{1}{16}\left(11+\sqrt{25+16\alpha}\right),\right. \times \left.\left.\frac{1}{8}\left(\sqrt{16\alpha+25}+8\right); u^4\right]\right). \quad (141)$$

One hence notices that $H_{x_1}^{(0)}(u=0,\alpha) \neq 0$ for $\alpha = 0$ for which we will henceforth write

$$\mathcal{H}_{x_1}^{(0)}(u) = -\frac{2}{5}c_1u^{5/2} + \frac{1}{4}i\pi c_1 + c_2 + \mathcal{O}(u^{\frac{13}{2}}). \quad (142)$$

In terms of $\tilde{c}_{1,2}, \mathcal{H}_{x_1}^{(0)}(u) = \tilde{c}_1 + \tilde{c}_2 e^{\frac{5i\pi}{8}}u^{\frac{5}{2}}\, {}_2F_1\left(\frac{5}{8}, 1, \frac{13}{8}; u^4\right)$.

Assuming $\alpha = 0$ henceforth, $\mathcal{H}_{x_1}^{(1)}(u)$ will be determined by the following differential equation:

$$(u-1)^2\mathcal{H}_{x_1}^{(1)\prime\prime}(u) - \frac{i}{2}(u-1)\mathcal{H}_{x_1}^{(0)\,\prime}(u) + \frac{i}{4}\mathcal{H}_{x_1}^{(0)}(u) - \frac{i}{4}\frac{[4-\frac{5}{2}(1-u^4)]}{u(u+1)(u^2+1)}\mathcal{H}_{x_1}^{(0)}(u) + \frac{i}{4}\frac{[4-\frac{5}{2}(1-u^4)](u-1)}{u(u+1)(u^2+1)}\mathcal{H}_{x_1}^{(1)\,\prime}(u) = 0. \quad (143)$$

Near $u = 0$, (143) is solved to yield

$$\mathcal{H}_{x_1}^{(1)}(u) = c_4 + \frac{1}{16}(-4c_2 - i\pi c_1)\, iu + \frac{2}{5}ie^{3/2}c_3u^{5/2} + u^{7/2}\left(-\frac{3c_1 i}{70} + \frac{6}{7}ie^{3/2}c_3\right) + \mathcal{O}(u^{\frac{9}{2}}). \quad (144)$$

Imposing the boundary condition on $H_{x_1}(u)$ at $u = 0$, assuming $c_{1,2} >> c_4$, yields $H_{x_1 0} \approx \frac{i\pi c_1}{4} + c_2$. Hence, setting $\kappa_{11}^2 \sim \mathcal{O}(1)(g_sN)^{\frac{3}{4}}$, and defining $\cot^3\langle\theta_2\rangle\sin\langle\theta_2\rangle f_2(\langle\theta_2\rangle)|_{\theta_2\sim\epsilon^{\frac{3}{2}}} \sim \mathcal{O}(1)\beta \equiv \frac{9}{2}$ for the MQGP limit [26]:

$$\lim_{w\to 0}\left[\lim_{q\to 0}\frac{1}{w}\Im m G^R_{x_1x_2,x_1x_2}(w,q)\right]_{u=0} \sim \mathcal{O}(1)T^3\frac{(4\pi)^2N^{\frac{5}{4}}}{\mathcal{O}(1)g_s^{\frac{3}{4}}}\epsilon^{-\beta^{(i)/(ii)}}\lim_{w\to 0}\frac{1}{u^3}\Im m \times\left[\left(\frac{H_{x_1,q}(u)}{\frac{i\pi c_1}{4}+c_2}\right)\left(\frac{H'_{x_1,-q}(u)}{\frac{i\pi c_1}{4}+c_2}\right)\right]_{u=0} \sim \frac{(4\pi)^2\mathcal{O}(1)N^{\frac{5}{4}}}{\mathcal{O}(1)\pi g_s^{\frac{3}{4}}}\epsilon^{-\beta^{(i)/(ii)}}\Im m\left(\frac{-\frac{1}{64}i\,(\pi c_1 - 4ic_2)^2}{\left[\frac{i\pi c_1}{4}+c_2\right]^2}\right)$$

$$T^3 = \frac{4\pi N^{\frac{5}{4}}\mathcal{O}(1)\epsilon^{-\beta^{(i)/(ii)}}}{\mathcal{O}(1).g_s^{\frac{3}{4}}}T^3. \quad (145)$$

Using $s = \mathcal{O}(1)r_h^3 = \mathcal{O}(1)(4\pi)^{\frac{3}{2}}(g_sN)^{\frac{3}{2}}T^3$ [26], we obtain $\frac{\eta}{s} = \frac{\mathcal{O}(1)\epsilon^{-\beta^{(i)/(ii)}}}{\sqrt{4\pi}g_s^{\frac{9}{4}}N^{\frac{1}{4}}(\mathcal{O}(1))^2}$, which, writing $g_s = \alpha_{g_s}^{(i)/(ii)}\epsilon$, $N = \alpha_N^{(i)/(ii)}\epsilon^{-19/-39}$ in the aforementioned two limits (i) and (ii), is $\frac{\mathcal{O}(1)\epsilon}{(\mathcal{O}(1))^2\left(\alpha_{g_s}^{(i)}\right)^{\frac{9}{4}}\left(\alpha_N^{(i)}\right)^{\frac{1}{4}}\sqrt{4\pi}}$ in the limit of [24]. If one assumes that the introduction of M fractional $D3$-branes and N_f flavour $D7$-branes does not have a significant effect on the 10D warp factor h, then in the limit of [24] effected as the first limit of (4), one can show that ϵ cannot be taken to be much smaller than around 0.01. One can choose the appropriate $\alpha_{g_s,N}^{(i)} \sim \frac{1}{\mathcal{O}(1)}$ such that $\frac{\mathcal{O}(1)\epsilon}{(\mathcal{O}(1))^2\left(\alpha_{g_s}^{(i)}\right)^{\frac{9}{4}}\left(\alpha_N^{(i)}\right)^{\frac{1}{4}}} = \frac{1}{\sqrt{4\pi}}$, implying one can generate $\frac{\eta}{s} = \frac{1}{4\pi}$.

In the more important MQGP limit, one obtains $\frac{\eta}{s} = \frac{\mathcal{O}(1)\epsilon^3}{(\mathcal{O}(1))^2\left(\alpha_{g_s}^{(ii)}\right)^{\frac{9}{4}}\left(\alpha_N^{(ii)}\right)^{\frac{1}{4}}}$. Now, in the MQGP limit, ϵ is less than but close to unity, hence yet again we can choose $\alpha_{g_s,N}^{(ii)} \sim \frac{1}{\mathcal{O}(1)}$ such that $\frac{\mathcal{O}(1)\epsilon^3}{(\mathcal{O}(1))^2\left(\alpha_{g_s}^{(ii)}\right)^{\frac{9}{4}}\left(\alpha_N^{(ii)}\right)^{\frac{1}{4}}} = \frac{1}{\sqrt{4\pi}}$ hence implying $\frac{\eta}{s} = \frac{1}{4\pi}$.

3.3.2 Metric tensor mode fluctuations

To obtain the correlations function corresponding to the tensor mode, we consider a fluctuation of $M3$-brane metric of the form $h_{x_2x_3} \neq 0$ with all other $h_{\mu\nu} = 0$ [20]. By Fourier decomposing them,

$$h_{x_2}^{x_3}(u,\vec{x}) = e^{-iwt+iqx_1}\phi(u). \quad (146)$$

Using Eqs. (124) and (125), the linearised Einstein EOM for $\phi(u)$ will be given as

$$\phi''(u) - \frac{(3+u^2)}{u(1-u^4)}\phi'(u) + \frac{1}{(1-u^4)} \times\left[w_3^2 - (1-u^4)q_3^2\right]\phi(u) = 0. \quad (147)$$

The horizon $u = 1$ is a regular singular point and the roots of the indicial equation around this are $\pm\frac{iw_3}{4}$; choosing the incoming-wave-boundary condition, $\phi(u) = (1-u)^{-\frac{iw_3}{4}}\Phi(u)$ and writing a double perturbative ansatz:

$$\Phi(u) = \Phi^{(0,0)}(u)+w_3\Phi^{(0,1)}(u)+q_3^2\Phi^{(2,0)}(u)+\cdots, \quad (148)$$

the solutions near $u = 0$, up to $\mathcal{O}(u^4)$, are given below:

$$\Phi^{(0,0)}(u) = c_1 + c_2 - \frac{c_1\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)}{\sqrt{2}} - \frac{1}{4}c_1u^4 + \mathcal{O}(u^5);$$
$$\Phi^{(0,1)}(u) = \left(c_4 - \frac{2e^3c_3}{9}\right) + \frac{1}{8}iu \times\left(-2c_1 - 2c_2 + \sqrt{2}c_1\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) -\frac{1}{4}\left(e^3c_3\right)u^4 + \mathcal{O}\left(u^5\right);$$
$$\Phi^{(2,0)}(u) = \frac{1}{8}u^2\left[c_1\left(\sqrt{2}\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)-2\right)-2c_2\right] +u^3\left[c_1+c_2-\frac{c_1\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)}{\sqrt{2}}\right] +\frac{1}{64}u^4\left[-108\left\{c_1\left(\sqrt{2}\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)-2\right)-2c_2\right\} \times\log(u)+342c_2-16e^3c_3 -171c_1\left(\sqrt{2}\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)-2\right)\right]+\mathcal{O}\left(u^5\right). \quad (149)$$

Writing

$$\phi(u) = (1-u)^{-\frac{iw_3}{4}}\left[a_1 + f_1u^4 + w_3\left(a_2 + b_2u + f_2u^4\right) +q_3^2\left(c_3u^2 + d_3u^3 + g_3u^4\ln u\right)\right], \quad (150)$$

the boundary condition yields $\phi_0 \approx a_1 + w_3a_2 \stackrel{c_{1,2}>>c_{3,4}}{\longrightarrow} a_1$. The kinetic term for ϕ is given by

$$\cot^3\langle\theta_2\rangle f_2(\langle\theta_2\rangle)\sin\langle\theta_2\rangle\left(\frac{r_h^4}{g_s^2\kappa_{11}^2}\right) \times\int du d^4x\frac{g_1(u)(\phi')^2}{u^3}. \quad (151)$$

Hence, using (151) and Kubo's formula:

$$\eta = \lim_{w\to 0}\frac{1}{w}\left(\lim_{q\to 0}\Im m\left(G^R_{x_2x_3,x_2x_3}\right)\right) \sim \frac{(4\pi)^2\mathcal{O}(1)N^{\frac{5}{4}}}{\mathcal{O}(1)g_s^{\frac{3}{4}}}\epsilon^{-\beta^{(i)/(ii)}}\lim_{w\to 0}\frac{1}{w} \times\left(\lim_{q\to 0}\frac{1}{u^3}\Im m\left(\phi_q(u)\phi'_{-q}(u)\right)\right)\Big|_{u=0} \sim T^3\frac{(4\pi)^2\mathcal{O}(1)N^{\frac{5}{4}}}{\mathcal{O}(1)\pi g_s^{\frac{3}{4}}}\epsilon^{-\beta^{(i)/(ii)}}\frac{4\Im m(f_2)}{\pi a_1}, \quad (152)$$

which utilising $s = \mathcal{O}(1)r_h^3 = \mathcal{O}(1)(4\pi)^{\frac{3}{2}}(g_sN)^{\frac{3}{2}}T^3$ [26], yields $\frac{\eta}{s} = -\frac{2\mathcal{O}(1)\epsilon^{-\beta^{(i)/(ii)}}}{\sqrt{\pi}g_s^{\frac{9}{4}}N^{\frac{1}{4}}(\mathcal{O}(1))^2}\frac{e^3\Im m(c_3)}{\pi\phi_0}$. Therefore for $\Im m(c_3) : -\frac{e^3\mathcal{O}(1)\Im m(c_3)\epsilon^{1,3}}{(\mathcal{O}(1))^2\left(\alpha_{g_s}^{(i),(ii)}\right)^{\frac{9}{4}}\left(\alpha_N^{(i),(ii)}\right)^{\frac{1}{4}}} = \frac{1}{4\sqrt{\pi}}$, one obtains $\frac{\eta}{s} = \frac{1}{4\pi}$.

4 Summary and outlook

Given the strong-coupling nature of QGP it is believed that this will be better described in the limit of finite gauge coupling (or string coupling from the string theory dual perspective) [27]. With this as the basic motivation, the black $M3$-branes in [26] were obtained as a solution to the M-theory uplift of resolved warped deformed conifold in the 'delocalised' limit of [25] (in conformity with the non-locality of T-duality transformations), constructed by using modified an 'OKS-BH' background [24] given in the context of type IIB string theory involving N D3-branes placed at the tip, N_f $D7$-branes wrapped around a four-cycle and M $D5$-branes wrapping an S^2 inside a resolved warped deformed conifold, in particular in the MQGP limit discussed in [26]: *finite* g_s, finite g_sM, N_f, $g_s^2MN_f$, very large g_sN, and very small $\frac{g_sM^2}{N}$. Given the finite string coupling, such a limit could have been meaningfully discussed only in M-theory, which is what we did in [26]. The thermodynamical stability of the M-theory uplift in this limit was demonstrated in [26] by showing positivity of the specific heat. Also, it was shown in [26] that the black $M3$-branes' near-horizon geometry near the $\theta_{1,2} = 0$ branches preserved $\frac{1}{8}$ supersymmetry. By using the KSS prescription [7], we had calculated in [26] the diffusion coefficient to be $\frac{1}{T}$ in both type IIB and type IIA backgrounds, and the η/s turned out to be $\frac{1}{4\pi}$ in the type IIB, type IIA at finite string coupling (as part of the MQGP limit).

The flow-chart of our calculations and results are summarised in Fig. 1. After having explicitly shown that the uplift obtained in [26], in the 'delocalised' limit of [25], is a solution to the $D = 11$ SUGRA EOMs in the MQGP limit, we have looked at the following two aspects at finite string coupling (and hence from an M-theory perspective) as part of the MQGP limit of [26]:

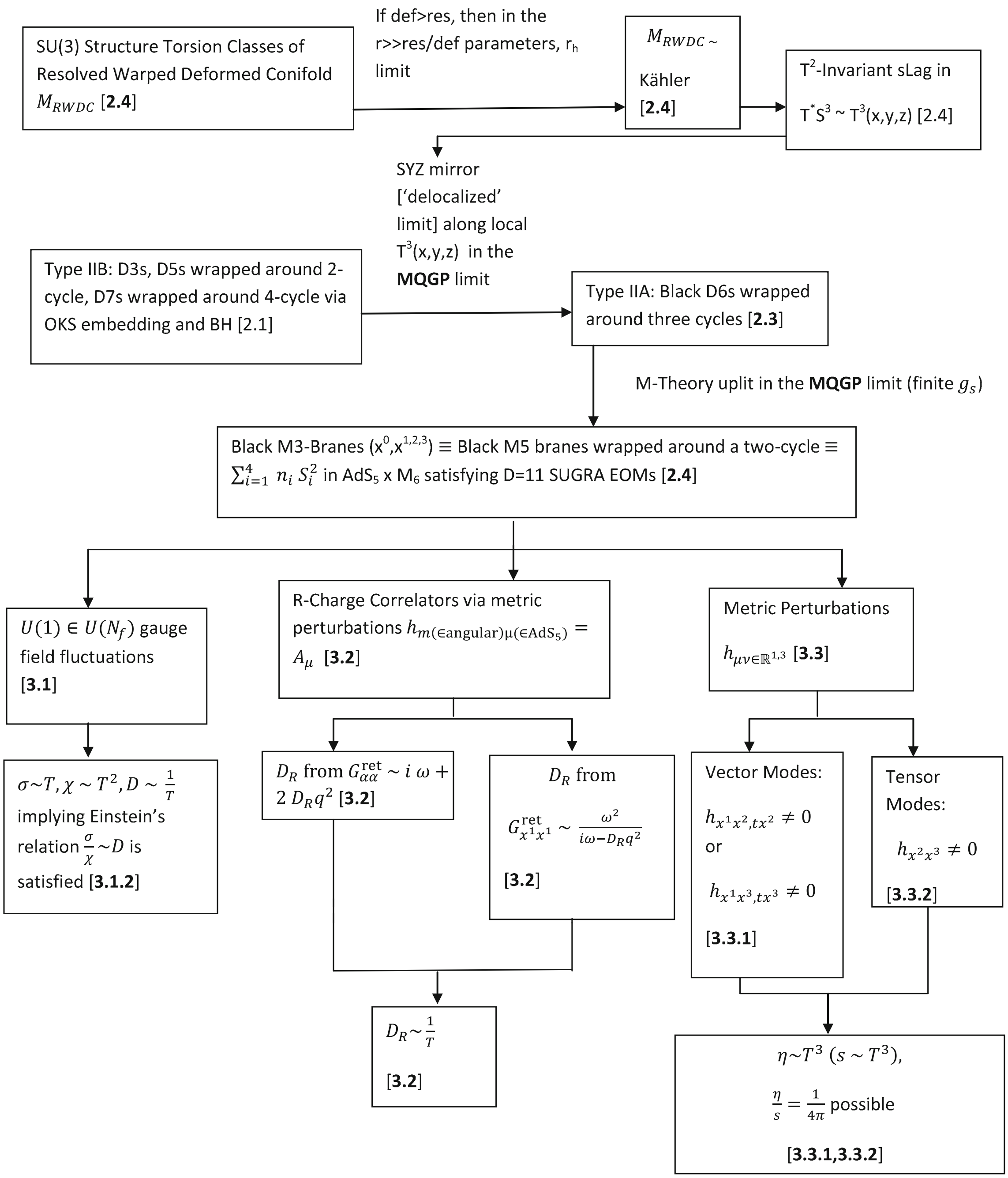

Fig. 1 Flow-chart of the paper with (sub-)section [mentioned in square brackets]-wise description

- **Geometry of black $M3$-branes in the MQGP limit of** [26]: By evaluating the flux (charge) of (corresponding to) G_4 by integrating over all (non-) compact four-cycles, the black $M3$-branes, asymptotically, were shown to be black $M5$-branes wrapping a two-cycle homologously given by a (large) integer sum of two-spheres in AdS$_5 \times M_6$. As shown in [31], the supersymmetry breaking, measured by violation of the ISD condition of the flux G_3, is proportional to the square of the resolution parameter. This in turn (turning off a bare resolution parameter, or assuming it to be extremely small) goes like $\mathcal{O}\left(\frac{g_s M^2}{N} r_h^2\right)$; in the MQGP limit of [26] we hence disregard this. By comparing the non-Kähler resolved warped deformed conifold (NKRWDC) metric with the one of [42–44] and hence evaluating the five $SU(3)$ torsion (τ) classes $W_{1,2,3,4,5}$, we show that in the MQGP limit of

[26], for extremely large radial coordinates, the NKR-WDC (i) is Kähler as $\tau \in W_5$, and (ii) is asymptotically Calabi–Yau as $W_{1,2,3,4,5} = 0$ (as expected). Further, to permit use of the SYZ symmetry, in addition to the large base of the T^3 used for triple T dualities in [26], one requires this three-cycle to be a special Lagrangian. We explicitly prove that the local three-torus T^3 of [26] satisfies the constraints satisfied by the maximal T^2-invariant special Lagrangian submanifold of a deformed conifold of [28] in the MQGP limit of [26].

- **Transport coefficients of black $M3$-branes in the MQGP limit of** [26]: Exploiting the aforementioned asymptotic $\text{AdS}_5 \times M_6$ background and based on the prescription of [4], we have evaluated at finite string coupling (as part of the MQGP limit), different transport coefficients by calculating fluctuations of the metric as well as gauge field corresponding to a $D7$-brane living on the world volume of a non-compact four-cycle in the (warped) deformed conifold and R-symmetry group present in the M-theory. However, to do this, we need to extract the five-dimensional AdS metric by integrating out all r-independent angular directions. Thus proceeding to calculate the two-point correlator/spectral functions, we evaluate the EOMs for $U(1)_{N_f}$-gauge field, the $U(1)_R$ gauge field and the vector and tensor modes and then evaluate the solutions by double perturbative ansatze up to $\mathcal{O}(w_3, q_3^2)$. The electrical conductivity, diffusion coefficient and charge susceptibility obtained due to the $U(1)_{N_f}$ gauge field satisfy Einstein's relation, which is a reasonable check of our results. Similarly, we show that one can calculate the shear viscosity from the vector and tensor modes by using Kubo's formula such that η/s turns out[3] to be $\frac{1}{4\pi}$, which is expected for any theory obeying the gauge–gravity correspondence.

Our results, we feel, are significant in the sense that these have been obtained in the context of M-theory background [26] that are valid even for a *finite coupling constant* g_s, which in fact could be more appealing to studying aspects of a 'strongly coupled quark gluon plasma' and might bring one closer to the results obtained using experimental data at RHIC. We repeat, as we wrote in Sect. 1, that we are not aware of previous attempts at evaluation of transport coefficients, such as shear viscosity (η), diffusion constant (D), electrical conductivity (σ), charge susceptibility (χ), etc. of large-N thermal QCD-like theories at finite gauge coupling or equivalently finite string coupling, and hence correctly addressable only from M-theory perspective. It is for this reason that we feel that our results $\eta \sim T^3$, $D, D_R \sim 1/T$, $\chi \sim T^2$ (such that $D = \sigma/\chi$), etc. (apart from scalings w.r.t. N, etc.) serve as M-theory predictions for sQGP.

For the future, it would be interesting to extend these calculations to compute the "second speed of sound" by working out the two-point correlator functions of the scalar modes of the stress–energy tensor. One can also calculate the thermal conductivity corresponding to R-charge correlators and check if the ratio of thermal conductivity and viscosity satisfies the Wiedemann–Franz law. Also, one should obtain the holographic spectral function by using a non-abelian $SU(N_f)$ gauge field background and produce the expected continuous meson spectra as a function of the non-zero chemical potential due to the presence of a black hole in the background [26].

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Appendix A: G_4 components

We now list the non-zero four-flux components G_4 in the limit of [24] and disregarding the asymmetry w.r.t. ϕ_1 and ϕ_2, which can be eliminated by symmetrising A_1^{IIB} w.r.t. them:

[3] The entropy was calculated in [26] from the partition function of an 11-dimensional M-theory background in the MQGP limit, or equivalently the horizon area.

$$(i)\quad G_{r\theta_1\phi_2\theta_2} \sim \left(\frac{81 g_s M N_f \sin\ \phi_2 \cot\left(\frac{\theta_1}{2}\right)\sin(\theta_2)\sqrt[4]{\frac{g_s N}{r^4}}}{4\sqrt{2}\pi^{5/4}\sqrt{g_s N}} + \frac{3\sqrt{3}N_f \sin\ \phi_1 r \sin(\theta_1)\sqrt[4]{\frac{g_s N}{r^4}}}{\sqrt{2}\pi\sqrt[4]{g_s N}}\right)$$

$$\times \frac{1}{96\pi^{3/2}r^2\sqrt{\frac{g_s N}{r^4}}\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)^2}\Big(g_s \sin(\theta_2)\Big(64\pi^2 N \cos(\theta_1)\cot(\theta_2) f_1{}'(\theta_1)$$

$$\times\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right) + 3 g_s{}^2 M_{\rm eff}^2 N_f f_1(\theta_1)\log(r)\cos(\theta_1)\cot\left(\frac{\theta_1}{2}\right)\cot(\theta_2)$$

$$\times\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right) - 64\pi^2 N f_1(\theta_1)\sin(\theta_1)\cot(\theta_2)\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)$$

$$+256\pi^2 N f_1(\theta_1)\cot^2(\theta_1)\csc(\theta_1)\cot(\theta_2)\Big)\Big), \Rightarrow \lim_{\epsilon_{\theta_2}\to 0}\int_{\epsilon_{\theta_2}}^{\pi-\epsilon_{\theta_2}} G_{r\theta_1\theta_2\phi_2} d\theta_2 = 0;$$

$$(ii)\quad G_{r\theta_2\phi_2\phi_1} \sim \frac{1}{3}\sqrt{\pi} r^2 \sin(\theta_1)\sin(\theta_2)\sqrt{\frac{g_s N}{r^4}}$$

$$\times\left(\frac{2\sqrt[4]{\pi} f_1(\theta_1)\cos(\theta_1)\cot(\theta_2)\sqrt[4]{\frac{g_s N}{r^4}}}{\sqrt{3}\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)} - \frac{2\sqrt[4]{\pi} g_s N f_1(\theta_1)\cos(\theta_1)\cot(\theta_2)}{\sqrt{3}r^4\left(\frac{g_s N}{r^4}\right)^{3/4}\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)}\right)$$

$$\times\left(\frac{\sqrt{\frac{3}{2}}N_f \sin\ \phi_2 \sin^2(\theta_2)\left(8\cos^3(\theta_1)\cos(\theta_2)\cot(\theta_2) - 16\cos^4(\theta_1)\cot(\theta_2)\right)}{\pi(\cos(2\theta_1)-5)\left(2\sin^2(\theta_1)\cos^2(\theta_2)+2\cos^2(\theta_1)\sin^2(\theta_2)\right)}\right.$$

$$\left.-\frac{\sqrt{\frac{2}{3}}N_f 2\sin\left(\frac{\psi}{2}\right)\csc(\theta_2)\left(-2\cos(\theta_1)\cos(\theta_2)+\sin^2(\theta_1)-\cos^2(\theta_1)+5\right)}{\pi(\cos(2\theta_1)-5)}\right),$$

$$\Rightarrow \int_0^{2\pi} G_{r\theta_2\phi_2\phi_1} d\phi_2 \sim \frac{N_f \mathcal{F}(\langle\theta_1\rangle,\theta_2)}{r}\sin\left(\frac{\langle\psi\rangle}{2}\right) \text{ which we disregard in the large } r \text{ limit;}$$

$$(iii)\quad G_{r\theta_2\psi\theta_1} \sim \frac{1}{2560\sqrt{2}\pi^{11/4}N^{3/4}r^2}$$

$$\times\left(g_s{}^{5/4} M N_f \theta_1 \cot\left(\frac{\theta_1}{2}\right)\left(9\log(r)\left(768\pi^2 a^2 N\ + g_s{}^2 M_{\rm eff}^2 N_f r \log\left(\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right) + 4\pi g_s M_{\rm eff}^2 r\right)\right.\right.$$

$$\left.\left.+2268 a^2 g_s{}^2 M_{\rm eff}^2 N_f \log^2(r)\log\left(\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right) + 32\pi^2 N r\right)\right),$$

$$\Rightarrow \lim_{\epsilon_{\theta_{1,2}}\to 0}\int_{\epsilon_{\theta_1}}^{\pi-\epsilon_{\theta_1}} d\theta_1 \int_{\epsilon_{\theta_2}}^{\pi-\epsilon_{\theta_2}} d\theta_2 G_{r\theta_2\psi\theta_1} \sim \epsilon^{-5}\frac{N_f}{r} \text{ which we disregard in the large } r \text{ limit;}$$

$$(iv)\quad G_{r\theta_2\psi\phi_1} \sim$$

$$-\left(\frac{81 g_s M N_f \sin\ \phi_2 \cot\left(\frac{\theta_1}{2}\right)\sin(\theta_2)\sqrt[4]{\frac{g_s N}{r^4}}}{4\sqrt{2}\pi^{5/4}\sqrt{g_s N}} + \frac{3\sqrt{3}N_f \sin\ \phi_1 r \sin(\theta_1)\sqrt[4]{\frac{g_s N}{r^4}}}{\sqrt{2}\pi\sqrt[4]{g_s N}}\right)$$

$$\times\frac{1}{27\left(3\sin^2(\theta_1)\sin^2(\theta_2)+2\sin^2(\theta_1)\cos^2(\theta_2)+2\cos^2(\theta_1)\sin^2(\theta_2)\right)^2}$$

$$\times\left(\sqrt{\frac{2\pi}{3}} r^2 \sin^2(\theta_1)\sin^3(\theta_2)\sqrt{\frac{g_s N}{r^4}}\left(4\cos^3(\theta_1)\left(6\sqrt{6}-4\cot(\theta_1)\right)\cot(\theta_2)\csc(\theta_2)\right.\right.$$

$$\left.\left.-12\sin(\theta_1)\cos^2(\theta_1)\sin(2\theta_2)\csc^3(\theta_2) + 36\sqrt{6}\sin^2(\theta_1)\cos(\theta_1)\cot(\theta_2)\csc(\theta_2)\right)\right),$$

$$\Rightarrow \lim_{\epsilon_{\theta_2}\to 0}\int_{\epsilon_{\theta_2}}^{\pi-\epsilon_{\theta_2}} d\theta_2 \int_0^{2\pi} d\phi_1 G_{r\theta_2\psi\phi_1} = 0;$$

$$(v)\quad G_{r\phi_2\psi\theta_1} \sim -\left(\frac{81 g_s M N_f \sin\ \phi_2 \cot\left(\frac{\theta_1}{2}\right)\sin(\theta_2)\sqrt[4]{\frac{g_s N}{r^4}}}{4\sqrt{2}\pi^{5/4}\sqrt{g_s N}} + \frac{3\sqrt{3}N_f \sin\ \phi_1 r \sin(\theta_1)\sqrt[4]{\frac{g_s N}{r^4}}}{\sqrt{2}\pi\sqrt[4]{g_s N}}\right)$$

$$\times\left(\frac{\sqrt{\pi}r^2\sin^3(\theta_2)\sqrt{\frac{g_s N}{r^4}}(4\cos(2(\theta_1-\theta_2))-4\cos(2(\theta_1+\theta_2)))}{12\left(3\sin^2(\theta_1)\sin^2(\theta_2)+2\sin^2(\theta_1)\cos^2(\theta_2)+2\cos^2(\theta_1)\sin^2(\theta_2)\right)^2}\right),$$

$$\Rightarrow \lim_{\epsilon_{\theta_1}\to 0}\int_{\epsilon_{\theta_1}}^{\pi-\epsilon_{\theta_1}} d\theta_1 \int_0^{2\pi} d\phi_1 G_{r\phi_2\psi\theta_1} = 0;$$

$$(vi)\quad G_{\theta_1\phi_2\psi\theta_2} \sim -\frac{\pi^{3/4}\sqrt[4]{N}r^2\theta_1\sin^3(\theta_2)\sqrt{\frac{g_s N}{r^4}}(4\cos(2(\theta_1-\theta_2))-4\cos(2(\theta_1+\theta_2)))}{360\sqrt{2}{g_s}^{3/4}\left(3\sin^2(\theta_1)\sin^2(\theta_2)+2\sin^2(\theta_1)\cos^2(\theta_2)+2\cos^2(\theta_1)\sin^2(\theta_2)\right)^2},$$

$$\lim_{\epsilon_{\theta_2}\to 0}\int_{\epsilon_{\theta_2}}^{\pi-\epsilon_{\theta_2}} d\theta_1 G_{\theta_1\phi_2\psi\theta_2} = 0;$$

$$(vii)\quad G_{\theta_2\phi_2\psi\phi_1} \sim \left(\frac{\sqrt{\frac{3}{2}}N_f \sin\ \phi_2 \sin^2(\theta_2)\left(8\cos^3(\theta_1)\cos(\theta_2)\cot(\theta_2)-16\cos^4(\theta_1)\cot(\theta_2)\right)}{\pi(\cos(2\theta_1)-5)\left(2\sin^2(\theta_1)\cos^2(\theta_2)+2\cos^2(\theta_1)\sin^2(\theta_2)\right)}\right.$$

$$\left. - \frac{\sqrt{\frac{2}{3}}N_f 2\sin\left(\frac{\psi}{2}\right)\csc(\theta_2)\left(-2\cos(\theta_1)\cos(\theta_2)+\sin^2(\theta_1)-\cos^2(\theta_1)+5\right)}{\pi(\cos(2\theta_1)-5)}\right)$$

$$\times\left(\frac{\pi^{3/4}r^3\sin^3(\theta_1)\sin(\theta_2)\left(\frac{g_s N}{r^4}\right)^{3/4}(3\cos(2(\theta_1-\theta_2))+3\cos(2(\theta_1+\theta_2))+2\cos(2\theta_1)-14\cos(2\theta_2)+6)}{12\sqrt{3}\left(3\sin^2(\theta_1)\sin^2(\theta_2)+2\sin^2(\theta_1)\cos^2(\theta_2)+2\cos^2(\theta_1)\sin^2(\theta_2)\right)^2}\right),$$

$$\Rightarrow \int_0^{2\pi} d\phi_2 \int_0^{4\pi} d\psi G_{\theta_2\phi_2\psi\phi_1} = 0;$$

$$(viii)\quad G_{\theta_1\theta_2\phi_2\phi_1} \sim \left(\frac{\sqrt{\frac{3}{2}}N_f \sin\ \phi_2 \sin^2(\theta_2)\left(8\cos^3(\theta_1)\cos(\theta_2)\cot(\theta_2)-16\cos^4(\theta_1)\cot(\theta_2)\right)}{\pi(\cos(2\theta_1)-5)\left(2\sin^2(\theta_1)\cos^2(\theta_2)+2\cos^2(\theta_1)\sin^2(\theta_2)\right)}\right.$$

$$\left. - \frac{\sqrt{\frac{2}{3}}N_f 2\sin\left(\frac{\psi}{2}\right)\csc(\theta_2)\left(-2\cos(\theta_1)\cos(\theta_2)+\sin^2(\theta_1)-\cos^2(\theta_1)+5\right)}{\pi(\cos(2\theta_1)-5)}\right)$$

$$\times\frac{1}{96\sqrt{3}\pi^{5/4}r\sqrt[4]{\frac{g_s N}{r^4}}\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)^2}\left(g_s\sin(\theta_1)\sin(\theta_2\left(64\pi^2 N\cos(\theta_1)\cot(\theta_2)f_1{}'(\theta_1)\right.\right.$$

$$\times\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)+3{g_s}^2 M_{\rm eff}^2 N_f f_1(\theta_1)\log(r)\cos(\theta_1)\cot\left(\frac{\theta_1}{2}\right)\cot(\theta_2)$$

$$\times\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)-64\pi^2 N f_1(\theta_1)\sin(\theta_1)\cot(\theta_2)\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)$$

$$\left.\left.+256\pi^2 N f_1(\theta_1)\cot^2(\theta_1)\csc(\theta_1)\cot(\theta_2)\right)\right), \Rightarrow \lim_{\epsilon_{\theta_{1,2}}\to 0}\int_{\epsilon_{\theta_1}}^{\pi-\epsilon_{\theta_1}} d\theta_1 \int_{\epsilon_{\theta_2}}^{\pi-\epsilon_{\theta_2}}$$

$$\times d\theta_2 \int_0^{2\pi} d\phi_2 G_{\theta_1\theta_2\phi_2\phi_1} \sim \epsilon^{-14}N_f \sin\left(\frac{\langle\psi\rangle}{2}\right);$$

$$(ix)\quad G_{r\theta_1\theta_2 11} \sim \frac{2g_s M}{r^3}\Bigg(2r^2\log(r)\Big(\frac{2f_2(\theta_2)\cot(\theta_1)\left(2\cot^2(\theta_1)+3\right)\sin^2(\theta_2)\cos(\theta_2)}{\left(2\cot^2(\theta_1)+3\right)\sin^2(\theta_2)+2\cos^2(\theta_2)}$$
$$-\frac{f_1(\theta_1)(\cos(2\theta_1)-5)\cot(\theta_1)\csc(\theta_1)\cot(\theta_2)}{2\cot^2(\theta_1)+2\cot^2(\theta_2)+3}\Big)\Bigg)$$
$$+\frac{2g_s M}{r^3}\Bigg(\sin(\theta_2)\Big(\frac{f_1(\theta_1)\left(r^2+2r^2\log(r)\right)(\cos(2\theta_1)-5)\cot(\theta_1)\csc(\theta_1)\cot(\theta_2)\csc(\theta_2)}{2\cot^2(\theta_1)+2\cot^2(\theta_2)+3}$$
$$+\frac{2f_2(\theta_2)\left(r^2+2r^2\log(r)\right)\cot(\theta_1)\left(-2\cot^2(\theta_1)-3\right)\sin(\theta_2)\cos(\theta_2)}{\left(2\cot^2(\theta_1)+3\right)\sin^2(\theta_2)+2\cos^2(\theta_2)}\Big)\Bigg),$$
$$\Rightarrow \lim_{\epsilon_{\theta_{1,2}}\to 0}\int_{\epsilon_{\theta_1}}^{\pi-\epsilon_{\theta_1}} d\theta_1 \int_{\epsilon_{\theta_2}}^{\pi-\epsilon_{\theta_2}} d\theta_2 G_{r\theta_1\theta_2 11}=0;$$

$$(x)\quad G_{r\theta_1\phi_1 11} \sim \frac{g_s M_{\rm eff}^2 r f_1(\theta_1)\sin^4(\theta_1)(\cos(2\theta_2)-5)\csc^2(\theta_2)\sqrt{\frac{g_s N}{r^4}}}{16\sqrt{\pi}N\left(\sin^2(\theta_1)\left(2\cot^2(\theta_2)+3\right)+2\cos^2(\theta_1)\right)},$$
$$\Rightarrow \lim_{\epsilon_{\theta_1}\to 0}\int_{\epsilon_{\theta_1}}^{\pi-\epsilon_{\theta_1}} d\theta_1 G_{r\theta_1\phi_1 11}=0;$$

$$(xi)\quad G_{r\theta_1\psi 11} \sim \frac{1}{256\pi^3 N r^2}$$
$$\times\Bigg(3{g_s}^2 M N_f \cot\left(\frac{\theta_1}{2}\right)\Bigg(-36\log^2(r)\left(96\pi^2 a^2 N - 63a^2{g_s}^2 M_{\rm eff}^2 N_f \log\left(\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right)\right)$$
$$+9\log(r)\Big(768\pi^2 a^2 N + {g_s}^2 M_{\rm eff}^2 N_f r \log$$
$$\times\left(\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right)+4\pi g_s M_{\rm eff}^2 r\Big)+32\pi^2 N r\Bigg)\Bigg) \sim \frac{1}{r}\left(\frac{g_s^2 M N_f}{N}\mathcal{F}_3(\theta_{1,2})\right);$$

$$(xii)\quad G_{r\theta_2\psi 11} \sim -\frac{1}{512\pi^3 N r^2(\cos(2\theta_1)-5)}\Bigg(3{g_s}^2 M N_f \csc\left(\frac{\theta_2}{2}\right)\Bigg(2\cos\left(\theta_1-\frac{3\theta_2}{2}\right)$$
$$+2\cos\left(\theta_1-\frac{\theta_2}{2}\right)+\cos\left(2\theta_1-\frac{\theta_2}{2}\right)+2\cos\left(\theta_1+\frac{\theta_2}{2}\right)+\cos\left(\frac{1}{2}(4\theta_1+\theta_2)\right)+2\cos\left(\theta_1+\frac{3\theta_2}{2}\right)$$
$$-10\cos\left(\frac{\theta_2}{2}\right)\Bigg)\Bigg(-36\log^2(r)\Big(a^2\left(-63{g_s}^2 M_{\rm eff}^2 N_f - 84\pi g_s M_{\rm eff}^2 + 32\pi^2 N\right)$$
$$-21a^2{g_s}^2 M_{\rm eff}^2 N_f \log\left(\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right) - {g_s}^2 M_{\rm eff}^2 N_f r\Big) + 9\log(r)\Big(256\pi^2 a^2 N$$
$$+{g_s}^2 M_{\rm eff}^2 N_f r \log\left(\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right) + 4\pi g_s M_{\rm eff}^2 r\Big)$$
$$-2{,}592a^2{g_s}^2 M_{\rm eff}^2 N_f \log^4(r) - 1728\pi a^2 g_s M_{\rm eff}^2 \log^3(r) + 32\pi^2 N r\Bigg)\Bigg)$$
$$\sim \frac{1}{r}\left(\frac{g_s^2 M N_f}{N}\mathcal{F}_1(\theta_{1,2}) + g_s^2 M N_f \mathcal{F}_2(\theta_{1,2})\right) \sim 0;$$

$$(xiii)\quad G_{\theta_1\phi_2\psi 11} \sim -\frac{\sqrt{\pi}r^2\sin^3(\theta_2)\sqrt{\frac{g_s N}{r^4}}(4\cos(2(\theta_1-\theta_2))-4\cos(2(\theta_1+\theta_2)))}{12\left(3\sin^2(\theta_1)\sin^2(\theta_2)+2\sin^2(\theta_1)\cos^2(\theta_2)+2\cos^2(\theta_1)\sin^2(\theta_2)\right)^2},$$
$$\Rightarrow \lim_{\epsilon_{\theta_1}\to 0}\int_{\epsilon_{\theta_1}}^{\pi-\epsilon_{\theta_1}} d\theta_1 G_{\theta_1\phi_2\psi 11}=0;$$

$$(xiv)\quad G_{\theta_2\phi_2\psi 11} \sim \frac{1}{12\left(3\sin^2(\theta_1)\sin^2(\theta_2)+2\sin^2(\theta_1)\cos^2(\theta_2)+2\cos^2(\theta_1)\sin^2(\theta_2)\right)^2}$$
$$\times\left(\sqrt{\pi}r^2\sin^2(\theta_1)\sin(\theta_2)\sqrt{\frac{g_sN}{r^4}}(3\cos(2(\theta_1-\theta_2))+3\cos(2(\theta_1+\theta_2))+2\cos(2\theta_1)-14\cos(2\theta_2)+6)\right),$$
$$\Rightarrow \lim_{\epsilon_{\theta_2}\to 0}\int_{\epsilon_{\theta_2}}^{\pi-\epsilon_{\theta_2}} \mathrm{d}\theta_2 G_{\theta_2\phi_2\psi 11}\Big|_{\theta_1\sim\epsilon_{\theta_1}} \sim \epsilon^{-\frac{17}{2}};$$

$$(xiv)\quad G_{\theta_1\phi_1\psi 11} \sim \frac{1}{27(\cos(2\theta_1)-5)^2\left(3\sin^2(\theta_1)\sin^2(\theta_2)+2\sin^2(\theta_1)\cos^2(\theta_2)+2\cos^2(\theta_1)\sin^2(\theta_2)\right)^2}$$
$$\times\Bigg(2\sqrt{\frac{2\pi}{3}}r^2\sin^3(\theta_1)\sin(\theta_2)\sqrt{\frac{g_sN}{r^4}}\Bigg(81\left(6\sqrt{6}\sin^6(\theta_1)\sin^3(\theta_2)+4\sqrt{6}\sin^6(\theta_1)\sin(\theta_2)\cos^2(\theta_2)\right)$$
$$+3\sin^3(\theta_1)\cos^3(\theta_1)(112\cos(2\theta_2)-14\cos(4\theta_2)+126)\csc(\theta_2)+54\sin^4(\theta_1)\cos^2(\theta_1)\Big(15\sqrt{6}\sin^3(\theta_2)$$
$$-8\sqrt{6}\cos^3(\theta_2)\cot(\theta_2)+2\sqrt{6}\sin(\theta_2)\cos^2(\theta_2)\Big)-8\cos^6(\theta_1)\Big(-4\cos^3(\theta_2)$$
$$\times\left(4\cot(\theta_1)\cot(\theta_2)-3\sqrt{6}\cot(\theta_2)\right)+9\left(2\sqrt{6}\cot^2(\theta_1)+5\sqrt{6}\right)\sin^3(\theta_2)$$
$$+6\left(-2\sqrt{6}\cot^2(\theta_1)-4\cot(\theta_1)+\sqrt{6}\right)\sin(\theta_2)\cos^2(\theta_2)\Big)+32\sin(\theta_1)\cos^5(\theta_1)\Big(16\cos^3(\theta_2)\cot(\theta_2)$$
$$+24\sin(\theta_2)\cos^2(\theta_2)\Big)-12\sin^2(\theta_1)\cos^4(\theta_1)\Big(-9\sqrt{6}\sin^3(\theta_2)+44\sqrt{6}\cos^3(\theta_2)\cot(\theta_2)$$
$$+30\sqrt{6}\sin(\theta_2)\cos^2(\theta_2)\Big)+9\sin^5(\theta_1)\cos(\theta_1)(16\cos(2\theta_2)-2\cos(4\theta_2)+18)\csc(\theta_2)\Bigg)\Bigg),$$
$$\Rightarrow \lim_{\epsilon_{\theta_1}\to 0}\int_{\epsilon_{\theta_1}}^{\pi-\epsilon_{\theta_1}} \mathrm{d}\theta_1 G_{\theta_2\phi_2\psi 11}\Big|_{\theta_2\sim\epsilon_{\theta_1}} \sim \epsilon^{-8};$$

$$(xv)\quad G_{\theta_2\phi_1\psi 11} \sim -\frac{1}{27\left(3\sin^2(\theta_1)\sin^2(\theta_2)+2\sin^2(\theta_1)\cos^2(\theta_2)+2\cos^2(\theta_1)\sin^2(\theta_2)\right)^2}$$
$$\times\Bigg(\sqrt{\frac{2\pi}{3}}r^2\sin^2(\theta_1)\sin^3(\theta_2)\sqrt{\frac{g_sN}{r^4}}\Big(4\cos^3(\theta_1)\left(6\sqrt{6}-4\cot(\theta_1)\right)\cot(\theta_2)\csc(\theta_2)$$
$$-12\sin(\theta_1)\cos^2(\theta_1)\sin(2\theta_2)\csc^3(\theta_2)+36\sqrt{6}\sin^2(\theta_1)\cos(\theta_1)\cot(\theta_2)\csc(\theta_2)\Big)\Bigg),$$
$$\Rightarrow \lim_{\epsilon_{\theta_2}\to 0}\int_{\epsilon_{\theta_2}}^{\pi-\epsilon_{\theta_2}} \mathrm{d}\theta_2 G_{\theta_2\phi_1\psi 11}=0;$$

$$(xvi)\quad G_{\theta_2\theta_1\psi 11} \sim -\frac{9{g_s}^4MM_{\mathrm{eff}}^2{N_f}^2\log^2(r)\cot\left(\frac{\theta_1}{2}\right)\cot\left(\frac{\theta_2}{2}\right)}{512\pi^3N} \sim (g_s^2MN_f)(g_sN_f)\left(\frac{g_sM^2}{N}\right)\mathcal{F}_4(\theta_{1,2})\sim 0;$$

$$(xvi)\quad G_{\theta_1\theta_2\phi_2 11} \sim \frac{1}{96\pi^{3/2}r^2\sqrt{\frac{g_sN}{r^4}}\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)^2}$$
$$\times\Bigg(g_s\sin(\theta_2)\Bigg(64\pi^2N\cos(\theta_1)\cot(\theta_2){f_1}'(\theta_1)\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)+3{g_s}^2M_{\mathrm{eff}}^2N_f f_1(\theta_1)\log(r)\cos(\theta_1)$$
$$\cot\left(\frac{\theta_1}{2}\right)\cot(\theta_2)\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)-64\pi^2Nf_1(\theta_1)\sin(\theta_1)\cot(\theta_2)$$
$$\times\left(2\cot^2(\theta_1)+2\cot^2(\theta_2)+3\right)+256\pi^2Nf_1(\theta_1)\cot^2(\theta_1)\csc(\theta_1)\cot(\theta_2)\Bigg)\Bigg),$$
$$\Rightarrow \lim_{\epsilon_{\theta_2}\to 0}\int_{\epsilon_{\theta_2}}^{\pi-\epsilon_{\theta_2}} \mathrm{d}\theta_2 G_{\theta_1\theta_2\phi_2 11}=0. \qquad \text{(A1)}$$

Appendix B: SU(3) torsion classes of resolved warped deformed conifolds

Based on the results of [43,44], the five SU(3) structure torsion classes for the resolved warped deformed conifold are given as follows:

$$W_1 = \frac{1}{6}e^{-g(\tau)-3p(\tau)-\frac{3x(\tau)}{2}} \times \Big(-\mathcal{B}(\tau)+a^2(\tau)\mathcal{B}(\tau)-2a(\tau)\mathcal{B}(\tau)e^{g(\tau)} - \mathcal{B}(\tau)e^{2g(\tau)} -2a\mathcal{B}(\tau)e^{6p(\tau)+2x(\tau)} - 2\mathcal{B}(\tau)e^{g(\tau)+6p(\tau)+2x(\tau)} -2a'e^{6p(\tau)+2x(\tau)} +2e^{g(\tau)+6p(\tau)+2x(\tau)}(\mathcal{B}(\tau)\mathcal{B}(\tau)' - \mathcal{B}(\tau)\mathcal{B}(\tau)')\Big) \sim e^{-\tau}(4\pi g_s N)^{-\frac{1}{4}};$$

$$W_2 = -\frac{2}{3}e^{-g(\tau)-3p-\frac{3x(\tau)}{2}}G_5\wedge G_6\big(-\mathcal{B}(\tau)+a^2(\tau)\mathcal{B}(\tau) -2a(\tau)\mathcal{B}(\tau)e^{g(\tau)} - \mathcal{B}(\tau)e^{2g(\tau)} +a\mathcal{B}(\tau)e^{6p(\tau)+2x(\tau)}\mathcal{B}(\tau)e^{g(\tau)+6p(\tau)+2x(\tau)} +e^{6p(\tau)+2x(\tau)}a'(\tau) -e^{g(\tau)+6p(\tau)+2x(\tau)}(\mathcal{B}(\tau)\mathcal{B}(\tau)' - \mathcal{B}(\tau)\mathcal{B}(\tau)')\big) +e^{-g(\tau)+3p(\tau)+\frac{x(\tau)}{2}}(G_2\wedge G_3 - G_1\wedge G_4)\big(a\mathcal{B}(\tau) +\mathcal{B}(\tau)\mathcal{B}(\tau)a'(\tau)+\mathcal{B}(\tau)^2e^{g(\tau)}g'(\tau)\big) -\frac{1}{3}e^{-g(\tau)-3p(\tau)-\frac{3x(\tau)}{2}}G_1\wedge G_2\big(\mathcal{B}(\tau)-a^2(\tau)\mathcal{B}(\tau) +2a(\tau)\mathcal{B}(\tau)e^{g(\tau)}+\mathcal{B}(\tau)e^{2g(\tau)} +2a\mathcal{B}(\tau)e^{6p(\tau)+2x(\tau)} - \mathcal{B}(\tau)e^{g(\tau)+6p(\tau)+2x(\tau)} +(3\mathcal{B}(\tau)^2-1)e^{6p(\tau)+2x(\tau)}a'(\tau) +e^{g(\tau)+6p(\tau)+2x(\tau)}(\mathcal{B}(\tau)\mathcal{B}(\tau)' -\mathcal{B}(\tau)\mathcal{B}(\tau)'+3\mathcal{B}(\tau)\mathcal{B}(\tau)g'(\tau))\big) +\frac{1}{3}e^{-g(\tau)-3p(\tau)-\frac{3x(\tau)}{2}}G_3\wedge G_4\big(-\mathcal{B}(\tau)+a^2(\tau)\mathcal{B}(\tau) -2a(\tau)\mathcal{B}(\tau)e^{g(\tau)}-\mathcal{B}(\tau)e^{2g(\tau)} +4a(\tau)\mathcal{B}(\tau)e^{6p(\tau)+2x(\tau)}+\mathcal{B}(\tau)e^{g(\tau)+6p(\tau)+2x(\tau)} +(3\mathcal{B}(\tau)^2+1)e^{6p(\tau)+2x(\tau)}a'(\tau) +e^{g(\tau)+6p(\tau)+2x(\tau)}(-\mathcal{B}(\tau)\mathcal{B}(\tau)'+\mathcal{B}(\tau)\mathcal{B}(\tau)' +3\mathcal{B}(\tau)\mathcal{B}(\tau)g'(\tau))\big) \sim -e^{-\tau}(4\pi g_s N)^{\frac{1}{4}} \times e_1\wedge e_2 + e^{-\tau}\epsilon_1\wedge\epsilon_2,$$

$$W_3 = -\frac{1}{4}e^{-g(\tau)-3p(\tau)-\frac{3x(\tau)}{2}}(G_1\wedge G_3 - G_2\wedge G_4)\wedge G_6 \times\big(-\mathcal{B}(\tau)+a^2(\tau)\mathcal{B}(\tau)-2a\mathcal{B}(\tau)e^{g(\tau)} -\mathcal{B}(\tau)e^{2g(\tau)}+2a\mathcal{B}(\tau)e^{6p(\tau)+2x(\tau)} +2\mathcal{B}(\tau)e^{g(\tau)+6p(\tau)+2x(\tau)} - 2e^{6p(\tau)+2x(\tau)}a'(\tau) +2e^{g(\tau)+6p(\tau)+2x(\tau)}(\mathcal{B}(\tau)\mathcal{B}(\tau)'-\mathcal{B}(\tau)\mathcal{B}(\tau)')\big) +\frac{\mathcal{B}(\tau)}{2}e^{-g(\tau)-3p(\tau)-\frac{3x(\tau)}{2}}(G_1\wedge G_2 - G_3\wedge G_4)\wedge G_5 \times\big(-1+a^2(\tau)+e^{2g(\tau)}-2\mathcal{B}(\tau)e^{6p(\tau)+2x(\tau)}a'(\tau) +2\mathcal{B}(\tau)e^{g(\tau)+6p(\tau)+2x(\tau)}g'(\tau)\big) +\frac{1}{4}e^{-g(\tau)-3p(\tau)-\frac{3x(\tau)}{2}}G_2\wedge G_3\wedge G_5 \times\big(\mathcal{B}(\tau)-a^2(\tau)\mathcal{B}(\tau)-2a(\tau)\mathcal{B}(\tau)e^{g(\tau)} -3\mathcal{B}(\tau)e^{2g(\tau)}+2a(\tau)\mathcal{B}(\tau)e^{6p(\tau)+2x(\tau)} +2\mathcal{B}(\tau)e^{g(\tau)+6p(\tau)+2x(\tau)}+2(\mathcal{B}(\tau)^2 -\mathcal{B}(\tau)^2)e^{6p(\tau)+2x(\tau)}a'(\tau)+2e^{g(\tau)+6p(\tau)+2x(\tau)}(\mathcal{B}(\tau) \times\mathcal{B}(\tau)'-\mathcal{B}(\tau)\mathcal{B}(\tau)'-2\mathcal{B}(\tau)\mathcal{B}(\tau)g'(\tau))\big) +\frac{1}{4}e^{-g(\tau)-3p(\tau)-\frac{3x(\tau)}{2}}G_1\wedge G_4\wedge G_5 \times\big(-3\mathcal{B}(\tau)+3a^2(\tau)\mathcal{B}(\tau)-2a(\tau)\mathcal{B}(\tau)e^{g(\tau)} +\mathcal{B}(\tau)e^{2g(\tau)}+2a(\tau)\mathcal{B}(\tau)e^{6p(\tau)+2x(\tau)} +2\mathcal{B}(\tau)e^{g(\tau)+6p(\tau)+2x(\tau)}-2(1+2\mathcal{B}(\tau)^2) \times e^{6p(\tau)+2x(\tau)}a'(\tau)+2e^{g(\tau)+6p(\tau)+2x(\tau)}(\mathcal{B}(\tau)\mathcal{B}(\tau)' -\mathcal{B}(\tau)\mathcal{B}(\tau)'+2\mathcal{B}(\tau)\mathcal{B}(\tau)g'(\tau))\big) \sim e^{-\tau}(g_sN)^{\frac{3}{4}}(e_1\wedge e_2+\epsilon_1\wedge e_2)\wedge d\tau;$$

$$W_4 = \frac{1}{2}e^{-g(\tau)-3p(\tau)-\frac{3x(\tau)}{2}} \times G_5\big(-\mathcal{B}(\tau)+a^2(\tau)\mathcal{B}(\tau)+2a(\tau)\mathcal{B}(\tau)e^{g(\tau)}-\mathcal{B}(\tau)e^{2g(\tau)} +2e^{g(\tau)+6p(\tau)+2x(\tau)}x'(\tau)\big) \sim e^{-\tau}(g_sN)^{\frac{1}{4}}d\tau;$$

$$W_5^{(\bar{3})} = \frac{1}{4}e^{-g(\tau)+3p(\tau)+\frac{x(\tau)}{2}}(G_5 - iG_6) \times\big(2a(\tau)\mathcal{B}(\tau)-2\mathcal{B}(\tau)e^{g(\tau)}-6e^{g(\tau)}p'(\tau)+e^{g(\tau)}x'(\tau)\big) \sim (g_sN)^{\frac{3}{4}}e^{-\tau} \times\left(d\tau - i\frac{(4\pi g_sN)^{\frac{1}{4}}}{3}[d\psi+\cos\theta_2 d\phi_2+\cos\theta_1 d\phi_1]\right). \quad (B1)$$

Appendix C: Embedding of T^2-invariant sLag in T^*S^3

Based on [28], the explicit embedding of the maximal T^2-invariant special Lagrangian three-cycle in a deformed conifold is given by the following:

$$r = \left(\frac{c_1}{\cos(\phi_1+\phi_2)}+\frac{c_2}{\cos(\phi_1-\phi_2)}\right)^{\frac{7}{3}};$$

In large r limit: $c_1 = c_2 \sim (r_\Lambda^<)^{\frac{7}{3}}, c_3 \sim (r_\Lambda^<)^6$

$\{\text{e.g.} r_\Lambda^< \sim r_\Lambda^{\frac{1}{4}}\}: -\cos\theta_1$

$$= \frac{1}{8}\Bigg\{\sec^2(\psi)\sec(\phi_1-\phi_2) \times\Big[32\cos(\phi_1-\phi_2-\psi)-8\cos(\phi_1+\phi_2-\psi) +32\cos(\phi_1-\phi_2+\psi)-8\cos(\phi_1+\phi_2+\psi) +\Big\{(-32\cos(\phi_1-\phi_2-\psi)+8\cos(\phi_1+\phi_2-\psi)$$

$$-32\cos(\phi_1-\phi_2+\psi)+8\cos(\phi_1+\phi_2+\psi)$$
$$+3\,2^{3/7}\cos(2\phi_1)\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$+3\,2^{3/7}\cos(2\phi_2)\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$+2^{3/7}\cos(2(2\phi_1+\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$+2^{3/7}\cos(2(\phi_1+2\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)\Big)^2$$
$$+64\cos(\psi)\cos(\phi_1-\phi_2)\,(-32\cos(\phi_1-\phi_2-\psi)$$
$$+16\cos(\phi_1+\phi_2-\psi)-32\cos(\phi_1-\phi_2+\psi)$$
$$+16\cos(\phi_1+\phi_2+\psi)+3\,2^{3/7}\cos(2\phi_1)\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$+3\,2^{3/7}\cos(2\phi_2)\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$+2^{3/7}\cos(2(2\phi_1+\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$+2^{3/7}\cos(2(\phi_1+2\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)\Big)\Big\}^{\frac{1}{2}}$$
$$-3\,2^{3/7}\cos(2\phi_1)\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$-3\,2^{3/7}\cos(2\phi_2)\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$-2^{3/7}\cos(2(2\phi_1+\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$-2^{3/7}\cos(2(\phi_1+2\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)\Big]\Big\}^{\frac{1}{2}}; \qquad \text{(C1)}$$

$$\cos\theta_2=\frac{1}{8}\Big\{\sec^2(\psi)(-\sec(\phi_1-\phi_2))$$
$$\times\Big[-32\cos(\phi_1-\phi_2-\psi)+8\cos(\phi_1+\phi_2-\psi)$$
$$-32\cos(\phi_1-\phi_2+\psi)+8\cos(\phi_1+\phi_2+\psi)$$
$$+\Big\{(-32\cos(\phi_1-\phi_2-\psi)+8\cos(\phi_1+\phi_2-\psi)$$
$$-32\cos(\phi_1-\phi_2+\psi)+8\cos(\phi_1+\phi_2+\psi)$$
$$+3\,2^{3/7}\cos(2\phi_1)\sec^{\frac{3}{7}}(\phi_1+\phi_2)+3\,2^{3/7}\cos(2\phi_2)$$
$$\times\sec^{\frac{3}{7}}(\phi_1+\phi_2)+2^{3/7}\cos(2(2\phi_1+\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$+2^{3/7}\cos(2(\phi_1+2\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)\Big)^2$$
$$+64\cos(\psi)\cos(\phi_1-\phi_2)\,(-32\cos(\phi_1-\phi_2-\psi)$$
$$+16\cos(\phi_1+\phi_2-\psi)\;-32\cos(\phi_1-\phi_2+\psi)$$
$$+16\cos(\phi_1+\phi_2+\psi)+3\,2^{3/7}\cos(2\phi_1)\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$+3\,2^{3/7}\cos(2\phi_2)\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$+2^{3/7}\cos(2(2\phi_1+\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$+2^{3/7}\cos(2(\phi_1+2\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)\Big)\Big\}^{\frac{1}{2}}$$
$$+3\,2^{3/7}\cos(2\phi_1)\sec^{\frac{3}{7}}(\phi_1+\phi_2)+3\,2^{3/7}\cos(2\phi_2)$$
$$\times\sec^{\frac{3}{7}}(\phi_1+\phi_2)+2^{3/7}\cos(2(2\phi_1+\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)$$
$$+2^{3/7}\cos(2(\phi_1+2\phi_2))\sec^{\frac{3}{7}}(\phi_1+\phi_2)\Big]\Big\}^{\frac{1}{2}}. \qquad \text{(C2)}$$

From [47], the Kähler form J and the nowhere-vanishing holomorphic three-form Ω for a deformed conifold are given by

$$J=-\frac{r^6\hat{\gamma}'+\mu^4\hat{\gamma}-r^2\mu^4\hat{\gamma}'}{2r^5\sqrt{1-\mu^4/r^4}}\,\mathrm{d}r\wedge(\mathrm{d}\psi+\cos\theta_1\,\mathrm{d}\phi_1+\cos\theta_2\mathrm{d}\phi_2)$$
$$+\frac{\hat{\gamma}}{4}\sqrt{1-\frac{\mu^4}{r^4}}\,(\sin\theta_1\,\mathrm{d}\theta_1\wedge\mathrm{d}\phi_1+\sin\theta_2\,\mathrm{d}\theta_2\wedge\mathrm{d}\phi_2), \qquad \text{(C3)}$$

and

$$\Omega=2\mathcal{T}\,\frac{\mathcal{S}\cos\psi-\imath\sin\psi}{r\mathcal{S}}\,\mathrm{d}r$$
$$\wedge\,(\sin\theta_1\,\mathrm{d}\theta_2\wedge\mathrm{d}\phi_1-\sin\theta_2\,\mathrm{d}\theta_1\wedge\mathrm{d}\phi_2)$$
$$+\frac{2\imath\mathcal{T}}{r\mathcal{S}}\,(\cos\psi-\imath\mathcal{S}\sin\psi)\,\mathrm{d}r$$
$$\wedge\,(\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2-\sin\theta_1\sin\theta_2\,\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2)$$
$$-\frac{2\mu^2\mathcal{T}}{r^3\mathcal{S}}\,\mathrm{d}r\wedge(\sin\theta_1\,\mathrm{d}\theta_1\wedge\mathrm{d}\phi_1-\sin\theta_2\,\mathrm{d}\theta_2\wedge\mathrm{d}\phi_2)$$
$$+\frac{\imath\mu^2\mathcal{T}}{r^2}\Big[\sin\theta_1\,\mathrm{d}\theta_1\wedge\mathrm{d}\phi_1\wedge(\mathrm{d}\psi+\cos\theta_2\,\mathrm{d}\phi_2)$$
$$-\sin\theta_2\,\mathrm{d}\theta_2\wedge\mathrm{d}\phi_2\wedge(\mathrm{d}\psi+\cos\theta_1\,\mathrm{d}\phi_1)\Big]$$
$$+\mathcal{T}(\imath\cos\psi+\mathcal{S}\sin\psi)$$
$$\times\Big[\sin\theta_2\,\mathrm{d}\theta_1\wedge\mathrm{d}\phi_2\wedge(\mathrm{d}\psi+\cos\theta_1\,\mathrm{d}\phi_1)$$
$$-\sin\theta_1\,\mathrm{d}\theta_2\wedge\mathrm{d}\phi_1\wedge(\mathrm{d}\psi+\cos\theta_2\,\mathrm{d}\phi_2)\Big]$$
$$+\mathcal{T}(\mathcal{S}\cos\psi-\imath\sin\psi)$$
$$\times\Big[\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\wedge(\mathrm{d}\psi+\cos\theta_1\,\mathrm{d}\phi_1+\cos\theta_2\,\mathrm{d}\phi_2)$$
$$-\sin\theta_1\sin\theta_2\,\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2\wedge\mathrm{d}\psi\Big], \qquad \text{(C4)}$$

where $\mathcal{S}=\sqrt{1-\mu^4/r^4}\xrightarrow{r>>1}1$ and $\mathcal{T}=\hat{\gamma}\sqrt{\hat{\gamma}+(r^2\hat{\gamma}'-\hat{\gamma})(1-\mu^4/r^4)}/8\xrightarrow{r>>1}r^2$ as $\hat{\gamma}\xleftarrow{r>>1}r^{\frac{4}{3}}$, and the deformation parameter μ is defined via $z_1^2+z_2^2+z_3^2+z_4^2=\mu^2$.

Appendix D: Some intermediate steps pertaining to *R*-charge gauge field EOM's solution

Plugging (102) into (101) yields for $\tilde{\mathcal{A}}_\alpha^{(0,0),(9,1),(2,0)}(u)$ relevant to the $\tilde{\mathcal{A}}_\alpha(u)$'s EOM, near $u=0$ up to $\mathcal{O}(u)$:

$$(1-u)^{-\frac{iw_3}{4}-1}\tilde{\mathcal{A}}_\alpha^{(0,0)}(u)=c_2+c_2\left(1+\frac{i}{4}w_3\right)u+\mathcal{O}(u^2);$$
$$(1-u)^{-\frac{iw_3}{4}-1}\tilde{\mathcal{A}}_\alpha^{(0,1)}(u)=\left(-\frac{1}{64}i\pi^2c_1-\frac{3\pi c_2}{16}+\frac{1}{4}i\pi c_3+c_4\right)$$
$$+\frac{1}{256}\Big[\pi^2(w_3w_3-4i)c_1-4i\pi(w_3w_3-4i)$$
$$\times(3c_2-4ic_3)+64i\Big((w_3w_3-4i)c_4-c_2\Big)\Big]$$
$$u+O\left(u^2\right);\,(1-u)^{-\frac{iw_3}{4}}\tilde{\mathcal{A}}_\alpha^{(2,0)}(u)=\frac{1}{24}$$
$$\times\Big[3\Big(2Li_3(-i)+2Li_3(i)+3\zeta(3)\Big)c_1$$

$$-i\pi^3 c_1 + \pi^2 c_2 + 6i\pi c_3 + 24c_4\Big] + \frac{1}{96}w_3\Big[\pi^3 c_1 + i\pi^2 c_2 - 6\pi c_3 + 3i \times \Big((2Li_3(-i) + 2Li_3(i) + 3\zeta(3))\,c_1 + 8c_4\Big)\Big] \times u + \mathcal{O}(u^2). \tag{D1}$$

One can show that $\tilde{\mathcal{A}}^{(0,1)\,\prime}_{x_1}(u)$, up to $\mathcal{O}(u)$, will be given by the expansion of the following up to $\mathcal{O}(u)$:

$$2\sqrt{2}e^{-\frac{4u-2}{4\sqrt{2}}}(2u-1)^{3/2} + e^{-\frac{4u-2}{4\sqrt{2}}}(4u-2)^{3/2}c_1 U \times\left(\frac{1}{8}\left(10+3\sqrt{2}\right), \frac{5}{2}, \sqrt{2}u - \frac{1}{\sqrt{2}}\right) + e^{-\frac{4u-2}{4\sqrt{2}}}(4u-2)^{3/2}c_2 L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}u - \frac{1}{\sqrt{2}}\right) \times\left[U\left(\frac{1}{8}\left(10+3\sqrt{2}\right), \frac{5}{2}, \sqrt{2}u - \frac{1}{\sqrt{2}}\right)\mathcal{I}_1 + \mathcal{I}_2 L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}u - \frac{1}{\sqrt{2}}\right)\right], \tag{D2}$$

where $U(a, b, z)$ are Tricomi confluent hypergeometric functions defined via $U(a, b; z) \equiv \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-B}(\tau)\ {}_1F_1(a-b+1; 2; 2-b; z) + \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\ {}_1F_1(a; b; z)$, b not being an integer; $L^a_b(z)$ are associated Laguerre polynomials, and $\mathcal{I}_{1,2}$ are defined in (D5). In the context of the equation of motion (106) for the R-charge gauge, the exact solutions being intractable, the same corresponding to $\tilde{\mathcal{A}}^{(0,0)\,\prime}_{x_1}$ in (107), near $u = 0$, is given by

$$\begin{aligned}
\tilde{\mathcal{A}}^{(0,0)\,\prime}_{x_1} = & -ie^{\frac{1}{2}i\left(i+\sqrt{11}+i\sqrt{15}\pi\right)} \\
& \times\left\{U\left(\frac{1}{2}+\frac{7i}{\sqrt{11}}+\frac{i\sqrt{15}}{2}, 1+i\sqrt{15}, -i\sqrt{11}\right)c_1 + c_2 L^{i\sqrt{15}}_{-\frac{1}{2}-\frac{7i}{\sqrt{11}}-\frac{i\sqrt{15}}{2}}\left(-i\sqrt{11}\right)\right\} \\
& + \frac{1}{2}e^{\frac{1}{2}i\left(i+\sqrt{11}+i\sqrt{15}\pi\right)}\Big[-\Big\{\left(-2i+\sqrt{11}+\sqrt{15}\right) \\
& \times U\left(\frac{1}{2}+\frac{7i}{\sqrt{11}}+\frac{i\sqrt{15}}{2}, 1+i\sqrt{15}, -i\sqrt{11}\right) + \left(14i+\sqrt{11}+i\sqrt{165}\right) \\
& \times U\left(\frac{3}{2}+\frac{7i}{\sqrt{11}}+\frac{i\sqrt{15}}{2}, 2+i\sqrt{15}, -i\sqrt{11}\right)\Big\}c_1 \\
& - c_2\Big\{2\sqrt{11}L^{1+i\sqrt{15}}_{-\frac{3}{2}-\frac{7i}{\sqrt{11}}-\frac{i\sqrt{15}}{2}}\left(-i\sqrt{11}\right) + \left(-2i+\sqrt{11}+\sqrt{15}\right)L^{i\sqrt{15}}_{-\frac{1}{2}-\frac{7i}{\sqrt{11}}-\frac{i\sqrt{15}}{2}}\left(-i\sqrt{11}\right)\Big\}\Big]u \\
& \times + \frac{1}{4}e^{\frac{1}{2}i\left(i+\sqrt{11}+i\sqrt{15}\pi\right)} \\
& \times\Big[\Big\{\left(14i+2\sqrt{11}+\sqrt{15}+i\sqrt{165}\right) \times U\left(\frac{1}{2}+\frac{7i}{\sqrt{11}}+\frac{i\sqrt{15}}{2}, 1+i\sqrt{15}, -i\sqrt{11}\right) \\
& + i\Big(\left(39+27i\sqrt{11}+25i\sqrt{15}+3\sqrt{165}\right) \times U\left(\frac{3}{2}+\frac{7i}{\sqrt{11}}+\frac{i\sqrt{15}}{2}, 2+i\sqrt{15}, -i\sqrt{11}\right) \\
& + 2\left(-82+14i\sqrt{11}+11i\sqrt{15}-7\sqrt{165}\right) \times U\left(\frac{5}{2}+\frac{7i}{\sqrt{11}}+\frac{i\sqrt{15}}{2}, 3+i\sqrt{15}, -i\sqrt{11}\right)\Big)\Big\}c_1 \\
& + c_2\Big\{22iL^{2+i\sqrt{15}}_{-\frac{5}{2}-\frac{7i}{\sqrt{11}}-\frac{i\sqrt{15}}{2}}\left(-i\sqrt{11}\right) + \left(22i+4\sqrt{11}+2i\sqrt{165}\right)L^{1+i\sqrt{15}}_{-\frac{3}{2}-\frac{7i}{\sqrt{11}}-\frac{i\sqrt{15}}{2}}\left(-i\sqrt{11}\right) \\
& + \left(14i+2\sqrt{11}+\sqrt{15}+i\sqrt{165}\right) \times L^{i\sqrt{15}}_{-\frac{1}{2}-\frac{7i}{\sqrt{11}}-\frac{i\sqrt{15}}{2}}\left(-i\sqrt{11}\right)\Big\}\Big]u^2 + O\left(u^3\right) \\
\approx & -(80.5-11.6i)c_2 + (136.7+32.7i)c_2 u - (28.1+22.2i)c_2 u^2 + \mathcal{O}(u^3) \\
\equiv & \ a + bu + cu^2 + \mathcal{O}(u^2).
\end{aligned} \tag{D3}$$

One can similarly show that $\tilde{\mathcal{A}}^{(2,0)\,\prime}_{x_1}(u)$, up to $\mathcal{O}(u)$, will be given by the expansion of the following up to $\mathcal{O}(u)$:

$$\tilde{\mathcal{A}}^{(2,0)\,\prime}_{x_1}(u) = 2\sqrt{2}e^{-\frac{4u-2}{4\sqrt{2}}}(2u-1)^{3/2} + e^{-\frac{4u-2}{4\sqrt{2}}}(4u-2)^{3/2}c_1 \times U\left(\frac{1}{8}\left(10+3\sqrt{2}\right), \frac{5}{2}, \sqrt{2}u - \frac{1}{\sqrt{2}}\right) + e^{-\frac{4u-2}{4\sqrt{2}}}(4u-2)^{3/2}c_2 L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}u - \frac{1}{\sqrt{2}}\right) \times\left(U\left(\frac{1}{8}\left(10+3\sqrt{2}\right), \frac{5}{2}, \sqrt{2}u - \frac{1}{\sqrt{2}}\right)\mathcal{I}'_1 + L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}u - \frac{1}{\sqrt{2}}\right)\mathcal{I}'_2\right), \tag{D4}$$

where $\mathcal{I}'_{1,2}$ are defined in (D6).

The integrals $\mathcal{I}_{1,2}$ appearing in (D2) and (D4) are defined as

$$\mathcal{I}_1 \equiv \int_1^u \mathrm{d}\lambda_1\Bigg[-\frac{iae^{-\frac{1-2\lambda_1}{2\sqrt{2}}}L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}\lambda_1 - \frac{1}{\sqrt{2}}\right)}{4\sqrt{2}(2\lambda_1-1)^{3/2}\mathcal{D}_1} - \frac{ice^{-\frac{1-2\lambda_1}{2\sqrt{2}}}L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}\lambda_1 - \frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_1-1)^{3/2}\mathcal{D}_1} - \frac{5iae^{-\frac{1-2\lambda_1}{2\sqrt{2}}}L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}\lambda_1 - \frac{1}{\sqrt{2}}\right)}{4\sqrt{2}(2\lambda_1-1)^{5/2}\mathcal{D}_1}$$

$$-\frac{ibe^{-\frac{1-2\lambda_1}{2\sqrt{2}}} L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}\lambda_1-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_1-1)^{5/2}\mathcal{D}_1} -\frac{ice^{-\frac{1-2\lambda_1}{2\sqrt{2}}} L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}\lambda_1-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_1-1)^{5/2}\mathcal{D}_1}\Bigg];$$

$$\mathcal{I}_2 \equiv \Bigg[\frac{iae^{-\frac{1-2\lambda_2}{2\sqrt{2}}} U\left(\frac{1}{8}\left(10+3\sqrt{2}\right),\frac{5}{2},\sqrt{2}\lambda_2-\frac{1}{\sqrt{2}}\right)}{4\sqrt{2}(2\lambda_2-1)^{3/2}\mathcal{D}_2} +\frac{ice^{-\frac{1-2\lambda_2}{2\sqrt{2}}} U\left(\frac{1}{8}\left(10+3\sqrt{2}\right),\frac{5}{2},\sqrt{2}\lambda_2-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_2-1)^{3/2}\mathcal{D}_2} +\frac{5iae^{-\frac{1-2\lambda_2}{2\sqrt{2}}} U\left(\frac{1}{8}\left(10+3\sqrt{2}\right),\frac{5}{2},\sqrt{2}\lambda_2-\frac{1}{\sqrt{2}}\right)}{4\sqrt{2}(2\lambda_2-1)^{5/2}\mathcal{D}_2} +\frac{ibe^{-\frac{1-2\lambda_2}{2\sqrt{2}}} U\left(\frac{1}{8}\left(10+3\sqrt{2}\right),\frac{5}{2},\sqrt{2}\lambda_2-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_2-1)^{5/2}\mathcal{D}_2} +\frac{ice^{-\frac{1-2\lambda_2}{2\sqrt{2}}} U\left(\frac{1}{8}\left(10+3\sqrt{2}\right),\frac{5}{2},\sqrt{2}\lambda_2-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_2-1)^{5/2}\mathcal{D}_2}\Bigg]; \tag{D5}$$

$$\mathcal{I}_1' \equiv \int_1^u \mathrm{d}\lambda_1 \Bigg[\frac{3ae^{-\frac{1-2\lambda_1}{2\sqrt{2}}} L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}\lambda_1-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_1-1)^{3/2}\mathcal{D}_1} -\frac{be^{-\frac{1-2\lambda_1}{2\sqrt{2}}} L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}\lambda_1-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_1-1)^{3/2}\mathcal{D}_1} +\frac{ae^{-\frac{1-2\lambda_1}{2\sqrt{2}}} L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}\lambda_1-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_1-1)^{5/2}\mathcal{D}_1} -\frac{be^{-\frac{1-2\lambda_1}{2\sqrt{2}}} L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\sqrt{2}\lambda_1-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_1-1)^{5/2}\mathcal{D}_1}\Bigg];$$

$$\mathcal{I}_2' \equiv \int_1^u \mathrm{d}\lambda_2 \Bigg[-\frac{3ae^{-\frac{1-2\lambda_2}{2\sqrt{2}}} U\left(\frac{1}{8}\left(10+3\sqrt{2}\right),\frac{5}{2},\sqrt{2}\lambda_2-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_2-1)^{3/2}\mathcal{D}_2} +\frac{be^{-\frac{1-2\lambda_2}{2\sqrt{2}}} U\left(\frac{1}{8}\left(10+3\sqrt{2}\right),\frac{5}{2},\sqrt{2}\lambda_2-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_2-1)^{3/2}\mathcal{D}_2} -\frac{ae^{-\frac{1-2\lambda_2}{2\sqrt{2}}} U\left(\frac{1}{8}\left(10+3\sqrt{2}\right),\frac{5}{2},\sqrt{2}\lambda_2-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_2-1)^{5/2}\mathcal{D}_2} +\frac{be^{-\frac{1-2\lambda_2}{2\sqrt{2}}} U\left(\frac{1}{8}\left(10+3\sqrt{2}\right),\frac{5}{2},\sqrt{2}\lambda_2-\frac{1}{\sqrt{2}}\right)}{\sqrt{2}(2\lambda_2-1)^{5/2}\mathcal{D}_2}\Bigg], \tag{D6}$$

wherein

$$c\mathcal{D}_i \equiv \Bigg(5\sqrt{2}U\left(\frac{3}{8}\left(6+\sqrt{2}\right),\frac{7}{2},\frac{2\lambda_i-1}{\sqrt{2}}\right)L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\frac{2\lambda_i-1}{\sqrt{2}}\right)c +3U\left(\frac{3}{8}\left(6+\sqrt{2}\right),\frac{7}{2},\frac{2\lambda_i-1}{\sqrt{2}}\right)L^{\frac{3}{2}}_{\frac{1}{8}\left(-10-3\sqrt{2}\right)}\left(\frac{2\lambda_i-1}{\sqrt{2}}\right) -4\sqrt{2}U\left(\frac{1}{8}\left(10+3\sqrt{2}\right),\frac{5}{2},\frac{2\lambda_i-1}{\sqrt{2}}\right)L^{\frac{5}{2}}_{-\frac{3}{8}\left(6+\sqrt{2}\right)}\left(\frac{2\lambda_i-1}{\sqrt{2}}\right)\Bigg). \tag{D7}$$

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