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Jeans instability in superfluids

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Abstract   We analyze the effect of a gravitational field on the sound modes of superfluids. We derive an instability condition that generalizes the well-known Jeans instability of the sound mode in normal fluids. We discuss potential experimental implications.

1 Introduction

The Jeans instability of the sound mode in normal fluids that is caused by a gravitational field is a well-known phenomenon [1]. It exhibits itself in astrophysical scenarios of aggregation of masses and galaxy formation. The Jeans dispersion relation of a normal fluid sound mode is obtained by linearizing the fluid equations in the presence of a gravitational field. It modifies the sound mode dispersion relation $\omega^2 = u_1^2 \vec{k}^2$ to

$$\omega^2 = u_1^2 \vec{k}^2 - 4\pi G \rho. \tag{1}$$

ω is the frequency, $\vec{k}$ is the momentum vector, u_1 is the speed of sound in a normal fluid $u_1^2 = \frac{\partial p}{\partial \rho}$ evaluated at fixed entropy, p is the pressure, ρ the local mass density and G is the gravitational coupling. Instability occurs when the RHS of (1) is negative,

$$\vec{k}^2 < \frac{4\pi G \rho}{u_1^2}. \tag{2}$$

It determines a Jeans length scale λ_J, all scales larger than that being unstable to a gravitational collapse.

In this letter we will analyze the effect of a gravitational field on the sound modes in superfluids. Superfluids are quantum fluids, i.e. fluids at temperatures close to zero where quantum effects are of primary importance [2]. They exhibit remarkable properties, such as the ability to flow without viscosity in narrow capillaries. Superfluidity/superconductivity

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is also expected to be realized in neutron stars (see e.g. [3]), as well as in high density phases of QCD [4].

The hydrodynamic description of superfluids consists of two separate motions, a normal flow and a super flow with densities ρ_n and ρ_s, respectively, $\rho_n + \rho_s = \rho$ [5,6]. The super flow moves without viscosity, the normal flow is viscous, and the two flows do not exchange momentum between them. We denote by $\vec{v}_n$ and $\vec{v}_s$ the velocities of the normal and super flows, respectively. The superfluid velocity $\vec{v}_s$ corresponds to the gradient of the condensate phase that breaks spontaneously the global/local symmetry, which results in superfluidity/superconductivity, respectively. Thus, the superfluid motion is irrotational

$$\vec{\nabla} \times \vec{v}_s = 0. \tag{3}$$

There are two sound modes in superfluids: the first sound u_1 which is a density wave as in normal fluids, and the second sound u_2, which is a temperature wave and is unique to superfluids. Its dispersion relation reads

$$\omega^2 = u_2^2 \vec{k}^2, \quad u_2^2 = \frac{\rho_s s^2}{\rho_n \frac{\partial s}{\partial T}}, \tag{4}$$

where T is the temperature, s is the entropy density per particle, and the derivative is taken at fixed pressure. As expected, the second sound vanishes in the limit $\rho_s \to 0$. In general the waves can be a superposition of the two.

As we will show, in the absence of density fluctuations the pure second sound (4) remains stable, while density fluctuations imply a stability criterion for both sound modes and their superpositions. It reads

$$\vec{k}^2 < \frac{4\pi G \rho}{u_1^2 + \frac{1}{(1-J)} u_2^2}, \tag{5}$$

where $J = \frac{\frac{\partial s}{\partial p} \frac{\partial \rho}{\partial T}}{\frac{\partial s}{\partial T} \frac{\partial \rho}{\partial p}}$. This condition generalizes the ordinary Jeans instability (2) and reduces to it in the limit $\rho_s \to 0$. The RHS of (5) depends on the details of the system's thermodynamic properties. An interesting limit to take is that of

very low temperatures $T \to 0$, where the super flow component is dominant. In this limit one has $u_1^2 = 3u_2^2$ [6] and $J \to 0$ [7]. Thus, at very low temperature we have a larger Jeans length than that of a normal fluid by a factor of $\frac{2}{\sqrt{3}}$.

The letter is organized as follows. We will introduce a gravitational potential to the superfluid hydrodynamics equations and study fluctuations at the linear order. We will then analyze the instability conditions and derive (5), which is our main result. Finally, we will discuss potential experimental implications.

2 Gravitational instability in superfluid hydrodynamics

We will consider the evolution equations of a superfluid in the presence of a gravitational field. We denote the gravitational potential by ϕ. It satisfies Gauss's law,

$$\nabla^2 \phi = 4\pi G \rho. \tag{6}$$

2.1 Self-gravitating superfluid hydrodynamics

We will consider an ideal superfluid. The superfluid density current reads

$$\vec{j} = \rho_n \vec{v}_n + \rho_s \vec{v}_s, \tag{7}$$

and it satisfies a continuity equation, which is not affected by the gravitational field

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0. \tag{8}$$

Similarly, the entropy conservation is not affected by the gravitational field and reads

$$\frac{\partial}{\partial t}(\rho s) + (\rho s)\vec{\nabla} \cdot \vec{v}_n = 0. \tag{9}$$

Note that entropy is carried only by the normal flow and not by the super flow.

The energy and momentum conservation equations, however, are modified in the presence of the gravitational field and take the form

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} + \rho \frac{\partial \phi}{\partial x_i} = 0, \tag{10}$$

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot \vec{Q} + \vec{j} \cdot \vec{\nabla}\phi = 0. \tag{11}$$

Π_{ik} is the momentum flux tensor and $\vec{Q}$ is the energy flux as calculated without a gravitational field in [6].

In order to derive the modified equations in the presence of the gravitational field, one uses thermodynamics and the Galilean principle [6]. One denotes by K_0 the reference frame, where the super flow velocity is zero. The velocity of the normal flow in this frame is $\vec{v}_n - \vec{v}_s$. In this frame one has

$$E_0 = -p + T\rho s + \mu\rho + \phi\rho + (\vec{v}_n - \vec{v}_s) \cdot \vec{j}_0 \tag{12}$$

and

$$d(\mu + \phi) = -s dT + \frac{1}{\rho}dp - \frac{\rho_n}{\rho}(\vec{v}_n - \vec{v}_s)d(\vec{v}_n - \vec{v}_s). \tag{13}$$

E_0 and $\vec{j}_0$ are the energy and density current in the system K_0, respectively. There are two effects of the gravitational field. First, to replace the chemical potential μ by $\mu + \phi$. Second, to introduce a new term in the fluid flow: in addition to the force term $-\vec{\nabla}p$, we have $-\rho\vec{\nabla}\phi$.

Finally, the super flow being a potential flow satisfies

$$\frac{\partial \vec{v}_s}{\partial t} + \vec{\nabla}\left(\frac{1}{2}\vec{v}_s^2 + \mu + \phi\right) = 0. \tag{14}$$

It is straightforward to see that the hydrodynamics equations together with the thermodynamic relations constitute a complete set that determines all the charge densities and velocities.

2.2 Self-gravitating superfluid sound

Linearizing the above ideal superfluid hydrodynamics equations in a presence of the gravitational field we get

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

$$\frac{\partial}{\partial t}(\rho s) + (\rho s)\vec{\nabla} \cdot \vec{v}_n = 0$$

$$\frac{\partial \vec{j}}{\partial t} + \vec{\nabla}p + \rho\vec{\nabla}\phi = 0 \tag{15}$$

$$\frac{\partial \vec{v}_s}{\partial t} + \vec{\nabla}\mu + \vec{\nabla}\phi = 0,$$

from which we derive

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p + \rho\nabla^2\phi,$$

$$\frac{\partial^2 s}{\partial t^2} = \frac{\rho_s}{\rho_n}s^2\nabla^2 T + \frac{\rho_s}{\rho_n}s\nabla^2\phi. \tag{16}$$

Perturbing[1] around (16), using Gauss's law for the gravitational potential, and expressing all quantities in terms of the pressure p and the temperature T we get

$$\frac{\partial \rho}{\partial p}\frac{\partial^2 \delta p}{\partial t^2} + \frac{\partial \rho}{\partial T}\frac{\partial^2 \delta T}{\partial t^2} = \nabla^2 \delta p + 4\pi G\rho\left(\frac{\partial \rho}{\partial p}\delta p + \frac{\partial \rho}{\partial T}\delta T\right)$$

$$\frac{\partial s}{\partial p}\frac{\partial^2 \delta p}{\partial t^2} + \frac{\partial s}{\partial T}\frac{\partial^2 \delta T}{\partial t^2}$$

$$= \frac{\rho_s}{\rho_n}s^2\nabla^2\delta T + 4\pi G\frac{\rho_s}{\rho_n}s\left(\frac{\partial \rho}{\partial p}\delta p + \frac{\partial \rho}{\partial T}\delta T\right). \tag{17}$$

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[1] As in the ordinary Jeans instability analysis, we perturb around a stable state and therefore ignore the zeroth order contribution of the Laplacian of the gravitational potential $\nabla^2\phi_0$.

This system of equations for a wave of the form $\exp(i\vec{k} \cdot \vec{x} - i\omega t)$ reads

$$\left(\frac{\partial \rho}{\partial p}(\omega^2 + 4\pi G\rho) - \vec{k}^2\right)\delta p + \frac{\partial \rho}{\partial T}(\omega^2 + 4\pi G\rho)\delta T = 0$$

$$\left(\frac{\partial s}{\partial p}\omega^2 + \frac{\partial \rho}{\partial p} \cdot 4\pi G\frac{\rho_s}{\rho_n}s\right)\delta p$$
$$+ \left(\frac{\partial s}{\partial T}\omega^2 - \frac{\rho_s}{\rho_n}s^2\vec{k}^2 + \frac{\partial \rho}{\partial T} \cdot 4\pi G\frac{\rho_s}{\rho_n}s\right)\delta T = 0. \quad (18)$$

In the absence of a gravitational field ($G \rightarrow 0$) the equations reduce to the well-known superfluid sound modes [6].

In the general case, for a solution of these system of equations we require the determinant to vanish,

$$H\omega^4 + \left(4\pi GH\rho - \left(\frac{\partial \rho}{\partial p}\frac{\rho_s}{\rho_n}s^2 + \frac{\partial s}{\partial T}\right)\vec{k}^2\right)\omega^2$$
$$+ \left(\frac{\rho_s}{\rho_n}s^2\vec{k}^2 - 4\pi G\frac{\rho_s}{\rho_n}s\left(\frac{\partial \rho}{\partial p}\rho s + \frac{\partial \rho}{\partial T}\right)\right)\vec{k}^2 = 0, \quad (19)$$

where $H = \frac{\partial(\rho, s)}{\partial(p, T)}$ is the Jacobian for changing variables from (ρ, s) to (p, T), and we assume that $H > 0$. Solving for ω^2 we get

$$2H\omega^2 = -\left(4\pi GH\rho - \left(\frac{\partial \rho}{\partial p}\frac{\rho_s}{\rho_n}s^2 + \frac{\partial s}{\partial T}\right)\vec{k}^2\right)$$
$$\pm \sqrt{\left(4\pi GH\rho + \left(\frac{\partial \rho}{\partial p}\frac{\rho_s}{\rho_n}s^2 - \frac{\partial s}{\partial T}\right)\vec{k}^2\right)^2 + 4\frac{\partial \rho}{\partial T}\frac{\rho_s}{\rho_n}s\vec{k}^2\left(4\pi GH + \frac{\partial s}{\partial p}s\vec{k}^2\right)}. \quad (20)$$

In the absence of a gravitational field, we have the plus and minus sign solutions reducing to first and second sound, respectively.

Consider the leading effect of the super flow in the dimensionless parameter $\frac{\rho_s}{\rho_n} \ll 1$. At first order we get

$$2H\omega^2 = -\left(4\pi GH\rho - \frac{\partial s}{\partial T}\vec{k}^2\right) \pm \left(4\pi GH\rho - \frac{\partial s}{\partial T}\vec{k}^2\right)$$
$$+ \frac{\rho_s}{\rho_n}\left(\frac{\partial \rho}{\partial p}s^2\vec{k}^2 \pm \frac{\partial \rho}{\partial p}s^2\vec{k}^2 \pm \frac{2\frac{\partial \rho}{\partial T}s\vec{k}^2\left(4\pi GH + \frac{\partial s}{\partial p}s\vec{k}^2\right)}{\left(4\pi GH\rho - \frac{\partial s}{\partial T}\vec{k}^2\right)}\right). \quad (21)$$

Therefore, we have the first sound solution

$$H\omega^2 = -\left(4\pi GH\rho - \frac{\partial s}{\partial T}\vec{k}^2\right)$$
$$- \frac{\rho_s}{\rho_n}\left(\frac{\frac{\partial \rho}{\partial T}s\vec{k}^2\left(4\pi GH + \frac{\partial s}{\partial p}s\vec{k}^2\right)}{\left(4\pi GH\rho - \frac{\partial s}{\partial T}\vec{k}^2\right)}\right) \quad (22)$$

and the second sound solution

$$H\omega^2 = \frac{\rho_s}{\rho_n}\left(\frac{\partial \rho}{\partial p}s^2\vec{k}^2 + \frac{\frac{\partial \rho}{\partial T}s\vec{k}^2\left(4\pi GH + \frac{\partial s}{\partial p}s\vec{k}^2\right)}{\left(4\pi GH\rho - \frac{\partial s}{\partial T}\vec{k}^2\right)}\right). \quad (23)$$

2.3 Comparison to the non-gravitational case

It is instructive to compare special solutions to the superfluid sound equations in the presence of the gravitational interaction (18) to the non-gravitating case.

2.3.1 The pure second sound limit

In the limit $\frac{\partial \rho}{\partial T} = 0$, the non-gravitational sound equations (setting $G = 0$ in (18)) read

$$\left(\frac{\partial \rho}{\partial p}\omega^2 - \vec{k}^2\right)\delta p = 0$$
$$\left(\frac{\partial s}{\partial p}\omega^2\right)\delta p + \left(\frac{\partial s}{\partial T}\omega^2 - \frac{\rho_s}{\rho_n}s^2\vec{k}^2\right)\delta T = 0. \quad (24)$$

One has a solution where $\delta p = 0$, δT arbitrary, and the dispersion relation (4) of the pure second sound describing the temperature wave. There is another solution, where the dispersion relation is that of a first sound mode, and the pressure and temperature variations are related by

$$\left(\frac{\partial s}{\partial p}\right)\delta p = \left(\frac{\rho_s}{\rho_n}s^2\frac{\partial \rho}{\partial p} - \frac{\partial s}{\partial T}\right)\delta T. \quad (25)$$

When we introduce gravity, still having $\frac{\partial \rho}{\partial T} = 0$, we get

$$\left(\frac{\partial \rho}{\partial p}(\omega^2 + 4\pi G\rho) - \vec{k}^2\right)\delta p = 0$$
$$\left(\frac{\partial s}{\partial p}\omega^2 + \frac{\partial \rho}{\partial p} \cdot 4\pi G\frac{\rho_s}{\rho_n}s\right)\delta p + \left(\frac{\partial s}{\partial T}\omega^2 - \frac{\rho_s}{\rho_n}s^2\vec{k}^2\right)\delta T = 0. \quad (26)$$

We see that the pure second sound solution (4) is not affected by gravity. This is expected since the pure second sound is characterized by no variation of the density, thus gravity does not affect it.

However, the first sound solution is changed. The dispersion relation is the same as the Jeans dispersion relation (1), but the pressure and temperature variations are related by a

different formula,

$$\left(\frac{\partial s}{\partial p}\omega^2 + \frac{\partial \rho}{\partial p} \cdot 4\pi G \frac{\rho_s}{\rho_n} s\right)\delta p$$
$$= \left(\left(\frac{\partial \rho}{\partial p}\frac{\rho_s}{\rho_n} s^2 - \frac{\partial s}{\partial T}\right)\omega^2 + \frac{\partial \rho}{\partial p} \cdot 4\pi G \frac{\rho_s}{\rho_n}\rho s^2\right)\delta T. \quad (27)$$

Comparing (25) and (27), we see a frequency dependence of the linear relation between pressure and temperature (27), which vanishes as $\rho_s \to 0$.

2.3.2 The pure first sound limit

Consider next the classical pure first sound limit, $\frac{\partial s}{\partial p} = 0$. The non-gravitational sound equations in this case are (setting $G = 0$ and $\frac{\partial s}{\partial p} = 0$ in (18)):

$$\left(\frac{\partial \rho}{\partial p}\omega^2 - \vec{k}^2\right)\delta p + \left(\frac{\partial \rho}{\partial T}\omega^2\right)\delta T = 0$$
$$\left(\frac{\partial s}{\partial T}\omega^2 - \frac{\rho_s}{\rho_n} s^2\vec{k}^2\right)\delta T = 0. \quad (28)$$

We have a solution where $\delta T = 0$, δp arbitrary, and the dispersion relation of the pure first sound. There is another solution, where the dispersion relation is (4), and the pressure and temperature variations are related by

$$\left(\frac{\partial s}{\partial T}\frac{\rho_n}{\rho_s}\frac{1}{s^2} - \frac{\partial \rho}{\partial p}\right)\delta p = \left(\frac{\partial \rho}{\partial T}\right)\delta T. \quad (29)$$

When we introduce gravity, still with $\frac{\partial s}{\partial p} = 0$, we get

$$\left(\frac{\partial \rho}{\partial p}(\omega^2 + 4\pi G\rho) - \vec{k}^2\right)\delta p + \frac{\partial \rho}{\partial T}\left(\omega^2 + 4\pi G\rho\right)\delta T = 0$$
$$\left(\frac{\partial \rho}{\partial p} \cdot 4\pi G \frac{\rho_s}{\rho_n} s\right)\delta p + \left(\frac{\partial s}{\partial T}\omega^2 - \frac{\rho_s}{\rho_n} s^2\vec{k}^2\right.$$
$$\left. + \frac{\partial \rho}{\partial T} \cdot 4\pi G \frac{\rho_s}{\rho_n} s\right)\delta T = 0. \quad (30)$$

We see that the pure first sound solution is not pure anymore. This is in contrast to the pure second sound solution, which was not modified by gravity.

2.4 General instability conditions

In the following we will analyze the instability conditions arising from the roots (20).

2.4.1 Identical roots

Consider the case where the two solutions (20) are identical. In this case we have the same dispersion relation for the first and second sound and we obtain the instability condition

$$\vec{k}^2 < \frac{4\pi GH\rho}{\left(\frac{\partial \rho}{\partial p}\frac{\rho_s}{\rho_n} s^2 + \frac{\partial s}{\partial T}\right)}. \quad (31)$$

Dividing by H, we can recast (31) as (5). The constraint (5) is a generalization of (2) to superflows, and it is valid for both the first and the second sounds. It reduces to (2) in the limit $\rho_s \to 0$. In limit $T \to 0$, $u_1^2 = 3u_2^2$ [6], and one can see from the data presented in [7] that $J \to 0$. Thus we see that the Jeans length of the superfluid is larger by a factor of $\frac{2}{\sqrt{3}}$ than that of a normal fluid.

2.4.2 Different roots

The conditions for instability of the two sound modes are the upper bound on the momentum (5) as well as a lower bound

$$\vec{k}^2 > 4\pi G\rho\left(\frac{\partial \rho}{\partial p} + \frac{\partial \rho}{\partial T}\frac{1}{\rho s}\right), \quad (32)$$

which together imply a constraint on the system independent of the gravitational field,

$$\left(\frac{\partial \rho}{\partial p} + \frac{\partial \rho}{\partial T}\frac{1}{\rho s}\right)\left(\frac{\partial \rho}{\partial p}\frac{\rho_s}{\rho_n} s^2 + \frac{\partial s}{\partial T}\right) < H. \quad (33)$$

2.5 Discussion

As the ordinary Jeans instability has been observed experimentally, it is natural to inquire how can we observe the generalized instability conditions for superfluids. It is probably unlikely that current condensed matter experiments can detect the gravitational instability of superfluids. A potential experimental laboratory could be neutron stars. Superfluidity is expected to occur both in the inner crust and the core of neutron stars as a consequence of the formation of Cooper pairs of neutrons that break spontaneously the $U(1)$ Baryon symmetry [3]. Superfluidity has been invoked in order to explain, for instance, pulsar glitches [8]. Therefore the gravitational instability seems relevant to the dynamics of neutron stars, such as its rotational and vibrational oscillations. The fluid velocities in neutron stars can be small compared to the speed of light. For instance, the velocity at the equator of the most rapidly spinning neutron stars is about 20 % of the speed of light, making the non-relativistic two fluid model of superfluid hydrodynamics a useful description (see e.g. [9]). It would, therefore, be interesting to work out the details of the gravitational instability effects in this framework. It would also be of interest to generalize our work to relativistic superfluid flows.

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