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Lagrangian perturbation theory: exact one-loop power spectrum in general dark energy models

Seokcheon Lee[a]

School of Physics, Korea Institute for Advanced Study, Heogiro 85, Seoul 130-722, Korea

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Abstract Recently, we found that the correction for the Einstein–de Sitter (EdS) assumption on the one-loop matter power spectrum for general dark energy models using the standard perturbation theory is not negligible (Lee et al., arXiv:1407.7325, 2014). Thus, we investigate the same problem by obtaining the exact displacement vector and kernels up to the third order for the general dark energy models in the Lagrangian perturbation theory (LPT). Using these exact solutions, we investigate the present one-loop matter power spectrum in the ΛCDM model with $\Omega_{m0} = 0.25(0.3)$ to obtain a 0.2(0.18) % error correction compared to that obtained from the EdS assumption for the $k = 0.1\,\mathrm{h\,Mpc}^{-1}$ mode. If we consider the total matter power spectrum, the correction is only 0.05(0.03) % for the same mode. It means that the EdS assumption is a good approximation for the ΛCDM model in LPT theory. However, one can use this method for general models where the EdS assumption is improper.

With the upcoming precision measurements of the large scale structure, accurate theoretical modeling is essential to interpret the observational data. It requires a huge number of mock catalogs and N-body simulations are too numerically expensive to be done. Fortunately, it seems that observable quantities at the quasi-linear scales might be accurately modeled semi-analytically. The Lagrangian perturbation theory (LPT) has been widely used to investigate this [2–6]. Also, the initial condition for the N-body simulation are generated using LPT [7–9].

In LPT, the fundamental object is the Lagrangian displacement vector $\mathbf{S}$, which displaces the particle from its initial position $\mathbf{q}$ to the final Eulerian position $\mathbf{x}$,

$$\mathbf{x}(\mathbf{q}, t) = \mathbf{q} + \mathbf{S}(\mathbf{q}, t). \tag{1}$$

The first order LPT solution is the Zel'dovich approximation [10] and higher order solutions have been obtained [11–15]. From the mass conservation, the matter density perturbation δ can be described by a function of $\mathbf{S}$,

$$\delta(\mathbf{x}, t) = \int d^3q\, \delta_D\Big(\mathbf{x} - \mathbf{q} - \mathbf{S}(\mathbf{q}, t)\Big) - 1. \tag{2}$$

One can expand the displacement vector $\mathbf{S}$ according to the Lagrangian perturbative prescription

$$\begin{aligned}\mathbf{S}(\mathbf{q}, t) &\equiv \sum_{n=1} \mathbf{S}^{(n)}(\mathbf{q}, t) = \sum_{n=1} D_{(n)}(t)\mathbf{S}^{(n)}(\mathbf{q}) \\ &= \sum_{n=1} I_n(t) D^n(t) \mathbf{S}^{(n)}(\mathbf{q}) \\ &\equiv D(t)\mathbf{S}^{(1)}(\mathbf{q}) + E(t)\mathbf{S}^{(2)}(\mathbf{q}) \\ &\quad + F_a(t)\mathbf{S}^{(3a)}(\mathbf{q}) + F_b(t)\mathbf{S}^{(3b)}(\mathbf{q}) + \cdots .\end{aligned} \tag{3}$$

This explicit separation with respect to the spatial and temporal coordinates for each order (i.e. I_n is a constant) is known to be a property of the perturbative Lagrangian description for an Einstein–de Sitter (EdS) universe [13]. However, the solution at each order can be a separable function of t and $\mathbf{q}$ even for general dark energy models by using $D_{(n)}(t)$ instead of $D^{(n)}(t)$. After one includes the time dependence of I_n in the each kernel, one can find the exact solution for each order. One can use $I_1 = 1$, $D_1 = D$ where $D_1(t)$ is the linear growth factor, and the $D_{(n)}(t) = I_n(t)D^n(t)$ are specified as

$$D_{(2)}(t) \equiv E(t) = I_2(t) D^2(t), \tag{4}$$
$$D_{(3a)}(t) \equiv F_a(t) = I_{3a}(t) D^3(t), \tag{5}$$
$$D_{(3b)}(t) \equiv F_b(t) = I_{3b}(t) D^3(t). \tag{6}$$

From (4)–(6), one can obtain the Lagrangian Poisson equation order by order (from the linear to the irrotational third orders),

$$\ddot{D} + 2H\dot{D} - 4\pi G\rho_m D = 0, \tag{7}$$

$$\ddot{E} + 2H\dot{E} - 4\pi G\rho_m E = -4\pi G\rho_m D^2, \quad \text{if } \mu_1(\mathbf{S}^{(2)}) = \mu_2(\mathbf{S}^{(1)}, \mathbf{S}^{(1)}), \tag{8}$$

[a] e-mail: skylee@kias.re.kr

$$\ddot{F}_a + 2H\dot{F}_a - 4\pi G\rho_m F_a = -8\pi G\rho_m D^3, \quad \text{if } \mu_1(\mathbf{S}^{(3a)}) = \mu_3(\mathbf{S}^{(1)}), \tag{9}$$

$$\ddot{F}_b + 2H\dot{F}_b - 4\pi G\rho_m F_b = -8\pi G\rho_m D(E - D^2), \quad \text{if } \mu_1(\mathbf{S}^{(3b)}) = \mu_2(\mathbf{S}^{(1)}, \mathbf{S}^{(2)}), \tag{10}$$

where dots represent the derivatives with respect to the cosmic time t and $\mu_2(S^{(1)}, S^{(2)}) = \mu_2(S^{(2)}, S^{(1)})$ is satisfied for any tensor [14]. $\mu_a(\mathbf{S}^{(n)})$ are defined as

$$\mu_1(\mathbf{S}^{(n)}) \equiv S^{(n)}_{ii}, \tag{11}$$

$$\mu_2(\mathbf{S}^{(n)}, \mathbf{S}^{(m)}) \equiv \frac{1}{2}\left(S^{(n)}_{ii}S^{(m)}_{jj} - S^{(n)}_{ij}S^{(m)}_{ji}\right), \tag{12}$$

$$\mu_3(\mathbf{S}^{(n)}) \equiv \det S^{(n)}_{ij}. \tag{13}$$

One can rewrite Eq. (3) in Fourier space, represented by using the linear matter density contrast, $\widetilde{\delta}_L(p)$,

$$\begin{aligned}
\widetilde{\mathbf{S}}^{(n)}(\mathbf{k}, t) &= -iD_{(n)}(t)\int \frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_n}{(2\pi)^3}(2\pi)^3\delta_D \\
&\quad\times(\mathbf{p}_{1\dots n} - \mathbf{k})\mathbf{F}^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n)\widetilde{\delta}_L(p_1)\dots\widetilde{\delta}_L(p_n) \\
&= -i\frac{D^n(t)}{n!}\int \frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_n}{(2\pi)^3}(2\pi)^3\delta_D \\
&\quad\times(\mathbf{p}_{1\dots n} - \mathbf{k})n!I_n(t)\mathbf{F}^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n)\widetilde{\delta}_L(p_1)\dots\widetilde{\delta}_L(p_n) \\
&\equiv -i\frac{D^n(t)}{n!}\int \frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_n}{(2\pi)^3}(2\pi)^3\delta_D \\
&\quad\times(\mathbf{p}_{1\dots n} - \mathbf{k})\mathbf{L}^{(n)}(t, \mathbf{p}_1, \dots, \mathbf{p}_n)\widetilde{\delta}_L(p_1)\dots\widetilde{\delta}_L(p_n).
\end{aligned} \tag{14}$$

In the second equality, we adopt the same notation as in [2–4]. We also use $\mathbf{p}_{1\dots n} = \mathbf{p}_1 + \cdots + \mathbf{p}_n$, $\mathbf{F}^{(n)} = (-1)^n \frac{\mathbf{p}_{1\dots n}}{p^2_{1\dots n}}\frac{\kappa^{(n)}(\mathbf{p}_1,\dots,\mathbf{p}_n)}{p_1^2\cdots p_n^2}$, and the nth order kernels $\kappa^{(n)}$ are the same as given in [13,14]. We neglect the transverse terms in $\mathbf{F}^{(n)}$, which do not appear in a one-loop correction. $I_n(a)$ should be obtained numerically from Eqs. (8)–(10) by using the EdS initial conditions given by Eqs. (31)–(33). In the literature, one uses the coefficients for the EdS solutions,

$$I_2 = -\frac{3}{7}, \quad I_{3a} = -\frac{1}{3}, \quad I_{3b} = \frac{10}{21}. \tag{15}$$

However, these values are approximate ones using the EdS assumption and we will use the exact values of them.

From (14), one can obtain the perturbative kernels in LPT up to the third order

$$L^{(1)}(\mathbf{p}_1) = \frac{\mathbf{k}}{k^2}, \tag{16}$$

$$L^{(2)}(a, \mathbf{p}_1, \mathbf{p}_2) = I_2(a)\frac{\mathbf{k}}{k^2}\left[1 - \left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{p_1p_2}\right)^2\right], \tag{17}$$

$$L^{(3a)}(a, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = I_{3a}(a)\frac{\mathbf{k}}{k^2}\left[1 - 3\left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{p_1p_2}\right)^2 + 2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{p}_2\cdot\mathbf{p}_3)(\mathbf{p}_3\cdot\mathbf{p}_1)}{p_1^2p_2^2p_3^2}\right], \tag{18}$$

$$L^{(3b)}(a, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = I_{3b}(a)\frac{\mathbf{k}}{k^2}\left[1 - \left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{p_1p_2}\right)^2\right]\times\left[1 - \left(\frac{(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{p}_3}{|p_1+p_2|p_3}\right)^2\right]. \tag{19}$$

The above kernels are identical to those of EdS given in [2,14] when I_2–I_{3b} are given by Eq. (15).

From the above consideration, one can obtain the non-linear power spectrum with one-loop correction by using a resummation scheme known as integrated perturbation theory [2,3],

$$\begin{aligned}
P(k) &= \exp\left[-k_ik_j\int\frac{d^3p}{(2\pi)^3}C_{ij}(\mathbf{p})\right] \\
&\quad\times\left[k_ik_jC_{ij}(\mathbf{k}) + k_ik_jk_k\int\frac{d^3p}{(2\pi)^3}C_{ijk}(\mathbf{k}, -\mathbf{p}, \mathbf{p}-\mathbf{k})\right. \\
&\quad\left.+\frac{1}{2}k_ik_jk_kk_l\int\frac{d^3p}{(2\pi)^3}C_{ij}(\mathbf{p})C_{kl}(\mathbf{k}-\mathbf{p})\right]
\end{aligned} \tag{20}$$

where the mixed polyspectrum of the linear density field and the displacement field is defined as

$$\begin{aligned}
&\left\langle\widetilde{\delta}_L(\mathbf{k}_1)\dots\widetilde{\delta}_L(\mathbf{k}_l)\widetilde{S}_{i_1}(\mathbf{p}_1)\dots\widetilde{S}_{i_m}(\mathbf{p}_m)\right\rangle_c \\
&\quad = (2\pi)^3\delta_D(\mathbf{k}_{1\dots l} + \mathbf{p}_{1\dots m})(-\imath)^mC_{i_i\dots i_m} \\
&\quad\times(\mathbf{k}_1, \dots, \mathbf{k}_l; \mathbf{p}_1, \dots, \mathbf{p}_m).
\end{aligned} \tag{21}$$

Also the mixed polyspectrum of each order in perturbations is given by

$$\begin{aligned}
&\left\langle\widetilde{\delta}_L(\mathbf{k}_1)\dots\widetilde{\delta}_L(\mathbf{k}_l)\widetilde{S}^{(n_1)}_{i_1}(\mathbf{p}_1)\dots\widetilde{S}^{(n_m)}_{i_m}(\mathbf{p}_m)\right\rangle_c \\
&\quad = (2\pi)^3\delta_D(\mathbf{k}_{1\dots l} + \mathbf{p}_{1\dots m})(-\imath)^mC^{(n_1\dots n_m)}_{i_i\dots i_m} \\
&\quad\times(\mathbf{k}_1, \dots, \mathbf{k}_l; \mathbf{p}_1, \dots, \mathbf{p}_m).
\end{aligned} \tag{22}$$

From Eqs. (20) and (22), one can obtain the matter power spectrum with one-loop correction,

$$\begin{aligned}
P(k) &= \exp\left[-k_ik_j\int\frac{d^3p}{(2\pi)^3}C^{(11)}_{ij}(\mathbf{p})\right] \\
&\quad\times\left(k_ik_j\left[C^{(11)}_{ij}(\mathbf{k}) + C^{(22)}_{ij}(\mathbf{k}) + C^{(13)}_{ij}(\mathbf{k}) + C^{(31)}_{ij}(\mathbf{k})\right]\right. \\
&\quad+k_ik_jk_k\int\frac{d^3p}{(2\pi)^3}\left[C^{(112)}_{ijk}(\mathbf{k}, -\mathbf{p}, \mathbf{p}-\mathbf{k})\right. \\
&\quad\left.+C^{(121)}_{ijk}(\mathbf{k}, -\mathbf{p}, \mathbf{p}-\mathbf{k}) + C^{(211)}_{ijk}(\mathbf{k}, -\mathbf{p}, \mathbf{p}-\mathbf{k})\right] \\
&\quad\left.+\frac{1}{2}k_ik_jk_kk_l\int\frac{d^3p}{(2\pi)^3}C^{(11)}_{ij}(\mathbf{p})C^{(11)}_{ij}(\mathbf{k}-\mathbf{p})\right).
\end{aligned} \tag{23}$$

After the analytic angular integration of Eq. (23), one obtains

$$\begin{aligned}
P(k) &= \exp\Big[-\frac{k^2}{6\pi^2}\int dp\, P_L(p)\Big] \\
&\quad\times\Big[P_L(k) + \frac{(2\pi)^{-2}k^3}{2}\int_0^\infty \mathrm{d}r\, P_L(kr) \\
&\quad\times\int_{-1}^{1}\mathrm{d}x\, P_L\Big(k\sqrt{1+r^2-2rx}\Big) \\
&\quad\times\Big[\frac{-I_2 r + x - (1-I_2)rx^2}{(1+r^2-2rx)}\Big]^2 \\
&\quad+\frac{(2\pi)^{-2}k^3}{48}P_L(k)\int_0^\infty \mathrm{d}r\, P_L(kr) \\
&\quad\times\Big(-6(2I_2+I_{3b})r^{-2} + 2(10I_2+11I_{3b}) \\
&\quad+2(-10I_2+11I_{3b})r^2 + 6(2I_2-I_{3b})r^4 + \frac{3}{r^3}(r^2-1)^3 \\
&\quad\times\Big((-2I_2+I_{3b})r^2 - (2I_2+I_{3b})\Big)\ln\Big|\frac{1+r}{1-r}\Big|\Big)\Big] \\
&\equiv \exp\Big[-\frac{k^2}{6\pi^2}\int dp\, P_L(p)\Big]\Big[P_L(k)+P_{22}(k)+P_{13}(k)\Big] \\
&\equiv \exp\Big[-\frac{k^2}{6\pi^2}\int dp\, P_L(p)\Big]\Big[P_{\rm NL}(k)\Big], \qquad (24)
\end{aligned}$$

where $r = \frac{p}{k}$ and $x = \frac{\mathbf{p}\cdot\mathbf{k}}{pk}$. The above equations are identical to Eqs. (36) of [2] when one replaces I_2 and I_{3b} with those of the EdS case given by Eq. (15). Thus, the terms with I_2 and I_{3b} represent the dark energy effect on the power spectrum. Also, both I_2 and I_{3b} depend on the time and their values are changed depending on the measuring epoch. One interesting feature is that I_{3a} does not contribute the one-loop correction in the matter power spectrum. When we generalize the power spectrum in the SPT without using the EdS assumption, we obtain a similar matter power spectrum to [1]:

$$\begin{aligned}
P^{\rm SPT}(k) &= P_L(k) + \frac{(2\pi)^{-2}k^3}{2}\int_0^\infty \mathrm{d}r\, P_L(kr) \\
&\quad\times\int_{-1}^{1}\mathrm{d}x\, P_L\Big(k\sqrt{1+r^2-2rx}\Big) \\
&\quad\times\Big[\frac{(c_{21}+2c_{22})r + (c_{21}-2c_{22})x - 2c_{21}rx^2}{(1+r^2-2rx)}\Big]^2 \\
&\quad+(2\pi)^{-2}k^3 P_L(k)\int_0^\infty \mathrm{d}r\, P_L(kr) \\
&\quad\times\Big[2c_{35}r^{-2} - \frac{1}{3}\Big(4c_{31}-8c_{32}+3c_{33}+24c_{35}-16c_{36}\Big) \\
&\quad-\frac{1}{3}\Big(4c_{31}-8c_{32}+12c_{33}-8c_{34}+6c_{35}\Big)r^2 \\
&\quad+c_{33}r^4 + \Big(\frac{r^2-1}{r}\Big)^3\ln\Big|\frac{1+r}{1-r}\Big|\Big(c_{35}-\frac{1}{2}c_{33}r^2\Big)\Big],
\end{aligned} \qquad (25)$$

where c_{21}–c_{36} are also given in the above reference. If we adopt the EdS assumption both for the LPT matter power spectrum and the SPT one, then P_{22} is the same for both approaches. However, the exact solutions will not be matched exactly for both cases. Also compared to SPT case where the magnitude of P_{13} is comparable to that to P_{13}, the magnitude of P_{13} is much smaller than that of P_{22} in LPT. This shows that the EdS approximation in LPT is quite accurate.

Now we obtain the one-loop power spectrum for ΛCDM model. We run the camb to obtain the linear power spectrum [16] using $\Omega_{b0} = 0.044$, $\Omega_{m0} = 0.26$, $h = 0.72$, $n_s = 0.96$, and the numerical integration range for p in Eq. (24) is $10^{-6} \le p \le 10^2$. In the left panel of Fig. 1, we show the linear power spectrum P_L (solid), the one-loop power spectrum P_{22} (dotted), $|P_{13}|$ (dashed), and the nonlinear power spectrum $P_{\rm NL} = P_L + P_{\text{1-loop}}$ (dot-dashed). Absolute magnitude of P_{13} is smaller than that of SPT. Thus, the one-loop correction is larger than that of SPT. The one-loop correction is mainly contributed from the P_{22} and the coefficient I_2 is not much deviant from that of EdS ($-\frac{3}{7}$) as shown in the appendix. That is why the EdS approximation is a good one for LPT. Also, there exists an additional exponential prefactor to get the total power spectrum. This is shown in the right panel of Fig. 1. The dot-dashed line represents $P_{\rm NL}$ and the dotted line indicates $P(k)$.

Now we investigate the effect of dark energy on the one-loop power spectrum compared to the one with EdS assumption. The difference in $P_{22} + P_{13}$ between them is shown in the left panel of Fig. 2. There exists only 0.2 % error in the $k = 0.1\,\mathrm{h\,Mpc}^{-1}$ mode at the present epoch. When we investigate them at the different zs, then the error is about the same. The error can be about 2 % at large scales but the one-loop power spectrum is much smaller than the linear power spectrum at these scales. When we consider the total $P_{\rm NL}$, the difference is even smaller. The error is less than 0.05 % for the same mode. This is shown in the right panel of Fig. 2 with the notation $\Delta P_{\rm NL} = P_{\rm NL} - P_{\rm NL}^{\rm (EdS)}$. This proves the goodness of the EdS assumption in LPT claimed in [17]. However, we need to pay attention to this EdS assumption when we consider more general models.

We show that the EdS assumption is a good approximation to calculate the ΛCDM one-loop power spectrum in Lagrangian perturbation theory. However, when we consider general dark energy models we need to consider the fully consistent method by using the fact I_n depends on time. This also makes it possible to separate the temporal and spatial parts of the solutions. We might be able to extend this method to the early dark energy or the modified gravity theories. The upcoming redshift surveys will provide observational data of the large scale structure of the universe in a larger volume with a higher density. We obtain the accurate Lagrangian perturbation theory without using any assumption and this matches the requirement from future surveys. The obtained

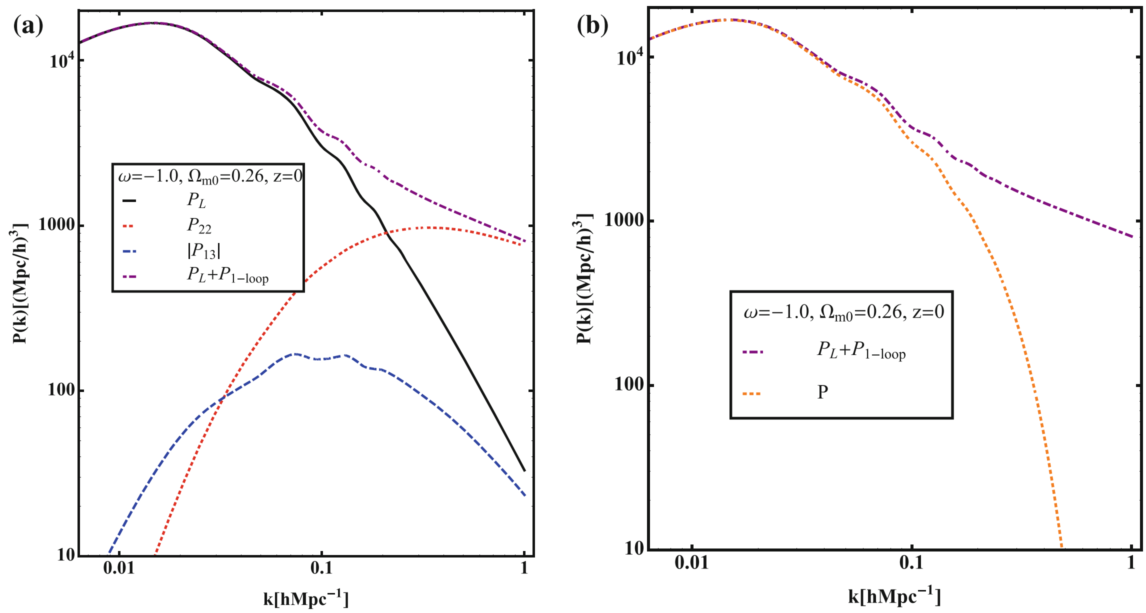

Fig. 1 $P_{\rm NL}(k)$ and $P(k)$. **a** *Solid*, *dotted*, *dashed*, and *dot-dashed line* represent P_L, P_{22}, $|P_{13}|$, and P_{NL}, respectively. **b** $P_{\rm NL}$ and $P(k)$ are indicated as *dot-dahed* and *dotted lines*, respectively

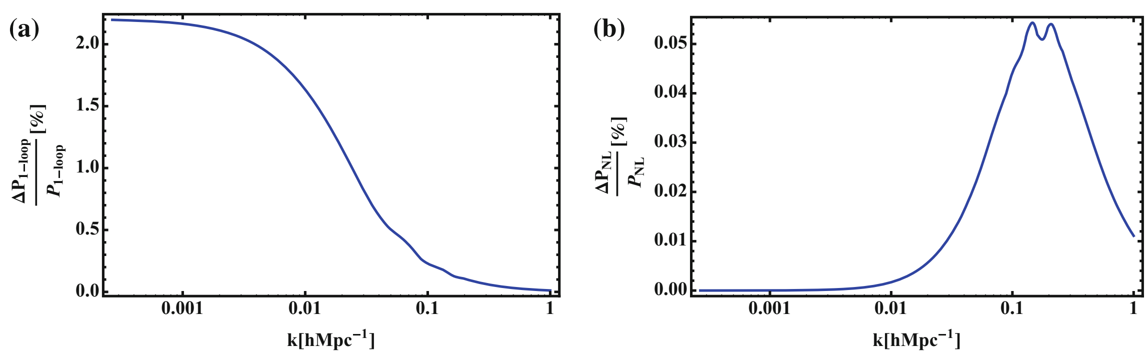

Fig. 2 Errors in $P_{1\text{-loop}}$ and $P_{\rm NL}$. **a** The percentage difference between the correct $P_{1\text{-loop}}$ and the one with EdS assumption. **b** The percentage difference between $P_{\rm NL}$ and $P_{\rm NL}^{\rm (EdS)}$

results are general for any background universe model including time varying dark energy models.

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Appendix

We need to obtain $I_n(a)$ of each order solution to calculate the higher order power spectrum. This can be obtained from Eqs. (7)–(10) by using the proper initial conditions. One can rewrite the above equations by using the scale factor a,

$$\frac{\mathrm{d}^2 D}{\mathrm{d}a^2} + \frac{3}{2a}\left(1 - w\Omega_{\rm DE}\right)\frac{\mathrm{d}D}{\mathrm{d}a} - \frac{3\Omega_{\rm m}}{2a^2}D = 0, \tag{26}$$

$$\frac{\mathrm{d}^2 E}{\mathrm{d}a^2} + \frac{3}{2a}\left(1 - w\Omega_{\rm DE}\right)\frac{\mathrm{d}E}{\mathrm{d}a} - \frac{3\Omega_{\rm m}}{2a^2}E = -\frac{3\Omega_{\rm m}}{2a^2}D^2, \tag{27}$$

$$\frac{\mathrm{d}^2 F_a}{\mathrm{d}a^2} + \frac{3}{2a}\left(1 - w\Omega_{\rm DE}\right)\frac{\mathrm{d}F_a}{\mathrm{d}a} - \frac{3\Omega_{\rm m}}{2a^2}F_a = -\frac{3\Omega_{\rm m}}{a^2}D^3, \tag{28}$$

$$\frac{\mathrm{d}^2 F_b}{\mathrm{d}a^2} + \frac{3}{2a}\left(1 - w\Omega_{\rm DE}\right)\frac{\mathrm{d}F_b}{\mathrm{d}a} - \frac{3\Omega_{\rm m}}{2a^2}F_b = -\frac{3\Omega_{\rm m}}{a^2}D(E - D^2). \tag{29}$$

One can obtain the fastest growing mode solution of each order by using the proper initial condition. At the early epoch, the background evolution should be identical to the EdS Uni-

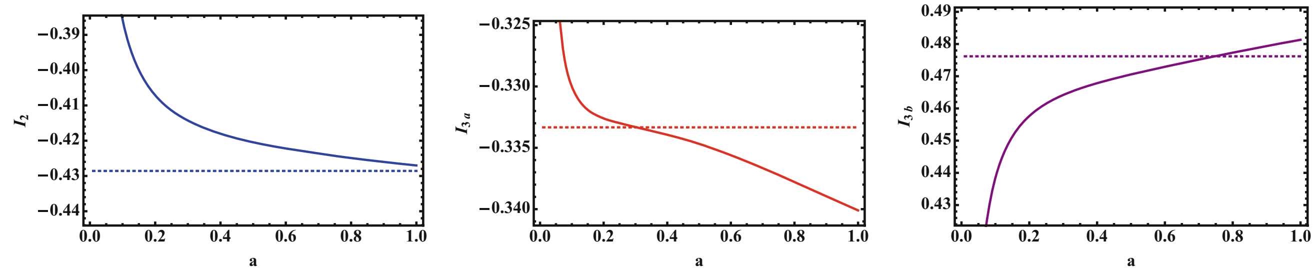

Fig. 3 The coefficients of E, F_a, and F_b as a function of time. The *dotted lines* are those of the EdS approximation

verse ($\Omega_{\mathrm{m}} = 1$) and the linear growing mode solution should be proportional to the scale factor and thus the initial conditions become $D_g(a_i) = a_i$ and $\frac{\mathrm{d}D}{\mathrm{d}a}|_{a=a_i} = 1$. Also, we assume initial Gaussianity for the higher order solutions. It means that higher solutions should be zero at an early epoch. From these, one can obtain the proper EdS fastest growing mode solutions for higher orders (E_g, F_{ag}, and F_{bg}),

$$D_g^{(\mathrm{EdS})}(a_i) = a_i, \quad \frac{\mathrm{d}D_g^{(\mathrm{EdS})}}{\mathrm{d}a}\Big|_{a=a_i} = 1, \tag{30}$$

$$E_g^{(\mathrm{EdS})}(a_i) = -\frac{3}{7}a^2 + \frac{3}{7}a_i a = 0, \frac{\mathrm{d}E_g^{(\mathrm{EdS})}}{\mathrm{d}a}\Big|_{a=a_i} = -\frac{3}{7}a_i, \tag{31}$$

$$F_{ag}^{(\mathrm{EdS})}(a_i) = -\frac{1}{3}a^3 + \frac{1}{3}a_i^2 a = 0, \frac{\mathrm{d}E_g^{(\mathrm{EdS})}}{\mathrm{d}a}\Big|_{a=a_i} = -\frac{2}{3}a_i^2, \tag{32}$$

$$F_{bg}^{(\mathrm{EdS})}(a_i) = \frac{70}{147}a^3 - \frac{54}{147}a_i a^2 - \frac{16}{147}a_i^2 a = 0,$$
$$\frac{\mathrm{d}E_g^{(\mathrm{EdS})}}{\mathrm{d}a}\Big|_{a=a_i} = \frac{86}{147}a_i^2. \tag{33}$$

From the above initial conditions Eqs. (30)–(33), one can find the higher order fastest growing mode solution for the general dark energy model and one can obtain $I_n(a)$ from the relation

$$I_2(a) = \frac{E}{D^2}, \quad I_{3a}(a) = \frac{F_a}{D^3}, \quad I_{3b}(a) = \frac{F_b}{D^3}. \tag{34}$$

We show the time evolutions of I_n in Fig. 3. In the first panel, we show the behavior of I_2. As time increases, I_2 approaches that of the EdS assumed one. Even though we show the behavior of I_{3a} in the second panel, this term does not contribute to the one-loop power spectrum as we show. I_{3b} increases as a does. This is shown in the last panel of Fig. 3.

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