

Resonance spectra of caged black holes

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Abstract Recent numerical studies of the coupled Einstein–Klein–Gordon system in a cavity have provided compelling evidence that *confined* scalar fields generically collapse to form black holes. Motivated by this intriguing discovery, we here use analytical tools in order to study the characteristic resonance spectra of the confined fields. These discrete resonant frequencies are expected to dominate the late-time dynamics of the coupled black-hole-field-cage system. We consider caged Reissner–Nordström black holes whose confining mirrors are placed in the near-horizon region $x_m \equiv (r_m - r_+)/r_+ \ll \tau \equiv (r_+ - r_-)/r_+$ (here r_m is the radius of the confining mirror and r_{\pm} are the radii of the black-hole horizons). We obtain a simple analytical expression for the fundamental quasinormal resonances of the coupled black-hole-field-cage system: $\omega_n = -i2\pi T_{\text{BH}} \cdot n [1 + O(x_m^n/\tau^n)]$, where T_{BH} is the temperature of the caged black hole and $n = 1, 2, 3, \dots$ is the resonance parameter.

1 Introduction

Caged black holes¹ have a long and broad history in general relativity. These composed objects were extensively studied in the context of black-hole thermodynamics [1–8]. In addition, the physics of caged black holes was studied with relation to the black-hole bomb mechanism of Press and Teukolsky [9–17].

Recently there is a renewed interest in the physics of caged black holes. This renewed interest stems from the important work of Bizoń and Rostworowski [18] who revealed that asymptotically anti-de Sitter (AdS) spacetimes are nonlinearly unstable. In particular, it was shown in [18] that the dynamics of massless, spherically symmetric scalar fields in

asymptotically AdS spacetimes generically leads to the formation of Schwarzschild–AdS black holes.

It is well known that the AdS spacetime can be regarded as having an infinite potential wall at asymptotic infinity² [19–23]. One therefore expects the dynamics of *confined* scalar fields³ to display a qualitatively similar behavior to the one observed in [18]. In an elegant work, Maliborski [24] (see also [25, 26]) has recently confirmed this physically motivated expectation. In particular, the recent numerical study by Okawa et al. [25] provides compelling evidence that spherically symmetric confined scalar fields generically collapse to form caged black holes.

The late-time dynamics of perturbation fields in a black-hole spacetime⁴ is characterized by quasinormal ringing, damped oscillations which reflect the dissipation of energy from the black-hole exterior region (see [27–29] for excellent reviews and detailed lists of references). The observation of these characteristic complex resonances may allow one to determine the physical parameters of the newly born black hole.

While there is a vast literature on the quasinormal spectra of black holes in asymptotically AdS spacetimes [27–29], much less is known about the corresponding resonances of caged black holes.⁵ The recent interest [24–26] in the dynamics and formation of caged black holes makes it highly important to study their characteristic resonance spectra. As we shall show below, the resonant frequencies of caged black holes can be determined *analytically* in the regime

$$\frac{r_m - r_+}{r_+ - r_-} \ll 1 \quad (1)$$

² It is worth mentioning that the black-hole bomb mechanism was also studied in the context of asymptotically AdS black holes, see: [19–23].

³ That is, scalar fields which are confined within finite-volume cavities.

⁴ That is, the dynamics of the fields well after the formation of the black-hole horizon.

⁵ It is worth emphasizing again that caged black holes may serve as a simple toy-model for the physically more realistic AdS black holes.

¹ We use the term “Caged black holes” to describe black holes which are confined within finite-volume cavities.

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of “tightly caged black holes”.⁶ Here r_m is the radius of the confining cage (mirror) and r_{\pm} are the radii of the black-hole horizons [see Eq. (4) below].

2 Description of the system

The physical system we explore consists of a massless scalar field Ψ linearly coupled to a Reissner–Nordström (RN) black hole of mass M and electric charge Q . In terms of the Schwarzschild coordinates (t, r, θ, ϕ) , the black-hole spacetime is described by the line element [30]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where⁷

$$f(r) \equiv 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \quad (3)$$

The radii of the black-hole (event and inner) horizons are determined by the zeros of $f(r)$:

$$r_{\pm} = M \pm (M^2 - Q^2)^{1/2}. \quad (4)$$

The dynamics of the scalar field Ψ in the RN spacetime is governed by the Klein–Gordon wave equation,

$$\nabla^\mu \nabla_\mu \Psi = 0. \quad (5)$$

Resolving the field Ψ into spherical harmonics:

$$\Psi(t, r, \theta, \phi) = \sum_{lm} Y_{lm}(\theta, \phi) R_{lm}(r) e^{-i\omega t} / r, \quad (6)$$

one obtains a Schrödinger-like wave equation for the radial part of the field [30–35]:⁸

$$\frac{d^2 R}{dy^2} + [\omega^2 - V(r)]R = 0, \quad (7)$$

where the “tortoise” radial coordinate y is defined by

$$dy = \frac{dr}{f(r)}. \quad (8)$$

⁶ We use the term “Tightly caged black holes” to reflect the fact that the boundary of the confining cavity is placed in the vicinity of the black-hole horizon: $r_m - r_+ \ll r_+ - r_-$.

⁷ We use natural units in which $G = c = \hbar = 1$.

⁸ We shall henceforth omit the indices l and m for brevity.

The effective scattering potential in (7) is given by

$$V[r(y)] = f(r) \left(\frac{\lambda}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4} \right); \quad \lambda \equiv l(l+1). \quad (9)$$

3 Boundary conditions

We shall be interested in solutions of the radial wave equation (7) with the physical requirement (boundary condition) of purely ingoing waves crossing the black-hole horizon [30]:

$$R \sim e^{-i\omega y} \quad \text{as } r \rightarrow r_+ \quad (y \rightarrow -\infty). \quad (10)$$

In addition, following [25] we shall consider two types of boundary conditions at the surface $r = r_m$ of the confining cavity:

(1) The Dirichlet-type boundary condition implies

$$R(r = r_m) = 0. \quad (11)$$

(2) The Neumann-type boundary condition implies

$$\frac{dR}{dr}(r = r_m) = 0. \quad (12)$$

4 The resonance conditions

The boundary conditions (11) and (12) single out two discrete families of complex resonant frequencies $\{\omega(M, Q, r_m, l; n)\}$ ⁹ which characterize the late-time dynamics of the composed black-hole-field-cavity system (these characteristic resonances are also known as “boxed quasinormal frequencies” [10, 11]). The main goal of the present paper is to determine these characteristic resonances *analytically*.

Defining the dimensionless variables

$$x \equiv \frac{r - r_+}{r_+}; \quad \tau \equiv \frac{r_+ - r_-}{r_+}, \quad (13)$$

one finds [see Eqs. (3) and (8)]

$$y = \frac{r_+}{\tau} \ln(x) + O(x) \quad (14)$$

in the near-horizon region (1), which implies¹⁰

$$x = e^{\tau y / r_+} [1 + O(e^{\tau y / r_+})]. \quad (15)$$

⁹ The integer n is the resonance parameter.

¹⁰ Note that the near-horizon region (1) corresponds to $x \leq x_m \ll \tau \leq 1$. This also implies [see Eq. (14)] $y \rightarrow -\infty$ (and thus $e^{\tau y / r_+} \rightarrow 0$) in the region (1).

Substituting (15) into Eqs. (3) and (9) one finds that, in the near-horizon region (1), the effective scattering potential can be approximated by¹¹

$$V(y) \rightarrow V_{\text{near}} \equiv \frac{\tau(\tau + \lambda)}{r_+^2} e^{\tau y/r_+} [1 + O(e^{\tau y/r_+})]. \quad (16)$$

Substituting (16) into (7), one obtains the Schrödinger-like wave equation

$$\frac{d^2 R}{d\tilde{y}^2} + \left[\omega^2 - \frac{4(\tau + \lambda)}{\tau} e^{2\tilde{y}} \right] R = 0, \quad (17)$$

where

$$\tilde{y} \equiv \frac{\tau y}{2r_+}; \quad \omega \equiv \frac{2\omega r_+}{\tau}. \quad (18)$$

Using equation 9.1.54 of [36], one finds that the general solution of Eq. (17) is given by

$$R(z) = AJ_{-i\omega} \left(2i\sqrt{(\tau + \lambda)/\tau} e^{\tilde{y}} \right) + BJ_{i\omega} \left(2i\sqrt{(\tau + \lambda)/\tau} e^{\tilde{y}} \right), \quad (19)$$

where A and B are normalization constants and $J_\nu(x)$ is the Bessel function of the first kind [36]. Using equation 9.1.7 of [36] one finds

$$R(r \rightarrow r_+) = A \frac{(i\sqrt{(\tau + \lambda)/\tau})^{-i\omega}}{\Gamma(-i\omega + 1)} e^{-i\omega y} + B \frac{(i\sqrt{(\tau + \lambda)/\tau})^{i\omega}}{\Gamma(i\omega + 1)} e^{i\omega y} \quad (20)$$

for the asymptotic near-horizon ($r \rightarrow r_+$ with $e^{\tilde{y}} \rightarrow 0$) behavior of the radial function (19). Taking cognizance of Eqs. (10) and (20), one concludes that the physically acceptable solution [the one which obeys the ingoing boundary condition (10) at the black-hole horizon] is characterized by $B = 0$. Thus, the physical solution of the radial Eq. (17) is given by¹²

$$R(x) = AJ_{-i\omega} \left(2i\sqrt{(\tau + \lambda)x/\tau} \right). \quad (21)$$

The Dirichlet-type boundary condition $R(x = x_m) = 0$ [see Eq. (11)] now reads

$$J_{-i\omega} \left(2i\sqrt{(\tau + \lambda)x_m/\tau} \right) = 0. \quad (22)$$

¹¹ Here we have used the relation $(\lambda + 2M/r - 2Q^2/r^2)/r^2 = (\lambda + \tau)/r_+^2 [1 + O(x)]$ in the near-horizon region (1).

¹² Here we have used the relation (15) in the near-horizon region (1).

Using equation 9.1.2 of [36], one can express this boundary condition in the form

$$\tan(i\omega\pi) = \frac{J_{i\omega} \left(2i\sqrt{(\tau + \lambda)x_m/\tau} \right)}{Y_{i\omega} \left(2i\sqrt{(\tau + \lambda)x_m/\tau} \right)}, \quad (23)$$

where $Y_\nu(x)$ is the Bessel function of the second kind [36]. In the near-horizon region [see Eq. (1)]

$$z_m \equiv (\tau + \lambda) \frac{x_m}{\tau} \ll 1 \quad (24)$$

one may use equations 9.1.7 and 9.1.9 of [36] in order to write the resonance condition (23) in the form¹³

$$\tan(i\omega\pi) = i \frac{\pi e^{-\pi\omega} z_m^{i\omega}}{\omega \Gamma^2(i\omega)} [1 + O(z_m)]. \quad (25)$$

The Neumann-type boundary condition $dR(x = x_m)/dx = 0$ [see Eq. (12)] now reads

$$\frac{d}{dx} \left[J_{-i\omega} \left(2i\sqrt{(\tau + \lambda)x/\tau} \right) \right]_{x=x_m} = 0. \quad (26)$$

Using equation 9.1.27 of [36], one can express (26) in the form

$$J_{-i\omega-1} \left(2i\sqrt{(\tau + \lambda)x_m/\tau} \right) - J_{-i\omega+1} \left(2i\sqrt{(\tau + \lambda)x_m/\tau} \right) = 0. \quad (27)$$

Using equation 9.1.2 of [36], one can express this boundary condition in the form

$$\tan(i\omega\pi) = \frac{J_{i\omega+1}(2i\sqrt{z_m}) - J_{i\omega-1}(2i\sqrt{z_m})}{Y_{i\omega+1}(2i\sqrt{z_m}) - Y_{i\omega-1}(2i\sqrt{z_m})}. \quad (28)$$

From equations 9.1.7 and 9.1.9 of [36] one finds $J_{i\omega+1}(2i\sqrt{z_m})/J_{i\omega-1}(2i\sqrt{z_m}) = O(z_m) \ll 1$ and $Y_{i\omega+1}(2i\sqrt{z_m})/Y_{i\omega-1}(2i\sqrt{z_m}) = O(z_m^{-1}) \gg 1$ in the near-horizon $z_m \ll 1$ region [see Eq. (24)]. Using these relations, one may write the resonance condition (28) in the form¹³ $\tan(i\omega\pi) = -J_{i\omega-1}(2i\sqrt{z_m})/Y_{i\omega+1}(2i\sqrt{z_m}) [1 + O(z_m)]$, which in the near-horizon region (24) implies (see equations 9.1.7 and 9.1.9 of [36])

$$\tan(i\omega\pi) = -i \frac{\pi e^{-\pi\omega} z_m^{i\omega}}{\omega \Gamma^2(i\omega)} [1 + O(z_m)]. \quad (29)$$

¹³ See equations 9.1.10 and 9.1.11 of [36] for the sub-leading correction terms.

5 The discrete resonance spectra of caged black holes

Taking cognizance of the near-horizon condition (24), one realizes that the r.h.s. of the resonance conditions (25) and (29) are small quantities. This observation follows from the fact that, for damped modes with $\Im\varpi < 0$ [$\Re(i\varpi) > 0$], one has $z_m^{i\varpi} \ll 1$ in the regime (24). We can therefore use an iteration scheme in order to solve the resonance conditions (25) and (29).

The zeroth-order resonance equation is given by $\tan(i\varpi^{(0)}\pi) = 0$ for both the Dirichlet-type boundary condition (11) and the Neumann-type boundary condition (12) [see Eqs. (24), (25), and (29)]. This yields the simple zeroth-order resonances

$$\varpi_n^{(0)} = -in; \quad n = 1, 2, 3, \dots \tag{30}$$

of the caged black holes.

Substituting (30) into the r.h.s. of (25) and (29), one obtains the first-order resonance condition

$$\tan(i\varpi_n^{(1)}\pi) = \mp \frac{\pi(-z_m)^n}{n\Gamma^2(n)}, \tag{31}$$

where the upper sign corresponds to the Dirichlet-type boundary condition (11) and the lower sign corresponds to the Neumann-type boundary condition (12). From (31) one finds¹⁴

$$\varpi_n = -in \left[1 \mp \frac{(-z_m)^n}{(n!)^2} \right]; \quad n = 1, 2, 3, \dots \tag{32}$$

for the characteristic resonance spectra of caged black holes in the regime (24).

6 Summary and discussion

Recent numerical studies of the Einstein–Klein–Gordon system in a cavity [25,26] have provided compelling evidence that confined scalar fields generically collapse to form (caged) black holes. Motivated by these intriguing studies, we have explored here the late-time dynamics⁴ of these confined fields in the background of caged black holes.

In particular, we have studied the characteristic resonance spectra of confined scalar fields in caged Reissner–Nordström black-hole spacetimes. It was shown that these resonances can be derived *analytically* for caged black holes whose confining mirrors are placed in the vicinity of the black-hole horizon [that is, in the regime $x_m \ll \tau$; see Eq.

(24)]. Remarkably, the resonant frequencies of these caged black holes can be expressed in terms of the Bekenstein–Hawking temperature T_{BH} of the black-hole.¹⁵

$$\omega_n = -i2\pi T_{\text{BH}} \cdot n \left\{ 1 \mp \frac{[-(\tau + \lambda)x_m/\tau]^n}{(n!)^2} \right\}. \tag{33}$$

Note that, for spherical field configurations (the ones studied in [24,25]), the characteristic resonant frequencies are given by the remarkably simple linear relation¹⁶

$$\omega_n = -i2\pi T_{\text{BH}} \cdot n \left[1 \mp \frac{(-x_m)^n}{(n!)^2} \right]. \tag{34}$$

Finally, it is worth mentioning other black-hole spacetimes which share this remarkable property (that is, black-hole spacetimes which are characterized by resonant frequencies whose imaginary parts¹⁷ scale linearly with the black-hole temperature):

- (1) Near-extremal asymptotically flat Kerr black holes [37–42].
- (2) Near-extremal asymptotically flat Reissner–Nordström black holes coupled to charged scalar fields [43].
- (3) Asymptotically flat charged black holes coupled to charged scalar fields in the highly charged regime $qQ \gg 1$ ¹⁸ [44,45].
- (4) The “subtracted” black-hole geometries studied in [46].
- (5) Asymptotically AdS black holes in the regime $r_+ \gg R$ ¹⁹ [47]. This last example, together with our result (34) for the resonant frequencies of caged black holes, provide an elegant demonstration of the analogy, already discussed in the Introduction, between asymptotically AdS black-hole spacetimes and caged black-hole spacetimes.

The confining cavity (mirror) of our analysis obviously restricts the dynamics of the fields to the near-horizon region $x \leq x_m \ll \tau$. It is therefore not surprising that the characteristic resonances (33) of these caged black holes are determined by the surface gravity²⁰ at the black-hole horizon. The fact that the black-hole spacetimes mentioned

¹⁴ Here we have used the relation $\tan(x + n\pi) = \tan(x)$. In addition, we have used the relation $\tan(x) = x + O(x^3)$ in the $x \ll 1$ regime [see Eq. (24)].

¹⁵ Here we have used Eqs. (18) and (32) together with the relation $T_{\text{BH}} = \tau/4\pi r_+$ for the Bekenstein–Hawking temperature of the black hole.

¹⁶ That is, the resonant frequencies scale linearly with the black-hole temperature T_{BH} .

¹⁷ It is worth emphasizing that, the characteristic relaxation time of generic field perturbations is determined by the (reciprocal of the) imaginary part of the fundamental ($n = 1$) resonance: $\tau = 1/\Im\omega_1$.

¹⁸ Here q is the charge coupling constant of the field.

¹⁹ Here R is the AdS radius.

²⁰ Note that the surface gravity is proportional to the black-hole temperature.

above [37–47] share this same property (namely, they are characterized by a linear scaling of their resonances with the black-hole temperature) suggests that the dynamics of perturbation fields in these black-hole spacetimes are mainly determined by the near-horizon properties of these geometries.

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