Letter

Propagation of gravitational waves in the nonperturbative spinor vacuum

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Abstract The propagation of gravitational waves on the background of a nonperturbative vacuum of a spinor field is considered. It is shown that there are several distinctive features in comparison with the propagation of plane gravitational waves through empty space: there exists a fixed phase difference between the $h_{yy,zz}$ and h_{yz} components of the wave; the phase and group velocities of gravitational waves are not equal to the velocity of light; the group velocity is always less than the velocity of light; under some conditions the gravitational waves are either damped or absent; for given frequency, there exist two waves with different wave vectors. We also discuss the possibility of an experimental verification of the obtained effects as a tool to investigate nonperturbative quantum field theories.

1 Introduction

Gravitational waves (GWs) are probably the most suitable object for studying the deep space (for a review with references on the subject see, e.g., Ref. [1]). It is usually assumed that GWs propagate in a classical vacuum, i.e., in empty space. But a quantum vacuum possesses the energy associated with the unavoidable quantum fluctuations of various fields when the vacuum expectation value of any quantum field is zero but the expectation value of the square of fluctuations is nonzero.

In this framework, of special interest is to study the question of the propagation of GWs in the case where fluctuations of a quantum spinor field are taken into account. The reason is that the energy-momentum tensor of a spinor field contains

the spin connection, which in turn contains first derivatives of tetrad components with respect to the coordinates. As a result, the Einstein equations yield the wave equation for a GW which contains second derivatives of the tetrad components on the left-hand side and their first derivatives on the right-hand side.

Such a situation is a reminder of the propagation of electromagnetic waves in a continuous conducting medium. The corresponding wave equation is

$$\Delta \vec{A} - \frac{\epsilon \mu}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \frac{\gamma \mu}{c^2} \frac{\partial \vec{A}}{\partial t},$$

where ϵ, μ are the dielectric permittivity and magnetic permeability, respectively, γ is the electrical conductivity. It is well known that the above equation describes damped waves.

Comparing both these situations, one may conclude that the propagation of GWs on the background of the spinor vacuum possesses some common features with the propagation of electromagnetic waves in a conducting medium (notice in this connection that the introduction of "Ohm's gravitational law" into the linearized Einstein equations is discussed in Ref. [2]). The problem in such studies is that one presumably has to consider a *nonperturbative* vacuum. The reason is that any perturbative calculations deal with zero-point quantum vacuum fluctuations of fundamental fields whose energy turns out to be infinite. This eventually results in a number of fundamental problems, including ultraviolet divergences and the well-known "cosmological constant problem" for the Universe [3]. This motivates one to go beyond the framework of perturbative theories in the hope that the use of the nonperturbative quantization would allow the possibility of avoiding these problems [4].

One possible way to consider a nonperturbative vacuum might be the approach adopted below, which suggests a phe-

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nomenological consideration of a nonperturbative vacuum of a spinor field. This approach is based on the following concepts:

(i) We make some physically reasonable assumptions about expectation values of the spinor field and its dispersion. Namely, we introduce an *ad hoc* ansatz for the dispersion of the spinor field and, evaluating the covariant divergence of the obtained right-hand part of the Einstein equations, check that the Bianchi identity is satisfied. In our opinion, this can be considered as some approximate way to cutoff the infinite system of differential equations for all Green functions of the nonperturbative quantum spinor field used in our calculations (for a more detailed discussion of this question, see Ref. [5]).

(ii) The right-hand side of the Einstein equations contains first derivatives of the tetrad with respect to the coordinates that is a consequence of the presence of the spin connection. As will be shown below, the presence of the derivatives results in a fixed phase difference between the components of a GW and modification of the dispersion relation. Also, under some conditions the damping of GWs may arise.

Within the framework of this approach, the paper considers the simplest case of a plane GW propagating through the nonperturbative spinor vacuum. In this case one might consider such a vacuum as consisting of a spinor condensate (a continuous medium) through which the GW propagates.

2 Perturbed Einstein equations

To begin with, we want to describe an exact formulation of the problem of gravitational waves propagation in a spinor vacuum. Strictly speaking, in describing this physical phenomenon, one needs to consider both a metric and a spinor field as quantum objects. For the nonperturbative quantization, we have to write the following equations (for details, see Ref. [5]):

$$\hat{R}_{\bar{a}\mu} - \frac{1}{2}\hat{e}_{\bar{a}\mu}\hat{R} = \varkappa\hat{T}_{\bar{a}\mu}, \tag{1}$$

$$\gamma^\mu \nabla_\mu \hat{\psi} - m\hat{\psi} = 0, \tag{2}$$

where $\hat{R}_{\bar{a}\nu}$ and $\hat{R}$ are, respectively, the operators of the Ricci tensor and the Ricci scalar; $\hat{e}_{\bar{a}\mu}$ is the vierbein operator; $\hat{T}_{\mu\nu}$ is the operator of the energy-momentum tensor; $\hat{\psi}$ is the operator of the spinor field; $\bar{a} = \bar{0}, \bar{1}, \bar{2}, \bar{3}$ is the vierbein index; $\mu = 0, 1, 2, 3$ is the coordinate index; $\nabla_\mu \hat{\psi} = \partial_\mu \hat{\psi} - \Gamma_\mu \hat{\psi} = \partial_\mu \hat{\psi} + \frac{1}{4}\hat{\omega}_{\bar{a}\bar{b}\mu}\gamma^{\bar{a}}\gamma^{\bar{b}}\hat{\psi}$ is the covariant derivative for the spinor with the operator of the

spin connection $\hat{\omega}_{\bar{a}\bar{b}\mu}$ [6]; $\gamma^{\bar{a}}$ are the Dirac matrices in flat Minkowski spacetime; $\varkappa = 8\pi\kappa/c^4$, κ is the gravitational constant.

As of now, a procedure of solving such an operator set of equations is unavailable. But we know that the properties of the operators are determined by all Green functions. For them we can write down an infinite set of equations (for details, see Ref. [5]). Such an *infinite* system of equations can be solved approximately by cutting it off to obtain a *finite* set of equations. Such a cutoff procedure is performed by applying some physically reasonable arguments.

Similar procedure is well known in modeling turbulence (see, for example, the textbook of Wilcox [7]). The situation there is as follows (we follow Ref. [7] in this paragraph): one can write a statistically averaged version of the Navier–Stokes equation (the Reynolds-averaged Navier–Stokes equation) for an averaged velocity. This equation contains six new unknown functions $\overline{\rho v_i v_j}$ (the Reynolds-stress tensor, where the overbar denotes statistical averaging). This means that our system is not yet closed. In quest of additional equations, we have to take moments of the Navier–Stokes equation. That is, we multiply the Navier–Stokes equation by a suitable quantity and statistically average the product. Using this procedure, we can derive a differential equation for the Reynolds-stress tensor. After such procedure we gained six new equations, one for each independent components of the Reynolds-stress tensor. However, we have also generated 22 new unknown functions: $\overline{\rho v_i v_j v_k}$, $\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$, $\overline{u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial p}{\partial x_i}}$. This situation illustrates the closure problem of turbulence theory (let us note that we have a similar problem for a nonperturbative quantization). Because of the nonlinearity of the Navier–Stokes equation, as we have higher and higher moments, we generate additional unknown functions at each level. As written in Ref. [7]: "The function of turbulence modeling is to derive approximations for the unknown correlations in terms of flow properties that are known so that a sufficient number of equations exist. In making such approximations, we close the system."

Following this scheme, we can rephrase the last sentence as applied to a nonperturbative quantization: the approximate approach for a nonperturbative quantization being suggested here is to derive approximations for the unknown Green functions using the properties of the quantum system under consideration so that a sufficient number of equations exists. In making such approximations, we close an infinite set of equations for the Green functions.

Here we employ some approximation, as described below. We evaluate the Bianchi identities, instead of solving the Dirac equation which contains the nonlinear term $\left\langle \hat{\bar{\psi}}\hat{\omega}_{\bar{a}\bar{b}\mu}\hat{\psi}\right\rangle$. The presence of this term prevents us from solving the

operator Dirac equation directly since, as follows from the above discussion, to do this we have to write an equation for the term $\left\langle \hat{\bar{\psi}} \hat{\omega}_{\bar{a}\bar{b}\mu} \hat{\psi} \right\rangle$, and so on *ad infinitum*. This makes us use the Bianchi identities instead of the Dirac equation. Nevertheless, if one wants to continue with calculations in the next approximation, the Dirac equation will necessarily appear.

The operator of the energy-momentum tensor of the spinor field is given by

$$
\hat{T}_{\bar{a}\bar{b}} = \frac{i}{2} \left[\hat{\bar{\psi}} \gamma_{(\bar{a}} \nabla_{\bar{b})} \hat{\psi} - \nabla_{(\bar{a}} \hat{\bar{\psi}} \gamma_{\bar{b})} \hat{\psi} \right]
$$
$$
+ \eta_{\bar{a}\bar{b}} \left(-\frac{i}{2} \hat{\bar{\psi}} \gamma^\mu \nabla_\mu \hat{\psi} + \frac{i}{2} \nabla_\mu \hat{\bar{\psi}} \gamma^\mu \hat{\psi} + m \hat{\bar{\psi}} \hat{\psi} \right), \quad (3)
$$

where $\hat{\bar{\psi}} \gamma_{(\bar{a}} \nabla_{\bar{b})}$ means the symmetrization over the indices $\bar{a}, \bar{b}$; m is the mass of the spinor field.

Equations (1) and (2) cannot be solved explicitly, and we have to use some approximation. First let us write down the expectation value of these equations:

$$
\left\langle Q \left| \hat{R}_{\bar{a}\mu} - \frac{1}{2} \hat{e}_{\bar{a}\mu} \hat{R} \right| Q \right\rangle = \varkappa \left\langle Q \left| \hat{T}_{\bar{a}\mu} \right| Q \right\rangle, \quad (4)
$$

$$
\left\langle Q \left| \gamma^\mu \nabla_\mu \hat{\psi} - m \hat{\psi} \right| Q \right\rangle = 0, \quad (5)
$$

where $| Q \rangle$ is a quantum state describing the propagation of a GW through a spinor vacuum. Let us note that as the consequence of (5), the expectation value of the term in the parentheses of Eq. (3) is exactly zero. Once again, we emphasize that we cannot use the Dirac equation (5) to calculate the expectation value of the spinor field $\left\langle \hat{\psi} \right\rangle$, since, as mentioned at the beginning of this section, strictly speaking, in performing such calculations we must also quantize a metric. In this case the expectation value of the Dirac equation will contain not only $\left\langle \hat{\psi} \right\rangle$ but also the term $\left\langle \hat{\omega}_{ab\mu} \hat{\psi} \right\rangle$. Then we will have to write down a new equation for this Green function, and so on *ad infinitum*. This is the main problem encountered in Heisenberg's nonperturbative quantization technique, discussed also in Ref. [5]. To avoid this problem, we employ the aforementioned approximation.

To solve Eqs. (4) and (5), we assume the following approximations: (a) the vierbein $e^{\bar{a}}{}_\mu$ and all geometrical quantities (the Ricci tensor, the Ricci scalar, and the spin connection) are the classical ones; (b) instead of solving the Dirac equation (5), we will check the validity of the Bianchi identities for the right-hand side of the Einstein equations (4) with the

shortened energy-momentum tensor; (c) we consider only weak GWs.

Within our approximation, we will consider the following set of equations:

$$
\delta R_{\bar{a}\bar{b}} - \frac{1}{2} \eta_{\bar{a}\bar{b}} \delta R = \varkappa \left\langle Q \left| \widehat{\delta T}_{\bar{a}\bar{b}} \right| Q \right\rangle, \quad (6)
$$

$$
\left\langle Q \left| \widehat{\delta T}_{\bar{a}}{}^\mu \right| Q \right\rangle_{;\mu} = 0, \quad (7)
$$

where $\delta R_{\bar{a}\bar{b}}$ and δR are the gravitational wave approximation for the Ricci tensor and the Ricci scalar, as given below by Eq. (14). In turn, the right-hand side of Eq. (6) is calculated in subsequent sections. To simplify the notation we will hereafter use $\langle \cdots \rangle$ instead of $\langle Q | \cdots | Q \rangle$.

2.1 The left-hand side of the perturbed Einstein equations

According to Ref. [8], let the vierbein perturbation $\phi_{\bar{a}}{}^{\bar{b}}$ be defined in the following manner:

$$
e^{\bar{a}}{}_\mu = \left(\delta^{\bar{a}}_{\bar{b}} - \phi^{\bar{a}}{}_{\bar{b}} \right) \overset{0}{e}{}^{\bar{b}}{}_\mu, \quad (8)
$$

$$
e_{\bar{a}}{}^\mu = \left(\delta^{\bar{b}}_{\bar{a}} + \phi_{\bar{a}}{}^{\bar{b}} \right) \overset{0}{e}{}_{\bar{b}}{}^\mu, \quad (9)
$$

where $\overset{0}{e}{}^{\bar{a}}{}_\mu$ is the unperturbed tetrad; $\overset{0}{e}{}_{\bar{a}}{}^\mu$ is the unperturbed inverse tetrad; $e^{\bar{a}}{}_\mu$ is the perturbed tetrad; $e_{\bar{a}}{}^\mu$ is the perturbed inverse tetrad; $-\phi^{\bar{a}}{}_{\bar{b}} \overset{0}{e}{}^{\bar{b}}{}_\mu$ is the perturbation of the tetrad. It is convenient to work with the covariant tetrad-frame components $\phi_{\bar{a}\bar{b}}$ of the tetrad perturbation,

$$
\phi_{\bar{a}\bar{b}} = \eta_{\bar{b}\bar{c}} \phi_{\bar{a}}{}^{\bar{c}}, \quad (10)
$$

where $\eta_{\bar{a}\bar{b}} = \mathrm{diag}\{+, -, -, -\}$ is the Minkowski metric. For a single Fourier mode whose wave vector $\vec{k}$ is taken to lie in the x-direction we have

$$
\phi_{\bar{a}\bar{b}} = \begin{pmatrix} \psi & \partial_x w & w_{\bar{2}} & w_{\bar{3}} \\ \partial_x \tilde{w} & \Phi + \partial_x^2 h & \partial_x h_{\bar{2}} & \partial_x h_{\bar{3}} \\ \tilde{w}_{\bar{2}} & \partial_x \tilde{h}_{\bar{2}} & \Phi + h_{\bar{2}\bar{2}} & h_{\bar{2}\bar{3}} + \partial_x \tilde{h} \\ \tilde{w}_{\bar{3}} & \partial_x \tilde{h}_{\bar{3}} & h_{\bar{3}\bar{2}} - \partial_x \tilde{h} & \Phi - h_{\bar{2}\bar{2}} \end{pmatrix}. \quad (11)
$$

One can introduce the gauge invariant functions Ψ and W_i

$$
\Psi = \psi - \partial_t \left(w + \tilde{w} - \partial_t h \right), \quad (12)
$$

$$
W_i = w_i + \tilde{w}_i - \partial_t \left(h_i + \tilde{h}_i \right), \quad (13)
$$

where $i = 1, 2, 3$ are the spacelike world indices. After that the perturbations of the Einstein tensor are

$$\delta G_{\bar{a}\bar{b}} = \begin{pmatrix} -2\partial_x^2 \Phi & 2\partial_x \dot{\Phi} & -\frac{1}{2}\partial_x^2 W_y & -\frac{1}{2}\partial_x^2 W_z \\ 2\partial_x \dot{\Phi} & -2\ddot{\Phi} & \frac{1}{2}\partial_x \dot{W}_y & \frac{1}{2}\partial_x \dot{W}_z \\ -\frac{1}{2}\partial_x^2 W_y & \frac{1}{2}\partial_x \dot{W}_y & -2\ddot{\Phi} - \partial_x^2(\Psi - \Phi) + \Box h_+ & \Box h_\times \\ -\frac{1}{2}\partial_x^2 W_z & \frac{1}{2}\partial_x \dot{W}_z & \Box h_\times & -2\ddot{\Phi} - \partial_x^2(\Psi - \Phi) - \Box h_+ \end{pmatrix}, \tag{14}$$

where the dot denotes differentiation with respect to $\tau = ct$; $\Box = \frac{\partial^2}{\partial \tau^2} - \nabla^2$ is the d'Alembertian and h_+ and $h_\times$ are the two polarizations of gravitational waves,

$$h_+ = h_{yy} = -h_{zz}, \ h_\times = h_{yz} = h_{zy}, \tag{15}$$

and Φ, $W_{\bar{i}}$, Ψ are vierbein components given by the formulas (11)–(13).

2.2 The right-hand side of the Einstein equations

To calculate the expectation value of the energy-momentum tensor of the spinor field, we state the following assumptions concerning the spinor field:

- The vacuum expectation value of the spinor field is zero:

$$\left\langle \hat{\psi}_{\mathfrak{a}} \right\rangle = 0. \tag{16}$$

- The vacuum expectation value of the product of the spinor field in two points x, y is nonzero:

$$\left\langle \hat{\psi}_{\mathfrak{a}}^*(x)\hat{\psi}_{\mathfrak{b}}(y) \right\rangle = \Upsilon_{\mathfrak{a}\mathfrak{b}}(x, y) \neq 0 \tag{17}$$

here $\hat{\psi}$ is the operator of the spinor field; $\mathfrak{a}$, $\mathfrak{b}$ are the spinor indices; $\Upsilon_{\mathfrak{a}\mathfrak{b}}$ is the 2-point Green function.
- Every component

$$|\Upsilon_{\mathfrak{a}\mathfrak{b}}(x, y)| = \text{const.} \tag{18}$$

- As a consequence of Eq. (18) we have

$$\left\langle \hat{\psi}_{\mathfrak{a}}^*(x)\partial_{y^\mu}\hat{\psi}_{\mathfrak{b}}(y) \right\rangle = 0. \tag{19}$$

The energy-momentum tensor contains the following unperturbed and perturbed contributions:

$$\hat{T}_{\bar{a}\bar{b}} = \overset{0}{\hat{T}}_{\bar{a}\bar{b}} + \widehat{\delta T}_{\bar{a}\bar{b}}, \tag{20}$$

where $\overset{0}{\hat{T}}_{\bar{a}\bar{b}}$ is calculated for unperturbed Minkowski spacetime with zero spin connection, $\omega_{\bar{a}\bar{b}\mu} = 0$. Consequently,

$$\overset{0}{\hat{T}}_{\bar{a}\bar{b}} = \frac{i}{2}\left[\overset{\hat{}}{\bar{\psi}}\gamma_{(\bar{a}}\partial_{\bar{b})}\hat{\psi} - \partial_{(\bar{a}}\overset{\hat{}}{\bar{\psi}}\gamma_{\bar{b})}\hat{\psi} \right]. \tag{21}$$

According to Eq. (19),

$$\left\langle \overset{0}{\hat{T}}_{\bar{a}\bar{b}} \right\rangle = 0. \tag{22}$$

Its physical meaning is that, since the expectation value of the energy-momentum tensor in unperturbed Minkowski spacetime is equal to zero, it does not affect the propagation of GWs. The perturbed energy-momentum tensor is calculated in Appendix.

3 Gravitational wave propagating on the background of the spinor vacuum

We consider a GW propagating along the x axis, described by the Einstein equations (6). It has to be emphasized that the right-hand side of these equations cannot be calculated by using a perturbative technique. The reason is that perturbative calculations give us an infinite energy of zero-point vacuum fluctuations. This energy acts as a source of gravitational field and, in general, cannot be excluded by using a renormalization procedure [9]. In fact, this is just an imprint of the well-known problem of the contradiction between gravity and the *perturbative* quantum paradigm.

To calculate a nonperturbative expectation value of $\left\langle \hat{T}_{\bar{a}\bar{b}} \right\rangle$, we will use the assumptions about expectation values of the spinor field and its dispersion as described in the previous section. In doing so, we will consider a particular case of GWs for which

$$\Phi = \Psi = W_i = 0. \tag{23}$$

Below we consider two different ansätze for the spinor field.

3.1 Case I

For the ansatz

$$\hat{\psi} = e^{-i(\omega t - kx)} \begin{pmatrix} \hat{A} \\ \hat{B} \\ \hat{B} \\ \hat{A} \end{pmatrix}, \tag{24}$$

where ω is the frequency and k is the x-component of the wave vector.

The algorithm for calculating the right-hand sides of Eqs. (6) is as follows. The first step is to evaluate them as classical quantities using (24) without hats over A, B, and then to restore the hats: $\hat{A}$, $\hat{B}$. These calculations give the $\bar{a}$, $\bar{b} \neq \bar{y}$, $\bar{z}$ components of the classical energy-momentum tensor $T_{\bar{a}\bar{b}}$, which contain only the terms

$$A^*B - AB^* + SV^* - S^*V \quad \text{and} \quad |A|^2 - |B|^2 - |S|^2 + |V|^2. \tag{25}$$

In calculating the components of the energy-momentum tensor, we have used the spinor in the general form $\psi^T = e^{-i(\omega t - kx)}(A, B, V, S)$. Taking into account the gauge (23), the left-hand side of the Einstein equations (6) is not zero only for the yy, zz, yz components. Therefore we have to choose $\hat{A}$, $\hat{B}$, $\hat{S}$, $\hat{V}$ in such a manner that the only nonzero components of the energy-momentum tensor would be $T_{\bar{y}\bar{y}, \bar{z}\bar{z}, \bar{y}\bar{z}}$. We see from (25) that the components $T_{\bar{a}\bar{b}}$, $(\bar{a}\bar{b} \neq \bar{y}\bar{y}, \bar{z}\bar{z}, \bar{y}\bar{z})$ of the energy-momentum tensor are equal to zero only when

Case I: $\quad V = B, S = A;$ $\tag{26}$

Case II: $\quad B = A, S = V.$ $\tag{27}$

In this Section we consider the first case, corresponding to the ansatz (24), and the second case, corresponding to the ansatz (43), will be studied in next subsection. We assume the following values of the 2-point Green functions of the spinor field ψ:

$$\Upsilon = \langle \psi_1^* \psi_2 \rangle = \langle \psi_1^* \psi_3 \rangle = \langle \psi_4^* \psi_2 \rangle$$
$$= \langle \psi_4^* \psi_3 \rangle = \langle A^*B \rangle = \Upsilon_1 + i\Upsilon_2, \tag{28}$$
$$\Upsilon^* = \langle \psi_2^* \psi_1 \rangle = \langle \psi_2^* \psi_4 \rangle = \langle \psi_3^* \psi_1 \rangle$$
$$= \langle \psi_3^* \psi_4 \rangle = \langle B^*A \rangle = \Upsilon_1 - i\Upsilon_2, \tag{29}$$

with $|\Upsilon_{1,2}| = $ const. By choosing $\hat{A}$, $\hat{B}$, $\hat{S}$, $\hat{V}$ in the form of (26), the $\bar{y}\bar{y}$, $\bar{z}\bar{z}$, and $\bar{y}\bar{z}$ components of the energy-momentum tensor (64) are

$$\left\langle \widehat{\delta T_{\bar{y}\bar{y}}} \right\rangle = -\left\langle \widehat{\delta T_{\bar{z}\bar{z}}} \right\rangle = 2\left(\left\langle \hat{A}^*\hat{B} + \hat{A}\hat{B}^* \right\rangle \right) h_{\bar{y}\bar{z}}, \tag{30}$$

$$\left\langle \widehat{\delta T_{\bar{y}\bar{z}}} \right\rangle = 2\left(\left\langle \hat{A}^*\hat{B} + \hat{A}\hat{B}^* \right\rangle \right) h_{\bar{y}\bar{y}}. \tag{31}$$

Equations (6) with the gauge (23) and the perturbed components of the energy-momentum tensor (30) and (31) give the following set of equations for the components (15):

$$h_{\bar{y}\bar{y}}'' - \ddot{h}_{\bar{y}\bar{y}} = -2\varkappa \left(\left\langle \hat{A}^*\hat{B} \right\rangle + \left\langle \hat{A}\hat{B}^* \right\rangle \right) h_{\bar{y}\bar{z}}, \tag{32}$$

$$h_{\bar{y}\bar{z}}'' - \ddot{h}_{\bar{y}\bar{z}} = 2\varkappa \left(\left\langle \hat{A}^*\hat{B} \right\rangle + \left\langle \hat{A}\hat{B}^* \right\rangle \right) h_{\bar{y}\bar{y}}, \tag{33}$$

where the prime denotes differentiation with respect to x, and the appearance of the derivatives of the components $h_{\bar{y}\bar{y}}$, $h_{\bar{y}\bar{z}}$ on the right-hand side of these equations is connected with the presence of the spin connection on the right-hand side of Einstein's equations.

We are looking for the x-plane wave solution in the form

$$h_{\bar{y}\bar{y}} = -h_{\bar{z}\bar{z}} = A_1 e^{-i(\omega t - kx)}, \tag{34}$$

$$h_{\bar{y}\bar{z}} = h_{\bar{z}\bar{y}} = A_2 e^{-i(\omega t - kx)}. \tag{35}$$

Substituting the solutions (34) and (35) into the wave equations (32) and (33) and using the expressions (28) and (29), we obtain the following relations (hereafter we work in natural units, where $\hbar = c = 1$):

$$A_1 \left(k^2 - \omega^2 \right) = -4i\varkappa A_2 \Upsilon_1 \omega, \tag{36}$$

$$A_2 \left(k^2 - \omega^2 \right) = 4i\varkappa A_1 \Upsilon_1 \omega. \tag{37}$$

From them one can immediately read off

$$A_2 = \pm i A_1 = A_1 e^{\pm i\frac{\pi}{2}}. \tag{38}$$

This means that the phase difference between $\bar{y}\bar{y}$, $\bar{z}\bar{z}$ and $\bar{y}\bar{z}$ components of the GW is $\pm\pi/2$. In turn, the dispersion relation is

$$k^2 = \omega^2 \pm 4\varkappa \Upsilon_1 \omega. \tag{39}$$

Thus, we see that there are two GWs with different wave vectors for the same frequency ω. But for the case $k = \sqrt{\omega^2 - 4\varkappa\Upsilon_1\omega}$ a situation may occur where the GW does not exist. This happens if

$$\omega < 4\varkappa\Upsilon_1. \tag{40}$$

The phase velocity of the GW is given by

$$v_p = \frac{\omega}{k} = \sqrt{\frac{1}{1 \pm \frac{4\varkappa\Upsilon_1}{\omega}}} \neq 1 \tag{41}$$

(recall that v is measured in units of c). We see that there are two branches: one with $v_p < 1$ and the other with $v_p > 1$.

The group velocity of the GW is

$$v_g = \frac{d\omega}{dk} = \frac{\sqrt{1 \pm \frac{4\varkappa\Upsilon_1}{\omega}}}{1 \pm \frac{2\varkappa\Upsilon_1}{\omega}} \neq 1. \tag{42}$$

It is interesting that if $\frac{\varkappa\Upsilon_1}{\omega} \ll 1$ then $v_g \approx 1$. It is also seen that the group velocity $v_g < 1$ for any value of $\varkappa\Upsilon_1/\omega$ and for any sign of Υ_1.

3.2 Case II

In this section we consider the following ansatz for the spinor field:

$$\hat{\psi} = e^{-i(\omega t - kx)} \begin{pmatrix} \hat{A} \\ \hat{A} \\ \hat{V} \\ \hat{V} \end{pmatrix}, \tag{43}$$

where $\langle \hat{A} \hat{A}^* \rangle$, $\langle \hat{V} \hat{V}^* \rangle$ are taken to be constant in accordance with the assumption (18).

For the ansatz (43), we assume the following values of the 2-point Green functions of the spinor field ψ:

$$\langle \psi_1^* \psi_2 \rangle = \langle \psi_1 \psi_2^* \rangle = \langle \psi_1^* \psi_1 \rangle$$
$$= \langle \psi_2^* \psi_2 \rangle = \langle \hat{A}^* \hat{A} \rangle = \Upsilon_1, \tag{44}$$

$$\langle \psi_3^* \psi_4 \rangle = \langle \psi_4^* \psi_3 \rangle = \langle \psi_3^* \psi_3 \rangle$$
$$= \langle \psi_4^* \psi_4 \rangle = \langle \hat{V}^* \hat{V} \rangle = \Upsilon_2, \tag{45}$$

with $|\Upsilon_{1,2}| = $ const. By choosing $\hat{A}$, $\hat{B}$, $\hat{S}$, $\hat{V}$ in the form of (27), the $\bar{y}\bar{y}$, $\bar{z}\bar{z}$ and $\bar{y}\bar{z}$ components of the energy-momentum tensor (64) are

$$\langle \widehat{\delta T_{\bar{y}\bar{y}}} \rangle = - \langle \widehat{\delta T_{\bar{z}\bar{z}}} \rangle = 2 \left[\left(\langle \hat{A}\hat{A}^* \rangle - \langle \hat{V}\hat{V}^* \rangle \right) h'_{\bar{y}\bar{z}} \right.$$
$$\left. + \left(\langle \hat{A}\hat{A}^* \rangle + \langle \hat{V}\hat{V}^* \rangle \right) \dot{h}_{\bar{y}\bar{z}} \right], \tag{46}$$

$$\langle \widehat{\delta T_{\bar{y}\bar{z}}} \rangle = -2 \left[\left(\langle \hat{A}\hat{A}^* \rangle - \langle \hat{V}\hat{V}^* \rangle \right) h'_{\bar{y}\bar{y}} \right.$$
$$\left. + \left(\langle \hat{A}\hat{A}^* \rangle + \langle \hat{V}\hat{V}^* \rangle \right) \dot{h}_{\bar{y}\bar{y}} \right]. \tag{47}$$

Substituting these expressions into Eq. (6) and taking into account the gauge (23), we have the following set of equations for the components (15):

$$h''_{\bar{y}\bar{y}} - \ddot{h}_{\bar{y}\bar{y}} = -2\varkappa \left[\left(\langle \hat{A}\hat{A}^* \rangle - \langle \hat{V}\hat{V}^* \rangle \right) h'_{\bar{y}\bar{z}} \right.$$
$$\left. + \left(\langle \hat{A}\hat{A}^* \rangle + \langle \hat{V}\hat{V}^* \rangle \right) \dot{h}_{\bar{y}\bar{z}} \right], \tag{48}$$

$$h''_{\bar{y}\bar{z}} - \ddot{h}_{\bar{y}\bar{z}} = 2\varkappa \left[\left(\langle \hat{A}\hat{A}^* \rangle - \langle \hat{V}\hat{V}^* \rangle \right) h'_{\bar{y}\bar{y}} \right.$$
$$\left. + \left(\langle \hat{A}\hat{A}^* \rangle + \langle \hat{V}\hat{V}^* \rangle \right) \dot{h}_{\bar{y}\bar{y}} \right]. \tag{49}$$

The algorithm for calculating the right-hand sides of these equations is the same as that for the case I from Sect. 3.1. The appearance of the derivatives of the components h_{yy}, h_{yz} on the right-hand side of these equations, as before, is connected with the presence of the spin connection on the right-hand side of Einstein's equations.

Again, we are looking for the x-plane wave solution in the form (34) and (35). Substituting them into the wave equations (48) and (49) and taking into account (44) and (45), we obtain the following relations:

$$A_1 \left(k^2 - \omega^2 \right) = 2i\varkappa A_2 \left[k \left(\Upsilon_1 - \Upsilon_2 \right) - \omega \left(\Upsilon_1 + \Upsilon_2 \right) \right], \tag{50}$$

$$A_2 \left(k^2 - \omega^2 \right) = -2i\varkappa A_1 \left[k \left(\Upsilon_1 - \Upsilon_2 \right) - \omega \left(\Upsilon_1 + \Upsilon_2 \right) \right], \tag{51}$$

which immediately give

$$A_2 = \pm i A_1 = A_1 e^{\pm i \frac{\pi}{2}}. \tag{52}$$

That is, the phase difference between $\bar{y}\bar{y}$, $\bar{z}\bar{z}$ and $\bar{y}\bar{z}$ components of the GW is again $\pm\pi/2$, as in the case I. In turn, the dispersion relation takes the form

$$k^2 \pm 2\varkappa k \left(\Upsilon_1 - \Upsilon_2 \right) + \left[-\omega^2 \mp 2\varkappa\omega \left(\Upsilon_1 + \Upsilon_2 \right) \right] = 0. \tag{53}$$

Here we have two cases:

(1) For $A_2 = i A_1$, the wave vector is

$$k_{+,1,2} = -\varkappa \left(\Upsilon_1 - \Upsilon_2 \right)$$
$$\pm \sqrt{\varkappa^2 \left(\Upsilon_1 - \Upsilon_2 \right)^2 + \omega^2 + 2\varkappa\omega \left(\Upsilon_1 + \Upsilon_2 \right)}. \tag{54}$$

(2) For $A_2 = -i A_1$, the wave vector is

$$k_{-,1,2} = \varkappa \left(\Upsilon_1 - \Upsilon_2 \right)$$
$$\pm \sqrt{\varkappa^2 \left(\Upsilon_1 - \Upsilon_2 \right)^2 + \omega^2 - 2\varkappa\omega \left(\Upsilon_1 + \Upsilon_2 \right)}. \tag{55}$$

Thus, we see that in both cases there are two GWs with different wave vectors for the same frequency ω. But in the second case a situation may occur where the GW is damped. This happens if

$$\omega^2 - 2\varkappa\omega \left(\Upsilon_1 + \Upsilon_2 \right) + \varkappa^2 \left(\Upsilon_1 - \Upsilon_2 \right)^2 < 0, \tag{56}$$

and the GW becomes damped when ω lies in the region

$$\varkappa \left(\sqrt{\Upsilon_1} - \sqrt{\Upsilon_2} \right)^2 < \omega < \varkappa \left(\sqrt{\Upsilon_1} + \sqrt{\Upsilon_2} \right)^2. \tag{57}$$

Let us consider the simplest case, when $\Upsilon_1 = \Upsilon_2 = \Upsilon$. In this case

$$k_{\pm,1,2} = \pm \sqrt{\omega^2 \pm 4\varkappa\omega\Upsilon}, \tag{58}$$

and for the sign $(-)$ a situation may occur where the GW does not exist. For this case the phase and group velocities of the GWs will be the same as those in the case I from Sect. 3.1.

4 Bianchi identities

Now check the Bianchi identities for Eq. (6),

$$
\left\langle \widehat{\delta T}_{\bar{a}}{}^{\mu} \right\rangle_{;\mu} = \frac{\partial \left\langle \delta \hat{T}_{\bar{a}}{}^{\mu} \right\rangle}{\partial x^{\mu}} = 0. \tag{59}
$$

Here we took into account that the covariant derivative $(\cdots)_{;\mu}$ is calculated in Minkowski spacetime. For the case I we have the following expression for $\widehat{\delta T}_{\bar{a}}{}^{\mu}$:

$$
\widehat{\delta T}_{\bar{a}}{}^{\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4\Upsilon_1 \dot{h}_{\bar{y}\bar{z}} & 4\Upsilon_1 \dot{h}_{\bar{y}\bar{y}} \\ 0 & 0 & 4\Upsilon_1 \dot{h}_{\bar{y}\bar{y}} & 4\Upsilon_1 \dot{h}_{\bar{y}\bar{z}} \end{pmatrix} \tag{60}
$$

with Υ_1 taken from Eqs. (28) and (29).

For the case II we have

$$
\widehat{\delta T}_{\bar{a}}{}^{\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2\left[(\Upsilon_1 - \Upsilon_2)\,h'_{\bar{y}\bar{z}} + (\Upsilon_1 + \Upsilon_2)\,\dot{h}_{\bar{y}\bar{z}}\right] & 2\left[(\Upsilon_1 - \Upsilon_2)\,h'_{\bar{y}\bar{y}} + (\Upsilon_1 + \Upsilon_2)\,\dot{h}_{\bar{y}\bar{y}}\right] \\ 0 & 0 & 2\left[(\Upsilon_1 - \Upsilon_2)\,h'_{\bar{y}\bar{y}} + (\Upsilon_1 + \Upsilon_2)\,\dot{h}_{\bar{y}\bar{y}}\right] & 2\left[(\Upsilon_1 - \Upsilon_2)\,h'_{\bar{y}\bar{z}} + (\Upsilon_1 + \Upsilon_2)\,\dot{h}_{\bar{y}\bar{z}}\right] \end{pmatrix} \tag{61}
$$

with $\Upsilon_{1,2}$ given by Eqs. (44) and (45). For both cases one can show by direct calculation that

$$
\frac{\partial \left\langle \delta \hat{T}_{\bar{a}}{}^{\mu} \right\rangle}{\partial x^{\mu}} = 0. \tag{62}
$$

It is interesting that in both cases operations of evaluating the covariant derivative and the quantum averaging commutate. To show this, let us calculate the averaged Bianchi identities

$$
\left\langle \left(\widehat{\delta T}_{\bar{a}}{}^{\mu}\right)_{;\mu} \right\rangle = \left\langle \frac{\partial \widehat{\delta T}_{\bar{a}}{}^{\mu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\nu\mu} \widehat{\delta T}_{\bar{a}}{}^{\nu} - \omega^{\bar{c}}{}_{\bar{a}\nu} \widehat{\delta T}_{\bar{c}}{}^{\nu} \right\rangle
$$
$$
= \left\langle \frac{\partial \widehat{\delta T}_{\bar{a}}{}^{\mu}}{\partial x^{\mu}} \right\rangle = \frac{\partial \left\langle \delta \hat{T}_{\bar{a}}{}^{\mu} \right\rangle}{\partial x^{\mu}} = 0. \tag{63}
$$

Here we took into account that both the unperturbed Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma} = 0$ and the unperturbed spin connection $\omega_{\bar{a}\bar{b}\mu} = 0$, since they are calculated for Minkowski spacetime.

5 Conclusions

We have considered the process of propagation of GWs on the background of the nonperturbative vacuum of spinor fields.

Using the simplifying assumptions from Sect. 2.2, it was shown that there are several distinctive features in comparison with the propagation of GWs through empty space:

- There exists a fixed phase difference of $\pm\pi/2$ between components $h_{yy,zz}$ and h_{yz}.
- The phase and group velocities of GWs are not equal to the velocity of light. Moreover, the group velocity is always less than the velocity of light.
- The components $h_{yy,zz}$ and h_{yz} exist together only.
- Depending on the properties of the spinor vacuum, the damping of GWs may occur for some frequencies ω of the spinor field, or no GW may exist.
- For given frequency ω, there exist two waves with different wave vectors k.

All features mentioned above can in principle be verified after the experimental detection of GWs. Then the simplest test will be to verify the existence of the phase difference. In addition, one might expect that GWs could be a fruitful tool for studying *nonperturbative* quantum field theories.

Appendix: Perturbed energy-momentum tensor

The perturbed component of the energy-momentum tensor is calculated as follows:

$$
\widehat{\delta T}_{\bar{a}\bar{b}} = -\frac{i}{2}\hat{\bar{\psi}} \left[\gamma_{(\bar{a}} \delta\Gamma_{\bar{b})} + \delta\Gamma_{(\bar{a}} \gamma_{\bar{b})} \right] \hat{\psi}, \tag{64}
$$

where the perturbed spinor connection is

$$
\delta\Gamma_{\bar{a}} = \delta\left(e_{\bar{a}}{}^{\mu} \Gamma_{\mu}\right) = -\frac{1}{4}\left(\delta e_{\bar{a}}{}^{\mu} \omega_{\bar{b}\bar{c}\mu} + e_{\bar{a}}{}^{\mu} \delta\omega_{\bar{b}\bar{c}\mu}\right) \gamma^{\bar{b}} \gamma^{\bar{c}}, \tag{65}
$$

where the spin connection $\Gamma_\mu = -\frac{1}{4}\omega_{\bar{a}\bar{b}\mu}\gamma^{\bar{a}}\gamma^{\bar{b}}$. The perturbed vierbein $\delta e_{\bar{a}}{}^\mu$ is

$$\delta e_{\bar{a}}{}^\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{\bar{y}\bar{y}} & h_{\bar{y}\bar{z}} \\ 0 & 0 & h_{\bar{z}\bar{y}} & h_{\bar{z}\bar{z}} \end{pmatrix}. \tag{66}$$

Using the standard definitions of the covariant derivative of a spinor, ∇_μ, and the spin connection, Γ_μ, $\omega_{\bar{a}\bar{b}\mu}$,

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi, \tag{67}$$

$$\Gamma_\mu = -\frac{1}{4}\omega_{\bar{a}\bar{b}\mu}\gamma^{\bar{a}}\gamma^{\bar{b}}, \tag{68}$$

$$\omega_{\bar{a}\bar{b}\mu} = -e_{\bar{a}}{}^\alpha e_{\bar{b}}{}^\beta \Delta_{\alpha\beta\mu} \tag{69}$$

with

$$\Delta_{\alpha\beta\gamma} = e_{\bar{a}\alpha}\Sigma^{\bar{a}}{}_{\beta\gamma} - e_{\bar{a}\beta}\Sigma^{\bar{a}}{}_{\alpha\gamma} - e_{\bar{a}\gamma}\Sigma^{\bar{a}}{}_{\alpha\beta}, \tag{70}$$

$$\Sigma^{\bar{a}}{}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial e^{\bar{a}}{}_\mu}{\partial x^\nu} - \frac{\partial e^{\bar{a}}{}_\nu}{\partial x^\mu}\right), \tag{71}$$

one can obtain the perturbed spin connection

$$\delta\omega_{\bar{a}\bar{b}\mu} = -\delta e_{\bar{a}}{}^\alpha e_{\bar{b}}{}^\beta \Delta_{\alpha\beta\mu} - e_{\bar{a}}{}^\alpha \delta e_{\bar{b}}{}^\beta \Delta_{\alpha\beta\mu} - e_{\bar{a}}{}^\alpha e_{\bar{b}}{}^\beta \delta\Delta_{\alpha\beta\mu} \tag{72}$$

with

$$\begin{aligned} \delta\Delta_{\alpha\beta\gamma} &= \delta e_{\bar{a}\alpha}\Sigma^{\bar{a}}{}_{\beta\gamma} + e_{\bar{a}\alpha}\delta\Sigma^{\bar{a}}{}_{\beta\gamma} - \delta e_{\bar{a}\beta}\Sigma^{\bar{a}}{}_{\alpha\gamma} \\ &\quad - e_{\bar{a}\beta}\delta\Sigma^{\bar{a}}{}_{\alpha\gamma} - \delta e_{\bar{a}\gamma}\Sigma^{\bar{a}}{}_{\alpha\beta} - e_{\bar{a}\gamma}\delta\Sigma^{\bar{a}}{}_{\alpha\beta}, \end{aligned} \tag{73}$$

$$\delta\Sigma^{\bar{a}}{}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \delta e^{\bar{a}}{}_\mu}{\partial x^\nu} - \frac{\partial \delta e^{\bar{a}}{}_\nu}{\partial x^\mu}\right). \tag{74}$$

Here $\Delta_{\alpha\beta\gamma}$ and $\delta\Delta_{\alpha\beta\gamma}$ are the unperturbed and perturbed Ricci coefficients; $\Sigma^{\bar{a}}{}_{\alpha\beta}$ and $\delta\Sigma^{\bar{a}}{}_{\alpha\beta}$ are the unperturbed and perturbed anholonomy coefficients; $\delta\omega_{\bar{a}\bar{b}\mu}$ are the perturbed spin connection.

Substituting (66) into (72) and taking into account (73) and (74), we have

$$\delta\omega_{\bar{t}\bar{y}y} = \delta\omega_{\bar{x}\bar{y}y} = -\delta\omega_{\bar{t}\bar{z}z} = -\delta\omega_{\bar{x}\bar{z}z} = \dot{h}_{\bar{y}\bar{y}}, \tag{75}$$

$$\delta\omega_{\bar{t}\bar{z}y} = \delta\omega_{\bar{x}\bar{z}y} = \delta\omega_{\bar{t}\bar{y}z} = \delta\omega_{\bar{x}\bar{y}z} = \dot{h}_{\bar{y}\bar{z}}. \tag{76}$$

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