

# Finite field-energy and interparticle potential in logarithmic electrodynamics

Patricio Gaete<sup>1,a</sup>, José Helayël-Neto<sup>2,b</sup>

<sup>1</sup> Departamento de Física and Centro Científico-Tecnológico de Valparaíso, Universidad Técnica Federico Santa María, Valparaíso, Chile

<sup>2</sup> Centro Brasileiro de Pesquisas Físicas (CBPF), Rio de Janeiro, RJ, Brasil

Received: 23 December 2013 / Accepted: 6 March 2014 / Published online: 18 March 2014

© The Author(s) 2014. This article is published with open access at Springerlink.com

**Abstract** We pursue an investigation of logarithmic electrodynamics, for which the field energy of a point-like charge is finite, as happens in the case of the usual Born–Infeld electrodynamics. We also show that, contrary to the latter, logarithmic electrodynamics exhibits the feature of birefringence. Next, we analyze the lowest-order modifications for both logarithmic electrodynamics and for its non-commutative version, within the framework of the gauge-invariant path-dependent variables formalism. The calculation shows a long-range correction ( $1/r^5$ -type) to the Coulomb potential for logarithmic electrodynamics. Interestingly enough, for its non-commutative version, the interaction energy is ultraviolet finite. We highlight the role played by the new quantum of length in our analysis.

## 1 Introduction

The photon–photon scattering of Quantum Electrodynamics (QED) and its physical consequences such as vacuum birefringence and vacuum dichroism have been of great interest since its earliest days [1–7]. Even though this subject has had a revival after recent results of the PVLAS collaboration [8,9], the issue remains as relevant as ever. We also point out alternative scenarios such as Born–Infeld theory [10], millicharged particles [11] or axion-like particles [12–14] in order to account for the results reported by the PVLAS collaboration.

We further note that recently considerable attention has been paid to the study of nonlinear electrodynamics due to its natural emergence from D-brane physics, where the Born–Infeld theory plays a prominent role. In addition to the string interest, nonlinear electrodynamics has also been investigated in the context of gravitational physics. In fact,

Hoffman [15] was the one who first considered the connection between gravity and nonlinear electrodynamics (Born–Infeld theory). In passing we recall that these nonlinear gauge theories are endowed with interesting features, like a finite electron self-energy and a regular point charge electric field at the origin. Very recently, in addition to Born–Infeld theory, other types of nonlinear electrodynamics have been studied in the context of black hole physics [16–19].

Let us also mention here that Lagrangian densities of nonlinear extensions of electrodynamics with a logarithmic function of the electromagnetic field strengths are a typical characteristic of QED effective actions. In the classical work by Euler and Heisenberg [20], in which the authors studied electrons in a background set up by a uniform electromagnetic field, a logarithmic term of the field strength came out as an exact 1-loop correction to the vacuum polarization. Furthermore, some years ago, Volovik [21] has worked out the action for an electromagnetic field that emerges as a collective field in superfluid  $^3\text{He} - A$ ; this 4-dimensional action exhibits a logarithmic factor whose argument is a function of the electromagnetic field strengths [22].

It is perhaps worth to better motivate the choice of the so-called logarithmic electrodynamics to investigate finiteness of the field energy and to work out the interparticle potential. If we are bound to considering the regime of slow-varying fields, namely, electric and magnetic fields (let us denote both generically by  $F$ ) such that

$$\frac{|\nabla F|}{|F|} \ll \frac{mc}{\hbar} \quad (1)$$

and

$$\frac{|d/dt F|}{|F|} \ll \frac{mc^2}{\hbar}, \quad (2)$$

where  $m$  stands for electron’s mass, we can assume that we are actually considering the physics of purely photonic processes. To justify our interest in pursuing our investigation

<sup>a</sup> e-mail: patricio.gaete@usm.cl

<sup>b</sup> e-mail: helayel@cbpf.br

in the framework of logarithmic electrodynamics, we stress that it is not a model aimed at purely mathematical purposes. As clearly reported in [23], the effective Lagrangian induced by radiative corrections in the regime of slowly varying fields increases logarithmically with the field strengths, in the limit of high field intensities, and this result holds true even if the **E**-and the **B**-fields are stronger than the well-known critical value  $\frac{m^2 c^3}{e\hbar}$ . So, logarithmic electrodynamics is of actual interest for the study of photonic processes in special regimes of the electromagnetic fields. Another interesting path, which attributes to nonlinearity magnetic properties of electric monopoles, is the main content of the paper in Ref. [24].

On the other hand, it is worth recalling here that the study of extensions of the Standard Model (SM) such as Lorentz invariance violation and fundamental length, have attracted much attention in the past years [25–29]. As is well known, this is mainly so because the SM does not include a quantum theory of gravitation. In fact, the necessity of a new scenario has been suggested to overcome difficulties theoretical in the quantum gravity research. Among these new scenarios, probably the most studied framework are quantum field theories allowing non-commuting position operators [30–35], where this non-commutativity is an intrinsic property of spacetime. We call attention to the fact that these studies have been achieved by using a star product (Moyal product). More recently, a novel way to formulate non-commutative field theory (or quantum field theory in the presence of a minimal length) has been proposed in [36–38]. Later, it has been shown that this approach can be summarized through the introduction of a new multiplication rule which is known as a Voros star product. Evidently, physics will turn out be independent from the choice of the type of product [39]. With these ideas in mind, in previous studies [40,41], we have considered the effect of the spacetime non-commutativity on a physical observable. In fact, we have computed the static potential for axionic electrodynamics both in (3 + 1) and (2 + 1) spacetime dimensions, in the presence of a minimal length. The point we wish to emphasize, however, is that our analysis leads to a well-defined non-commutative interaction energy. Indeed, in both cases we have obtained a fully ultraviolet finite static potential. Later, we have extended our analysis for both Yang–Mills theory and gluodynamics in curved spacetime, where we have obtained a finite string tension [42].

Given the ongoing experiments related to photon–photon interaction physics [43–45], it is desirable to have some additional understanding of the physical consequences presented by a particular nonlinear electrodynamics, that is, logarithmic electrodynamics. Of special interest will be the study of aspects of birefringence as well as the computation of physical observables. In particular, we mention the static potential between two charges, using the gauge-invariant but path-

dependent variables formalism, which is an alternative to the Wilson loop approach. We further note that the model under consideration satisfies the criteria of causality and unitarity as studied in [46].

Our work is organized according to the following outline: in Sect. 2, we present general aspects of logarithmic electrodynamics, show that it yields birefringence and compute the finite electrostatic field energy of a point-like charge. In Sect. 3, we analyze the interaction energy for a fermion–antifermion pair in the usual logarithmic electrodynamics and its version in the presence of a minimal length. Finally, in Sect. 4, we make final remarks.

## 2 The model under consideration

The model under consideration is described by the Lagrangian density:

$$\mathcal{L} = -\beta^2 \ln \left[ 1 + \frac{\mathcal{F}}{\beta^2} - \frac{\mathcal{G}^2}{2\beta^4} \right], \tag{3}$$

where  $\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$  and  $\mathcal{G} = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}$ . As usual,  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the electromagnetic field strength tensor and  $\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\lambda} F_{\rho\lambda}$  is the dual electromagnetic field strength tensor.

The equations of motion following from the Lagrangian density (3) read

$$\partial_\mu \left[ \frac{1}{\Gamma} \left( F^{\mu\nu} - \frac{1}{\beta^2} \mathcal{G} \tilde{F}^{\mu\nu} \right) \right] = 0, \tag{4}$$

while the Bianchi identities are

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \tag{5}$$

where

$$\Gamma = 1 + \frac{\mathcal{F}}{\beta^2} - \frac{\mathcal{G}^2}{2\beta^4}. \tag{6}$$

It follows from the above discussion that Gauss’ law takes the form

$$\nabla \cdot \mathbf{D} = 0, \tag{7}$$

where **D** is given by

$$\mathbf{D} = \frac{\mathbf{E} + \frac{1}{\beta^2} (\mathbf{E} \cdot \mathbf{B}) \mathbf{B}}{1 - \frac{(\mathbf{E}^2 - \mathbf{B}^2)}{2\beta^2} - \frac{1}{2\beta^4} (\mathbf{E} \cdot \mathbf{B})^2}. \tag{8}$$

For  $J^0(t, \mathbf{r}) = e\delta^{(3)}(\mathbf{r})$ , the electric field follows as

$$\mathbf{E} = \frac{\beta^2}{Q} \left( \sqrt{r^4 + \frac{2Q^2}{\beta^2}} - r^2 \right) \hat{r}, \tag{9}$$

or, what is the same,

$$\mathbf{E} = 2Q \frac{1}{\left(\sqrt{r^4 + 2\left(\frac{Q}{\beta}\right)^2 + r^2}\right)} \hat{r}. \tag{10}$$

$\hat{r} = \frac{r}{|r|}$  and  $Q \equiv \frac{e}{4\pi}$ . From this expression it should be clear that the electric field of a point-like charge is maximum at the origin,  $E_{\max} = \sqrt{2}\beta$ ; in the usual Born–Infeld electrodynamics,  $E_{\max} = \beta$ .

In order to write the dynamical equations in a more compact and convenient form, we shall introduce the vectors  $\mathbf{D} = \partial\mathcal{L}/\partial\mathbf{E}$  and  $\mathbf{H} = -\partial\mathcal{L}/\partial\mathbf{B}$ , in analogy to the electric displacement and magnetic field strength. We then have

$$\mathbf{D} = \frac{1}{\Gamma} \left( \mathbf{E} + \frac{\mathbf{B}(\mathbf{E} \cdot \mathbf{B})}{\beta^2} \right) \tag{11}$$

and

$$\mathbf{H} = \frac{1}{\Gamma} \left( \mathbf{B} - \frac{\mathbf{E}(\mathbf{E} \cdot \mathbf{B})}{\beta^2} \right). \tag{12}$$

As in the case of usual Born–Infeld electrodynamics, it is worthwhile to invert Eq. (11), so that we can express  $\mathbf{E}$  in terms of  $\mathbf{D}$  (and  $\mathbf{B}$ ), in analogy with the Hamiltonian treatment ( $\mathbf{E}$  could be thought of as being the velocity, whereas  $\mathbf{D}$  plays the role of the momentum). Lengthy algebraic manipulations yield

$$\mathbf{E} = \xi\mathbf{D} + \tilde{\xi}\mathbf{B}, \tag{13}$$

where

$$\xi = \frac{-\beta^2(\beta^2 + \mathbf{B}^2)}{[\beta^2\mathbf{D}^2 + (\mathbf{B} \times \mathbf{D})^2]} + \frac{\sqrt{\beta^4(\beta^2 + \mathbf{B}^2)^2 + (\beta^2 + \mathbf{B}^2)(2\beta^2 + \mathbf{D}^2)[\beta^2\mathbf{D}^2 + (\mathbf{B} \times \mathbf{D})^2]}}{[\beta^2\mathbf{D}^2 + (\mathbf{B} \times \mathbf{D})^2]} \tag{14}$$

and

$$\tilde{\xi} \equiv \frac{1}{\sqrt{\beta^2 + \mathbf{B}^2}} \sqrt{\mathbf{D}^2\xi^2 + 2\beta^2\xi - (2\beta^2 + \mathbf{B}^2)}. \tag{15}$$

Now that we have inverted  $\mathbf{E}$  in terms of  $\mathbf{D}$ , let us also re-express  $\mathbf{H}$  in terms of  $\mathbf{B}$  and  $\mathbf{D}$ . We arrive at

$$\mathbf{H} = \frac{1}{\xi} \left( 1 + \tilde{\xi}^2 \right) \mathbf{B} + \xi\mathbf{D}. \tag{16}$$

With this, we can write the corresponding equations of motion as

$$\nabla \cdot \mathbf{D} = 0, \quad \frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = 0, \tag{17}$$

and

$$\nabla \cdot \mathbf{B} = 0, \quad \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0. \tag{18}$$

Now, employing (11) and (12), we obtain the electric permittivity  $\varepsilon_{ij}$  and the inverse magnetic permeability  $(\mu^{-1})_{ij}$  tensors of the vacuum, that is,

$$\varepsilon_{ij} = \frac{1}{\Gamma} \left( \delta_{ij} + \frac{1}{\beta^2} B_i B_j \right), \tag{19}$$

$$(\mu^{-1})_{ij} = \frac{1}{\Gamma} \left( \delta_{ij} - \frac{1}{\beta^2} E_i E_j \right),$$

with  $D_i = \varepsilon_{ij} E_j$  and  $B_i = \mu_{ij} H_j$ .

It is now important to notice that the complicated field problem can be greatly simplified if Eq. (19) are linearized. As is well known, this procedure is justified for the description of a weak electromagnetic wave  $(\mathbf{E}_p, \mathbf{B}_p)$  propagating in the presence of a strong constant external field  $(\mathbf{E}_0, \mathbf{B}_0)$ . For computational simplicity our analysis will be developed in the case of a purely magnetic field, that is,  $\mathbf{E}_0 = 0$ . This then implies that

$$\mathbf{D} = \frac{1}{\left(1 + \frac{\mathbf{B}_0^2}{2\beta^2}\right)} \left[ \mathbf{E}_p + \frac{1}{\beta^2} (\mathbf{E}_p \cdot \mathbf{B}_0) \mathbf{B}_0 \right] \tag{20}$$

and

$$\mathbf{H} = \frac{1}{\left(1 + \frac{\mathbf{B}_0^2}{2\beta^2}\right)} \left[ \mathbf{B}_p - \frac{1}{\beta^2 \left(1 + \frac{\mathbf{B}_0^2}{2\beta^2}\right)} (\mathbf{B}_p \cdot \mathbf{B}_0) \mathbf{B}_0 \right], \tag{21}$$

where we have keep only linear terms in  $\mathbf{E}_p, \mathbf{B}_p$ .

Next, without loss of generality we take the  $z$  axis as the direction of the magnetic field,  $\mathbf{B}_0 = B_0\mathbf{e}_3$ , and we assume that the light wave moves along the  $x$  axis. We further make a plane wave decomposition for the fields  $E_p$  and  $B_p$ , that is,

$$\mathbf{E}_p(\mathbf{x}, t) = \mathbf{E}e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})}, \quad \mathbf{B}_p(\mathbf{x}, t) = \mathbf{B}e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})}, \tag{22}$$

so that the Maxwell equations become

$$\left( \frac{k^2}{w^2} - \varepsilon_{22}\mu_{33} \right) E_2 = 0 \tag{23}$$

and

$$\left( \frac{k^2}{w^2} - \varepsilon_{33}\mu_{22} \right) E_3 = 0. \tag{24}$$

As a consequence, we have two different situations: First, if  $\mathbf{E} \perp \mathbf{B}_0$  (perpendicular polarization), from (24)  $E_3 = 0$ , and from (23) we get  $\frac{k^2}{w^2} = \varepsilon_{22}\mu_{33}$ . Hence we see that the dispersion relation of the photon takes the form

$$n_{\perp} = \sqrt{\frac{1 + B_0^2/2\beta^2}{1 - B_0^2/2\beta^2}}. \tag{25}$$

Second, if  $\mathbf{E} \parallel \mathbf{B}_0$  (parallel polarization), from (23)  $E_2 = 0$ , and from (24) we get  $\frac{k^2}{w^2} = \epsilon_{33}\mu_{22}$ . In this case, the corresponding dispersion relation becomes

$$n_{\parallel} = \sqrt{1 + B_0^2/\beta^2}. \tag{26}$$

This implies that the electromagnetic waves with different polarizations have different velocities or, more precisely, the vacuum birefringence phenomenon is present. Before concluding this section, we should comment on our result. The above result give us an opportunity to compare our result with the related nonlinear electrodynamics, that is, Born–Infeld theory. In this case, the theory is written with a square root instead of a logarithm as in (3), and the phenomenon of birefringence is absent. However, in the case of generalized Born–Infeld electrodynamics [47], which contains two different parameters, again the phenomenon of birefringence is present.

Another relevant aspect to compare in both Born–Infeld and logarithmic electrodynamics is the calculation of the finite energy stored in the electrostatic field of a point-like charge; in the case of logarithmic electrodynamics, this field is given by Eqs. (9) and (10). With the general expression for the energy density (the  $\Theta^{00}$ -component of the energy-momentum tensor,  $\Theta^{\mu\nu}$ ):

$$\Theta^{\mu}_{\nu} = \frac{\partial \mathcal{L}}{\partial \mathcal{F}} F^{\mu\rho} F_{\nu\rho} + \frac{\partial \mathcal{L}}{\partial \mathcal{G}} \tilde{F}^{\mu\rho} F_{\nu\rho} - \delta^{\mu}_{\nu} \mathcal{L}, \tag{27}$$

$$\Theta^{00} = \frac{1}{\Gamma} \mathbf{E}^2 + \frac{1}{\beta^2 \Gamma} \mathbf{E} \cdot \mathbf{B} + \beta^2 \ln \Gamma \tag{28}$$

( $\Gamma$  is given by Eq. (6)), in our particular case,

$$\Theta^{00} = \frac{\mathbf{E}^2}{1 - \frac{\mathbf{E}^2}{2\beta^2}} + \beta^2 \ln \left( 1 - \frac{\mathbf{E}^2}{2\beta^2} \right). \tag{29}$$

From this, we are able to write down the overall (finite) stored electrostatic energy:

$$E_{\text{fin}} = 2\pi Q^{3/2} \beta^{1/2} (I_1 + I_2), \tag{30}$$

where

$$I_1 \equiv \int_0^{\infty} d\lambda \sqrt{\lambda} \frac{(\sqrt{2 + \lambda^2} - \lambda)^2}{1 - \frac{1}{2}(\sqrt{2 + \lambda^2} - \lambda)^2} \tag{31}$$

and

$$I_2 \equiv \int_0^{\infty} d\lambda \sqrt{\lambda} \ln \left[ 1 - \frac{1}{2}(\sqrt{2 + \lambda^2} - \lambda)^2 \right]. \tag{32}$$

Both integrals are finite:  $I_1 = 4.157$  and  $I_2 = -1.385$ ; from that, we get  $E_{\text{fin}}$  as given below:

$$E_{\text{fin}} = 0.391 \sqrt{e^3 \beta}, \tag{33}$$

to be compared with the corresponding value in the usual Born–Infeld case [48]:

$$E_{\text{fin}}^{BI} = 1.236 \sqrt{e^3 \beta}. \tag{34}$$

By virtue of the logarithmic form of our action (instead of the square root in the Born–Infeld case), it becomes clear why the stored electrostatic energy is smaller, in comparison with the case of Born–Infeld. To get an estimate of the coupling parameter  $\beta$ , we could identify the maximal electrostatic field,

$$|E_{\text{max}}| = \sqrt{2}\beta, \tag{35}$$

with the natural fundamental field that appears in terms of the electron’s charge and mass,  $m_e$ , and the fundamental constants  $c$  and  $\hbar$ :

$$\mathcal{E}_{\text{fund}} = \frac{m_e^2 c^3}{e \hbar}. \tag{36}$$

In natural units ( $\hbar = c = 1$ ), its value is

$$\mathcal{E}_{\text{fund}} = 5.981 \text{ MeV}^2, \tag{37}$$

which corresponds to  $2.590 \times 10^{18} \text{ N/C}$ .

If we assume that  $\beta$  is fixed by  $\mathcal{E}_{\text{fund}}$ ,

$$\beta = \frac{m^2}{\sqrt{2}e}, \tag{38}$$

then we may compute the total amount of electrostatic energy,  $U$ , stored in a domain whose radius is the electron’s Compton length ( $R = \frac{1}{m}$ ). We get

$$U = 4\pi \int_0^{1/m} dr r^2 \Theta^{00} = 8.67 \times 10^{-4} m_e, \tag{39}$$

after we take  $\beta$  given by Eq. (38) and the integrals  $I_1$  and  $I_2$  of Eqs. (31) and (32) are carried out over the region that corresponds to the electron’s Compton length.

At this point, we would like to draw the reader’s attention to the recent work by Costa et al. [49], where these authors investigate a nonlinear gauge-invariant extension of classical electrodynamics, quartic in the field strength (they consider an  $F^2$ -term) and also attain a finite value for the field energy of a point-like charge.

### 3 Interaction energy

As already stated, our principal purpose is to calculate explicitly the interaction energy between static point-like sources for logarithmic electrodynamics. To this end we will calculate the expectation value of the energy operator  $H$  in the

physical state  $|\Phi\rangle$ , which we will denote by  $\langle H \rangle_\Phi$ . The starting point is the Lagrangian (3), that is,

$$\mathcal{L} = -\beta^2 \ln \left[ 1 + \frac{\mathcal{F}}{\beta^2} - \frac{\mathcal{G}^2}{2\beta^4} \right]. \tag{40}$$

Now, in order to handle the logarithm in the Lagrangian density (40), we will introduce an auxiliary field  $v$  such that its equation of motion gives back the original theory. Expressed in terms of this field, the corresponding Lagrangian takes the form

$$\mathcal{L} = \beta^2 - \beta^2 \ln \beta^4 + \beta^2 \ln v - \frac{v}{\beta^2} \left[ 1 + \frac{1}{4\beta^2} F_{\mu\nu}^2 - \frac{1}{32\beta^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right]. \tag{41}$$

This equation may look peculiar, but this is nothing but the expression (40). In fact, since the  $v$ -field is an auxiliary one, it can be readily eliminated by means of its (algebraic) field equation. In doing so, we get

$$v = \frac{\beta^4}{1 + \frac{1}{4\beta^2} F_{\mu\nu}^2 - \frac{1}{32\beta^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2}, \tag{42}$$

and using it we recover Eq. (40).

With this at hand, the canonical momenta are  $\Pi^\mu = -\frac{v}{\beta^4} (F^{0\mu} - \frac{v}{4\beta^2} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \tilde{F}^{0\mu})$ , and one immediately identifies the two primary constraints  $\Pi^0 = 0$  and  $p \equiv \frac{\partial \mathcal{L}}{\partial v} = 0$ . The canonical Hamiltonian of the model can be worked out as usual and is given by the expression

$$H_C = \int d^3x \left\{ \Pi_i \partial^i A_0 + \frac{\beta^4}{2v} \Pi^2 + \frac{v}{\beta^2} \left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right) - \frac{\beta^2 (\boldsymbol{\Pi} \cdot \mathbf{B}^2)}{2v \left( 1 + \frac{\mathbf{B}^2}{\beta^2} \right)} + \beta^2 - \beta^2 \ln \beta^4 + \beta^2 \ln v \right\}. \tag{43}$$

Now, requiring the primary constraint  $\Pi_0$  to be preserved in time yields the secondary constraint (Gauss' law)  $\Gamma_1(x) \equiv \partial_i \Pi^i = 0$ . Similarly, the consistency condition for the constraint  $p$  yields no further constraints and just determines the  $v$ -field,

$$v = \frac{\beta^4}{2} \left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right) \times \left\{ 1 + \sqrt{1 + \frac{2}{\beta^2} \left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right) \left[ \Pi^2 - \frac{(\mathbf{B} \cdot \boldsymbol{\Pi})^2}{\beta^2 \left( 1 + \frac{\mathbf{B}^2}{\beta^2} \right)} \right]} \right\}. \tag{44}$$

The extended Hamiltonian that generates translations in time then reads  $H = H_C + \int d^3x (c_0(x)\Pi_0(x) + c_1(x)\Gamma_1(x))$ , where  $c_0(x)$  and  $c_1(x)$  are Lagrange multipliers. In addition, neither  $A_0(x)$  nor  $\Pi_0(x)$  is of interest in describing

the system and may be discarded from the theory. Thus we are left with the following expression for the Hamiltonian:

$$H = \int d^3x \left[ c'(x) \partial_i \Pi^i + \Pi^2 \frac{\left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right)}{\left\{ 1 + \sqrt{1 + \frac{2}{\beta^2} \left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right) \left[ \Pi^2 - \frac{(\mathbf{B} \cdot \boldsymbol{\Pi})^2}{\beta^2 \left( 1 + \frac{\mathbf{B}^2}{\beta^2} \right)} \right]} \right\}} \right] + \frac{\mathbf{B}^2}{4} \frac{1}{\left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right)} \left\{ 1 + \sqrt{1 + \frac{2}{\beta^2} \left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right) \left[ \Pi^2 - \frac{(\mathbf{B} \cdot \boldsymbol{\Pi})^2}{\beta^2 \left( 1 + \frac{\mathbf{B}^2}{\beta^2} \right)} \right]} \right\} - \frac{(\mathbf{B} \cdot \boldsymbol{\Pi})^2}{4\beta^2 \left( 1 + \frac{\mathbf{B}^2}{\beta^2} \right)} \frac{\left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right)}{\left\{ 1 + \sqrt{1 + \frac{2}{\beta^2} \left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right) \left[ \Pi^2 - \frac{(\mathbf{B} \cdot \boldsymbol{\Pi})^2}{\beta^2 \left( 1 + \frac{\mathbf{B}^2}{\beta^2} \right)} \right]} \right\}} + \frac{\beta^2 \left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right)}{2} \left\{ 1 + \sqrt{1 + \frac{2}{\beta^2} \left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right) \left[ \Pi^2 - \frac{(\mathbf{B} \cdot \boldsymbol{\Pi})^2}{\beta^2 \left( 1 + \frac{\mathbf{B}^2}{\beta^2} \right)} \right]} \right\} - \beta^2 + \beta^2 \ln \beta^4 - \beta^2 \ln \left[ \frac{\beta^4}{2} \left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right) \right] - \beta^2 \ln \left\{ 1 + \sqrt{1 + \frac{2}{\beta^2} \left( 1 + \frac{\mathbf{B}^2}{2\beta^2} \right) \left[ \Pi^2 - \frac{(\mathbf{B} \cdot \boldsymbol{\Pi})^2}{\beta^2 \left( 1 + \frac{\mathbf{B}^2}{\beta^2} \right)} \right]} \right\}. \tag{45}$$

where  $c'(x) = c_1(x) - A_0(x)$ .

Next, since there is one first class constraint  $\Gamma_1(x)$  (Gauss' law), we choose one gauge fixing condition that will make the full set of constraints become second class. We choose the gauge fixing condition so as to correspond to

$$\Gamma_2(x) \equiv \int_{C_{\xi x}} dz^v A_v(z) \equiv \int_0^1 d\lambda x^i A_i(\lambda x) = 0, \tag{46}$$

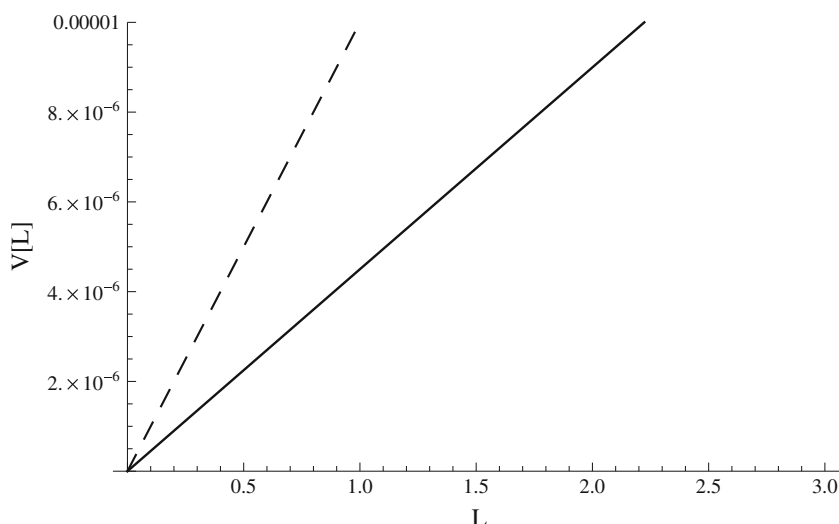
where  $\lambda$  ( $0 \leq \lambda \leq 1$ ) is the parameter describing the space-like straight path  $x^i = \xi^i + \lambda(x - \xi)^i$ , and  $\xi$  is a fixed point (reference point). There is no essential loss of generality if we restrict our considerations to  $\xi^i = 0$ . The choice (46) leads to the Poincaré gauge [50,51]. As a consequence, we can now write down the only nonvanishing Dirac bracket for the canonical variables,

$$\{A_i(x), \Pi^j(y)\}^* = \delta_i^j \delta^{(3)}(x - y) - \partial_i^x \int_0^1 d\lambda x^i \delta^{(3)}(\lambda x - y). \tag{47}$$

We are now in a position to compute the potential energy for static charges in this theory. To do this, we consider use of the gauge-invariant scalar potential which is given by

$$V \equiv e (\mathcal{A}_0(\mathbf{0}) - \mathcal{A}_0(\mathbf{L})), \tag{48}$$

**Fig. 1** Shape of the potential, Eqs. (52) (solid line) and (55) (dashed line)



where the physical scalar potential is given by

$$\mathcal{A}_0(t, \mathbf{r}) = \int_0^1 d\lambda r^i E_i(t, \lambda \mathbf{r}). \tag{49}$$

This equation follows from the vector gauge-invariant field expression

$$\mathcal{A}_\mu(x) \equiv A_\mu(x) + \partial_\mu \left( - \int_\xi^x dz^\mu A_\mu(z) \right), \tag{50}$$

where the line integral is along a spacelike path from the point  $\xi$  to  $x$ , on a fixed slice time. It should be noted that the gauge-invariant variables (50) commute with the sole first constraint (Gauss' law), showing in this way that these fields are physical variables.

Having made these observations, we see that Gauss' law for the present theory (obtained from the Hamiltonian formulation above) leads to  $\partial_i \Pi^i = J^0$ , where we have included the external current  $J^0$  to represent the presence of two opposite charges. For  $J^0(t, \mathbf{x}) = Q\delta^{(3)}(\mathbf{x})$ , the electric field then becomes

$$\mathbf{E} = \frac{Q}{2\pi} \frac{1}{\left( \sqrt{r^4 + 2 \left( \frac{Q}{\beta 4\pi} \right)^2} + r^2 \right)} \hat{r}. \tag{51}$$

As a consequence, Eq. (49) becomes

$$\mathcal{A}_0 = -\frac{Q}{2\pi} \left\{ \frac{2\sqrt{2}\beta}{Q} {}_2F_1 \left( -\frac{1}{2}, \frac{1}{4}, \frac{5}{4}, -\frac{8\pi^2\beta^2}{Q^2} r^4 \right) r - \frac{8\pi^2\beta^2}{3Q^2} r^3 \right\}, \tag{52}$$

where  ${}_2F_1$  is the hypergeometric function. In terms of  $\mathcal{A}_0(t, \mathbf{r})$ , the potential for a pair of static point-like opposite charges located at  $\mathbf{0}$  and  $\mathbf{L}$ , is given by

$$\begin{aligned} V &\equiv Q (\mathcal{A}_0(\mathbf{0}) - \mathcal{A}_0(\mathbf{L})) \\ &= \frac{Q^2}{2\pi} \left\{ \frac{2\sqrt{2}\beta}{Q} {}_2F_1 \left( -\frac{1}{2}, \frac{1}{4}, \frac{5}{4}, -\frac{8\pi^2\beta^2}{Q^2} L^4 \right) L - \frac{8\pi^2\beta^2}{3Q^2} L^3 \right\}, \end{aligned} \tag{53}$$

with  $L = |\mathbf{L}|$ .

The above analysis give us an opportunity to compare logarithmic electrodynamics with related Born–Infeld electrodynamics. In this case, the electric field is given by

$$\mathbf{E} = \frac{Q}{4\pi} \frac{1}{\sqrt{r^4 + \frac{Q^2}{(4\pi\beta)^2}}} \hat{r}, \tag{54}$$

from which it follows that

$$V = Q\beta {}_2F_1 \left( \frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{4\pi\beta}{Q} L^2 \right) L. \tag{55}$$

The plot of Eqs. (52) and (55) is showed in Fig. 1.

We further note that the scalar potential for logarithmic electrodynamics, at leading order in  $\beta$ , takes the form

$$\mathcal{A}_0(t, \mathbf{r}) = -\frac{Q}{8\pi r} \int_0^1 d\lambda \left\{ \frac{1}{\lambda^2} - \frac{1}{4} \frac{a^4}{\lambda^6} \right\}, \tag{56}$$

where  $a^4 \equiv \frac{\rho_0^2}{r^4} = \frac{Q^2}{8\beta^2\pi^2 r^4}$ . At first sight Eq. (56) indicates the presence of an infrared divergence. However, in our case, we are not interested only in the quantity  $\mathcal{A}_0$ ; what actually matters for us is the difference  $\mathcal{A}_0(\mathbf{0}) - \mathcal{A}_0(\mathbf{L})$ , as is stated right below. As far as this difference is concerned, the infrared infinities present in the individual integrals cancel against

each other. In this way, by employing Eq. (56), the potential for a pair of static point-like opposite charges located at  $\mathbf{0}$  and  $\mathbf{L}$  is given by

$$V \equiv Q (\mathcal{A}_0(\mathbf{0}) - \mathcal{A}_0(\mathbf{L})) = -\frac{Q^2}{8\pi} \frac{1}{L} \left( 1 - \frac{Q^2}{160\pi^2 \beta^2} \frac{1}{L^4} \right). \tag{57}$$

Thus, to  $\mathcal{O}\left(\frac{1}{\beta^2}\right)$ , logarithmic electrodynamics displays a marked qualitative departure from the usual Maxwell theory. More importantly, this is exactly the profile obtained for Born–Infeld electrodynamics. Accordingly, logarithmic electrodynamics also has a rich structure reflected by its long-range correction to the Coulomb potential.

At this point an interesting issue becomes clear. Although logarithmic electrodynamics has a finite electric field at the origin, the interaction energy between two test charges at leading order in  $\beta$  is not finite at the origin. In view of this situation, we now proceed to examine the behavior of logarithmic electrodynamics defined in non-commutative geometry, along the lines of references [40,41]. Basically, our goal is to explore the behavior of the interaction energy at short distances. In this case, Gauss’ law reads

$$\partial_i \Pi^i = e e^{\theta \nabla^2} \delta^{(3)}(\mathbf{x}). \tag{58}$$

This then implies that

$$\Pi^i = -\frac{2e}{\sqrt{\pi}} \frac{\hat{r}^i}{r^2} \gamma\left(3/2, r^2/4\theta\right), \tag{59}$$

with  $r = |\mathbf{r}|$ . Here  $\gamma(3/2, r^2/4\theta)$  is the lower incomplete Gamma function defined by the following integral representation:

$$\gamma(a/b, x) \equiv \int_0^x \frac{du}{u} u^{a/b} e^{-u}. \tag{60}$$

Next, from the expression for the electric field, we have

$$E_i = e \frac{1}{\left[1 + \sqrt{1 + \frac{2}{\beta^2} \Pi^2}\right]} \partial_i \left( -\frac{e^{\theta \nabla^2} \delta^{(3)}(\mathbf{x})}{\nabla^2} \right); \tag{61}$$

in this last line we have considered the static case ( $\mathbf{B} = 0$ ). At leading order in  $\beta$ , the electric field follows as

$$E_i = \frac{e}{2} \left( 1 - \frac{\Pi^2}{2\beta^2} \right) \partial_i \left( -\frac{e^{\theta \nabla^2} \delta^{(3)}(\mathbf{x})}{\nabla^2} \right), \tag{62}$$

where  $\Pi$  is given by expression (59).

Using this result, the physical scalar potential, Eq. (49), takes the form

$$\mathcal{A}_0(t, \mathbf{r}) = \frac{e}{2} \int_0^1 d\lambda r^i \partial_i^{\lambda r} \tilde{G}(\lambda \mathbf{r}) - \frac{e}{4\beta^2} \int_0^1 d\lambda \Pi^2(\lambda \mathbf{r}) r^i \partial_i^{\lambda r} \tilde{G}(\lambda \mathbf{r}), \tag{63}$$

where  $\tilde{G}(\mathbf{r}) = \frac{1}{4\pi^{3/2}} \frac{1}{r} \gamma(1/2, r^2/4\theta)$ . By employing Eq. (59) we can reduce Eq. (64) to

$$\mathcal{A}_0(\mathbf{r}) = \frac{e}{8\pi^{3/2}} \frac{1}{r} \gamma\left(1/2, r^2/4\theta\right) + \frac{2e^3}{(\pi)^{3/2} \beta^2} \hat{r}^i \int_0^{\mathbf{r}} dy^i \frac{1}{y^6} \gamma^3\left(3/2, y^2/4\theta\right). \tag{64}$$

Finally, replacing this result in (48), the potential for a pair of point-like opposite charges  $e$ , located at  $\mathbf{0}$  and  $\mathbf{L}$ , takes the form

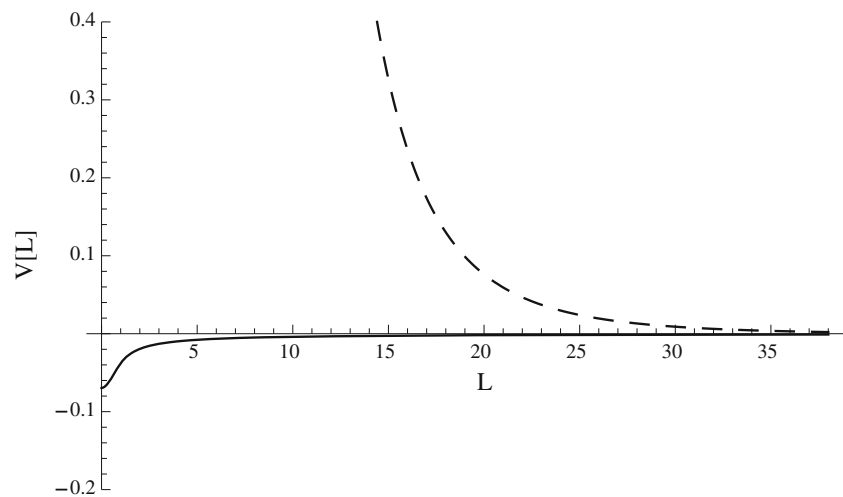
$$V = -\frac{e^2}{8\pi^{3/2}} \frac{1}{L} \left[ \gamma\left(1/2, L^2/4\theta\right) + \frac{16e^2}{\beta^2} L \hat{r}^i \int_0^{\mathbf{L}} dy^i \frac{1}{y^6} \gamma^3\left(3/2, y^2/4\theta\right) \right]. \tag{65}$$

One immediately observes that the introduction of the non-commutative space induces a finite static potential for  $L \rightarrow 0$  (See Fig. 2). This then implies that the self-energy and the electromagnetic mass of a point-like particle are finite in this version of non-commutative of logarithmic electrodynamics. It is also important to note that in the limit  $\theta \rightarrow 0$ , we recover our previous result (56).

#### 4 Final remarks

In summary, within the gauge-invariant but path-dependent variables formalism, we have considered the confinement versus screening issue for logarithmic electrodynamics. Once again, a correct identification of the physical degrees of freedom has been fundamental for understanding the physics hidden in gauge theories. We should highlight the different behaviors of the potentials associated to each of the models. In the logarithmic electrodynamics case, the static potential profile is similar to that encountered in Born–Infeld electrodynamics. Interestingly enough, its non-commutative version displays an ultraviolet finite static potential. The above analysis reveals the key role played by the new quantum of length in our analysis. In a general perspective, the benefit of considering the present approach is to provide unifications

**Fig. 2** Shape of the potential, Eqs. (65) (solid line) and (57) (dashed line)



among different models, as well as exploiting the equivalence in explicit calculations, as we have illustrated in the course of this work.

Finally, we should not conceive the electron simply as an electric monopole. The electron's electric dipole moment has recently been re-measured and its upper bound has been improved by a factor around 12 [52]:

$$d_e \leq 10^{-29} \text{ e.cm} \quad (66)$$

This means that, at distances of the order of  $10^{-29}$  cm, one can think of the electron's charge being non-symmetrically distributed around the electron's spin. Moreover, the electron is also a magnetic dipole. So, a very natural path to delve deeper into the study of logarithmic electrodynamics would be the investigation of the electron's magnetic dipole moment in terms of the magnetic field induced, through the nonlinearity, by the electrostatic field of Eqs. (9) and (10). A step toward this investigation was taken in the paper of Ref. [53], where the authors attempt to gain understanding of the electron's magnetic moment as a nonlinear effect induced by its own electrostatic field in the usual Born–Infeld scenario. We should now focus on the electron's electric and magnetic dipoles in the framework of logarithmic electrodynamics. The results of our pursuit shall be reported elsewhere.

**Acknowledgments** This work was partially supported by Fondecyt (Chile) Grant 1130426.

**Open Access** This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

Funded by SCOAP<sup>3</sup> / License Version CC BY 4.0.

## References

1. S.L. Adler, *Ann. Phys. (N.Y.)* **67**, 599 (1971)

2. V. Constantini, B. De Tollis, G. Pistoni, *Nuovo Cimento A* **2**, 733 (1971)
3. S. Biswas, K. Melnikov, *Phys. Rev. D* **75**, 053003 (2007)
4. D. Tommasini, A. Ferrando, H. Michinel, M. Seco, *J. High Energy Phys.* **0911**, 043 (2009)
5. A. Ferrando, H. Michinel, M. Seco, D. Tommasini, *Phys. Rev. Lett.* **99**, 150404 (2007)
6. D. Tommasini, A. Ferrando, H. Michinel, M. Seco, *Phys. Rev. A* **77**, 042101 (2008)
7. S.I. Kruglov, *Phys. Rev. D* **75**, 117301 (2007)
8. E.A. Zavattini et al., *Phys. Rev. D* **77**, 032006 (2008)
9. M. Bregant et al., *Phys. Rev. D* **78**, 032006 (2008)
10. M. Born, L. Infeld, *Proc. R. Soc. Lond. Ser. A* **144**, 425 (1934)
11. H. Gies, J. Jaeckel, A. Ringwald, *Phys. Rev. Lett.* **97**, 140402 (2006)
12. E. Masso, R. Toldra, *Phys. Rev. D* **52**, 1755 (1995)
13. P. Gaete, E.I. Guendelman, *Mod. Phys. Lett. A* **5**, 319 (2005)
14. P. Gaete, E. Spallucci, *J. Phys. A: Math. Gen.* **39**, 6021 (2006)
15. B. Hoffmann, *Phys. Rev.* **47**, 877 (1935)
16. S.H. Hendi, *Ann. Phys.* **333**, 282 (2013)
17. Z. Zhao, Q. Pan, S. Chen, J. Jing, *Nucl. Phys. B* **871**, 98 (2013)
18. O. Mišković, R. Olea, *Phys. Rev. D* **83**, 024011 (2011)
19. S. Habib Mazharimousavi, M. Halilsoy, *Phys. Lett. B* **678**, 407 (2009)
20. H. Euler, W. Heisenberg, *Z. Phys.* **98**, 714 (1936); translation: H. Kleinert and W. Korolevski, [arXiv:physics/0605038](https://arxiv.org/abs/physics/0605038)
21. G.E. Volovik, *The Universe in a Helium Droplet* (Clarendon Press, Oxford, 2003)
22. M.I. Katsnelson, G.E. Volovik, Quantum electrodynamics with anisotropic scaling: Heisenberg–Euler action and Schwinger pair production in the bilayer graphene. *Pis'ma ZhETF* **95**, 457 (2012). [arXiv:1203.1578](https://arxiv.org/abs/1203.1578)
23. A.I. Akhiezer V.B. Berestetskii, *Quantum Electrodynamics*, Chapter 54 (Interscience Publishers, New York, 1965)
24. F. Cotton, in *Spin as a Manifestation of a Nonlinear Constitutive Tensor and a Non-Riemannian Geometry*. APS Bulletin of the American Physical Society, APS April Meeting, vol 58, Number 4 (Denver, Colorado, 2013)
25. G. Amelino-Camelia, *Nature* **418**, 34 (2002)
26. T. Jacobson, S. Liberati, D. Mattingly, *Phys. Rev. D* **67**, 124011 (2003)
27. T.J. Konopka, S.A. Major, *New J. Phys.* **4**, 57 (2002)
28. S. Hossenfelder, *Phys. Rev. D* **73**, 105013 (2006)
29. P. Nicolini, *Int. J. Mod. Phys. A* **24**, 1229 (2009)
30. E. Witten, *Nucl. Phys. B* **268**, 253 (1986)
31. N. Seiberg, E. Witten, *JHEP* **9909**, 032 (1999)



32. M.R. Douglas, N.A. Nekrasov, *Rev. Mod. Phys.* **73**, 977–1029 (2001)
33. R.J. Szabo, *Phys. Rept.* **378**, 207–299 (2003)
34. J. Gomis, K. Kamimura, T. Mateos, *JHEP* **0103**, 010 (2001)
35. A.A. Bichl, J.M. Grimstrup, L. Popp, M. Schweda, R. Wulkenhaar, *Int. J. Mod. Phys. A* **17**, 2219 (2002)
36. A. Smailagic, E. Spallucci, *J. Phys. A* **36**, L517 (2003)
37. A. Smailagic, E. Spallucci, *J. Phys. A* **36**, L467 (2003)
38. A. Smailagic, E. Spallucci, *J. Phys. A* **37**, 1 (2004) [Erratum-ibid. *A* **37**, 7169 (2004)]
39. A.B. Hammou, M. Lagraa, M.M. Sheikh-Jabbari, *Phys. Rev. D* **66**, 025025 (2002)
40. P. Gaete, E. Spallucci, *J. Phys. A* **45**, 065401 (2012)
41. P. Gaete, J. Helayel-Neto, E. Spallucci, *J. Phys. A* **45**, 215401 (2012)
42. P. Gaete, *J. Phys. A* **46**, 475402 (2013)
43. J.M. Dávila, C. Schubert, M.A. Trejo, Photonic processes in Born-Infeld theory. [arXiv:1310.8410](https://arxiv.org/abs/1310.8410) [hep-ph]
44. D. d'Enterria, G.G. Silveira, *Phys. Rev. Lett.* **111**, 080405 (2013)
45. N. Kanda, Light-Light Scattering. [arXiv:1106.0592](https://arxiv.org/abs/1106.0592) [hep-ph]
46. A.E. Shabad, V.V. Usov, *Phys. Rev. D* **83**, 105006 (2011)
47. S.I. Kruglov, *J. Phys. A* **43**, 375402 (2010)
48. I. Bialynicki-Birula, in *75 Years of Born-Infeld Electrodynamics*. Non-linear theory of the electromagnetic field (Center for Theoretical Physics, Warsaw, 2008)
49. C.V. Costa, D.M. Gitman, A.E. Shabad, Finite field-energy of a point charge in QED. [arXiv:1312.0447](https://arxiv.org/abs/1312.0447) [hep-th]
50. P. Gaete, *Z. Phys. C* **76**, 355 (1997)
51. P. Gaete, *Phys. Rev. D* **59**, 127702 (1999)
52. J. Baron et al., ACME Collaboration. Order of magnitude smaller limit on the electric dipole moment of the electron. [arXiv:1310.7534](https://arxiv.org/abs/1310.7534) [phys. atom-phs]
53. S.O. Vellozo, J.A. Helayël, A.W. Smith, L.P.G. De Assis, *Int. J. Theor. Phys.* **48**, 1905 (2009)