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Exact solutions in modified massive gravity and off-diagonal wormhole deformations

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Abstract We explore off-diagonal deformations of 'prime' metrics in Einstein gravity (for instance, for wormhole configurations) into 'target' exact solutions in $f(R, T)$-modified and massive/bi-metric gravity theories. The new classes of solutions may, or may not, possess Killing symmetries and can be characterized by effective induced masses, anisotropic polarized interactions, and cosmological constants. For nonholonomic deformations with (conformal) ellipsoid/ toroid and/or solitonic symmetries and, in particular, for small eccentricity rotoid configurations, we can generate wormhole-like objects matching an external black ellipsoid—de Sitter geometries. We conclude that there are nonholonomic transforms and/or non-trivial limits to exact solutions in general relativity when modified/massive gravity effects are modeled by off-diagonal and/or nonholonomic parametric interactions.

1 Introduction

The bulk of physically important exact solutions in gravity theories (for instance, defining black holes and wormholes) are described by metrics with two Killing symmetries, see for summaries of the results the monographs [1,2]. For such solutions, there are certain 'canonical' frames of reference, when the coefficients of fundamental geometric/physical objects depend generically on one or two (from maximum four, in four dimensional, 4-d, theories) spacetime coordinates. This class of metrics can be diagonalized by coordinate transformations or contain off-diagonal terms generated by rotations. To construct generic off-diagonal solutions parameterized by metrics with six independent coefficients depending generically on three and/or, in general, on all spacetime coordinates is a very difficult technical and geometric task and the physical meaning of such generalized/modified, or Einstein, spacetimes is less clear.

In our work, see [3,4] and references therein, we elaborated a geometric method which allows us to deform nonholonomically any 'prime' diagonal metric into various classes of 'target' off-diagonal solutions with one Killing and/or non-Killing symmetries. For deformations on a small parameter, the new classes of target solutions may preserve certain important physical properties of a prime metric (for instance, of a black hole/ring one, or for a wormhole) but may also possess new characteristics related to anisotropic polarizations of constants, nonlinear off-diagonal interactions with new symmetries etc.

Wormhole configurations with spacetime handles (shortcuts), non-trivial topology and exotic matter [5] have attracted attention for theoretical probes of foundations of gravity theories and as possible objects of nature (for reviews, see [6–8] and references therein). Such solutions are determined in reverse direction when some tunneling metrics of prescribed (for instance, spherical and/or conformal) symmetry are considered and then one could try to find some corresponding exotic matter sources. A number of interesting and/or peculiar solutions were found when time-like curves and respective causality violations are allowed, for stress–energy tensors with possible violation of the null energy conditions. The wormhole subjects were revived some times in connection to black hole solutions, coupling with gauge interactions, singularities, generalized/modified gravity theories etc.

We studied locally anisotropic wormhole and/or flux tubes in five dimensional (5-d) gravity [9–11]. Such objects can be determined by extra dimensional or warped/ trapped configurations and/or possible ellipsoidal, toroidal, bipolar, solitonic etc. gravitational polarizations of vacuum and/or gravitational constants. The corresponding spacetime geometries are described by generic off-diagonal metrics[1] with coeffi-

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[1] These cannot be diagonalized by coordinate transformations.

cients depending on three or four coordinates and various types of (pseudo) Riemannian or non-Riemannian connections.

In the present paper, we address the problem of constructing deformations of prime wormhole metrics in general relativity, GR, resulting in generic off-diagonal solutions in modified gravity, MG, and theories with nonholonomically induced torsion, effective masses and bi-metric and bi-connection structures. We shall work with two equivalent connections (the Levi–Civita and an auxiliary one) defined by the same metric structure and apply and extend the anholonomic frame deformation method (AFDM, see details in [3,4], and references therein) of constructing exact solutions in gravity theories.

The idea of the AFDM is to find certain classes of nonholonomic (equivalently, anholonomic/ non-integrable) frames with conventional $2+2+\cdots$, or $3+2+\cdots$, splitting of dimensions on (pseudo) Riemannian spacetime when the (in general, modified) Einstein equations decouple for a correspondingly defined 'auxiliary' connection. This results in systems of nonlinear partial differential equations (PDE) which can be integrated in very general forms. The corresponding solutions are with generic off-diagonal metrics and generalized connections. They may depend on all spacetime coordinates via generating and integration functions. The formalism is different from that with a 'simpler' diagonal ansatz when the Einstein equations are transformed into certain systems of nonlinear ordinary differential equations (ODE). For instance, for the second order ODE, we get only integration constants which are related to certain physical ones like the gravitational constant, a point particle mass, and/or an electric charge etc. following certain asymptotic/boundary conditions.

We argue that it is possible to impose such constraints on a nonholonomic frame structure, via corresponding classes of generating/integration functions, when the 'auxiliary' torsion vanishes and we can 'extract' solutions for the Einstein gravity theory and various modifications. To provide a physical interpretation of certain off-diagonal exact solutions with one Killing symmetry or non-Killing symmetries is usually a very difficult task. In general, it is not clear if any physical meaning/importance can be found for a newly derived class of generalized solutions. Nevertheless, it is possible to elaborate realistic physical models with nonholonomically constrained nonlinear off-diagonal gravitational and matter field interactions if we consider deformations on a small parameter (for instance, small eccentricities for ellipsoid/rotoid configurations). This allows us to construct new classes of off-diagonal solutions determining parametric deformations of wormhole and black hole physical objects resulting in new observable physical effects and more complex spacetime configurations.

The article is organized as follows: We formulate a geometric approach to modified massive gravity theories in Sect. 2. A proof that the corresponding gravitational field equations can be decoupled and integrated in general forms with respect to certain classes of nonholonomic frames of references is provided in Sect. 3. The method of off-diagonal deformations of wormhole–de Sitter configurations is outlined in Sect. 4. Small parametric deformations are considered, resulting in physically interesting solutions. In Sect. 5, four classes of 'locally anisotropic' deformations of original wormhole metrics are constructed. We deduce spacetime metrics for rotoid deformations of wormholes, consider solitonic waves on such wormholes and (if possible) black ellipsoids, and we explore a model with a torus ringing the throat of a wormhole. In a more general context, massive gravity and f-modifications to configurations with nonholonomically induced (by metric coefficients) torsions are considered. Section 6 is devoted to concluding remarks.

2 Nonholonomic deformations in modified massive gravity

We outline certain geometric methods on nonholonomic 2+2 spacetime splitting provided in detail in Refs. [3,4].

2.1 Geometric preliminaries

We shall refer to gravity theories formulated on a four dimensional, 4-d, generalized pseudo-Riemannian manifold $\mathbf{V}$ endowed with metric structure $\mathbf{g}$ and a metric compatible linear connection $\mathbf{D}$, $\mathbf{Dg}=0$. There will be considered distortion relations of type

$$\mathbf{D} = \nabla + \mathbf{Z}, \tag{1}$$

when both 'auxiliary', $\mathbf{D}$, and Levi–Civita, $\nabla = \{\Gamma^{\alpha}_{\ \beta\gamma}\}$, connections and the distortion tensor, $\mathbf{Z} = \{\mathbf{Z}^{\alpha}_{\ \beta\gamma}\}$, are completely defined by the coefficients $\mathbf{g} = \{g_{\alpha\beta}(u^{\gamma})\}$. To construct a natural splitting (1) following a well-defined geometric principle we can introduce a conventional horizontal (h) and vertical (v) splitting of the tangent space $T\mathbf{V}$, when a non-integrable (equivalently, nonholonomic, or anholonomic) distribution

$$\mathbf{N} : T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V} \tag{2}$$

is determined locally via a set of coefficients $\mathbf{N} = \{N^a_i(x, y)\}$; a 2+2 splitting can be parameterized by local coordinates $u = (x, y)$, $u^{\mu} = (x^i, y^a)$, where the indices run over values $i, j, \ldots = 1, 2$ and $a, b, \ldots = 3, 4$.[2]

[2] The coefficients $\Gamma^{\alpha}_{\ \beta\gamma}$, $Z^{\alpha}_{\ \beta\gamma}$ and $g_{\alpha\beta}$ are computed with respect to certain (co) frames of reference, $e_{\alpha} = e^{\alpha'}_{\ \alpha}(u)\partial_{\alpha'}$ and $e^{\beta} = e^{\beta}_{\ \beta'}(u)\, du^{\beta'}$, for $\partial_{\alpha'} := \partial/\partial u^{\alpha'}$. The Einstein rule on summation on 'up–low' cross indices will be applied if the contrary is not stated. For convenience, 'primed', 'underlined' etc. indices will be used. The local pseudo-

A h–v splitting (2) results in a structure of N-adapted local bases, $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$, and cobases, $\mathbf{e}^\mu = (e^i, \mathbf{e}^a)$, when

$$\mathbf{e}_i = \partial/\partial x^i - N_i^a(u)\partial/\partial y^a, \; e_a = \partial_a = \partial/\partial y^a, \quad (3)$$
$$\text{and } e^i = dx^i, \; \mathbf{e}^a = dy^a + N_i^a(u)\, dx^i. \quad (4)$$

For such frames, the nonholonomy relations are satisfied:

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma, \quad (5)$$

with non-trivial anholonomy coefficients

$$W_{ia}^b = \partial_a N_i^b, \; W_{ji}^a = \Omega_{ij}^a = \mathbf{e}_j\left(N_i^a\right) - \mathbf{e}_i(N_j^a). \quad (6)$$

We can distinguish the coefficients of geometric objects on $\mathbf{V}$ with respect to N-adapted (co) frames (3) and (4) and call them, in short, d-objects. For instance, a vector $Y(u) \in T\mathbf{V}$ can be parameterized as a d-vector, $\mathbf{Y} = \mathbf{Y}^\alpha \mathbf{e}_\alpha = \mathbf{Y}^i \mathbf{e}_i + \mathbf{Y}^a e_a$, or $\mathbf{Y} = (hY, vY)$, with $hY = \{\mathbf{Y}^i\}$ and $vY = \{\mathbf{Y}^a\}$.

Any metric structure on $\mathbf{V}$ can be written (up to general frame/coordinate transformations) in two equivalent forms: with respect to a dual local coordinate basis,

$$\mathbf{g} = \underline{g}_{\alpha\beta}\, du^\alpha \otimes du^\beta,$$

where

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b g_{ab} & N_j^e g_{ae} \\ N_i^e g_{be} & g_{ab} \end{bmatrix}, \quad (7)$$

or as a d-metric,

$$\mathbf{g} = g_\alpha(u)\mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_i(x)\, dx^i \otimes dx^i + g_a(x,y)\mathbf{e}^a \otimes \mathbf{e}^a. \quad (8)$$

On a nonholonomic manifold $(\mathbf{V}, \mathbf{N})$, we can consider a subclass of linear connections called *distinguished connections, d-connections,* $\mathbf{D} = (hD, vD)$, preserving under parallelism the N-connection splitting (2). Any $\mathbf{D}$ defines a covariant derivative operator, $\mathbf{D}_\mathbf{X}\mathbf{Y}$, for a d-vector field $\mathbf{Y}$ in the direction of a d-vector $\mathbf{X}$. With respect to N-adapted frames (3) and (4), the value $\mathbf{D}_\mathbf{X}\mathbf{Y}$ can be computed as in GR but with the coefficients of the Levi–Civita connection substituted by $\mathbf{D} = \{\Gamma^\gamma_{\ \alpha\beta} = (L^i_{\ jk}, L^a_{\ bk}, C^i_{\ jc}, C^a_{\ bc})\}$. The respective coefficients are computed for the h–v-components of $\mathbf{D}_{\mathbf{e}_\alpha}\mathbf{e}_\beta := \mathbf{D}_\alpha \mathbf{e}_\beta$ using $\mathbf{X} = \mathbf{e}_\alpha$ and $\mathbf{Y} = \mathbf{e}_\beta$.

A d-connection is characterized by three fundamental geometric objects: the d-torsion, $\mathcal{T}$, the nonmetricity, $\mathcal{Q}$, and the d-curvature, $\mathcal{R}$, all defined by the standard equations

$$\mathcal{T}(\mathbf{X}, \mathbf{Y}) := \mathbf{D}_\mathbf{X}\mathbf{Y} - \mathbf{D}_\mathbf{Y}\mathbf{X} - [\mathbf{X}, \mathbf{Y}], \; \mathcal{Q}(\mathbf{X}) := \mathbf{D}_\mathbf{X}\mathbf{g},$$
$$\mathcal{R}(\mathbf{X}, \mathbf{Y}) := \mathbf{D}_\mathbf{X}\mathbf{D}_\mathbf{Y} - \mathbf{D}_\mathbf{Y}\mathbf{D}_\mathbf{X} - \mathbf{D}_{[\mathbf{X},\mathbf{Y}]}.$$

Footnote 2 continued
Euclidean signature is fixed in the form $(+++-)$. We shall write boldface letters in order to emphasize that a nonlinear connection, N-connection, structure (2) is fixed on a spacetime manifold $\mathbf{V}$.

We can compute the corresponding N-adapted coefficients,

$$\mathcal{T} = \left\{\mathbf{T}^\gamma_{\ \alpha\beta} = \left(T^i_{\ jk}, T^i_{\ ja}, T^a_{\ ji}, T^a_{\ bi}, T^a_{\ bc}\right)\right\}, \mathcal{Q} = \left\{\mathbf{Q}^\gamma_{\ \alpha\beta}\right\},$$
$$\mathcal{R} = \left\{\mathbf{R}^\alpha_{\ \beta\gamma\delta} = \left(R^i_{\ hjk}, R^a_{\ bjk}, R^i_{\ hja}, R^c_{\ bja}, R^i_{\ hba}, R^c_{\ bea}\right)\right\},$$

of these geometric objects by introducing $\mathbf{X} = \mathbf{e}_\alpha$ and $\mathbf{Y} = \mathbf{e}_\beta$, and $\mathbf{D} = \{\Gamma^\gamma_{\ \alpha\beta}\}$ into the above equations; see details in [3,4].

It should be noted that the Levi–Civita connection ∇ (in brief, LC,[3]) is not a d-connection because it does not preserve under general frame/coordinate transformations the N-connection splitting (2). Nevertheless, *there is a canonical d-connection* $\widehat{\mathbf{D}}$ also uniquely determined by any geometric data $(\mathbf{g}, \mathbf{N})$ following two similar but a bit 'relaxed' conditions: (1) it is metric compatible, $\widehat{\mathbf{D}}\mathbf{g} = \mathbf{0}$, and (2) with zero h-torsion, $h\widehat{\mathcal{T}} = \{\widehat{T}^i_{\ jk}\} = 0$, and zero v-torsion, $v\widehat{\mathcal{T}} = \{\widehat{T}^a_{\ bc}\} = 0$. This allows us to construct a canonical distortion relation of type (1) with respective splitting of N-adapted coefficients $\widehat{\Gamma}^\gamma_{\ \alpha\beta} = \Gamma^\gamma_{\ \alpha\beta} + \widehat{\mathbf{Z}}^\gamma_{\ \alpha\beta}$. We can work equivalently with the two metric compatible connections $\widehat{\mathbf{D}}$ and ∇, because both such geometric objects are completely defined by the same metric structure $\mathbf{g}$.[4] For the canonical d-connection, there are non-trivial d-torsions coefficients,

$$\widehat{T}^i_{\ jk} = \widehat{L}^i_{\ jk} - \widehat{L}^i_{\ kj}, \widehat{T}^i_{\ ja} = \widehat{C}^i_{\ jb}, \widehat{T}^a_{\ ji} = -\Omega^a_{\ ji}, \widehat{T}^c_{\ aj}$$
$$= \widehat{L}^c_{\ aj} - e_a(N^c_j), \widehat{T}^a_{\ bc} = \widehat{C}^a_{\ bc} - \widehat{C}^a_{\ cb}. \quad (9)$$

The geometric meaning of such a nonholonomically induced torsion is different from that, for instance, in Riemann–Cartan geometry because in our approach $\widehat{\mathcal{T}}$ is completely defined by the metric structure.

A (pseudo) Riemannian geometry can be formulated alternatively in 'geometric variables' $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$ computing in standard form, respectively, the Riemann, $\widehat{\mathcal{R}} = \{\widehat{\mathbf{R}}^\alpha_{\ \beta\gamma\delta}\}$, and the Ricci, $\widehat{\mathcal{R}}ic = \{\widehat{\mathbf{R}}_{\ \beta\gamma}\}$, d-tensors. For instance, the nonsymmetric d-tensor $\widehat{\mathbf{R}}_{\alpha\beta} := \widehat{\mathbf{R}}^\gamma_{\ \alpha\beta\gamma}$ of $\widehat{\mathbf{D}}$ is characterized by four h–v N-adapted coefficients,

[3] It is uniquely defined by the metric structure $\mathbf{g}$ if there are imposed two conditions: $\mathcal{T} = 0$ and $\mathcal{Q} = 0$, if $\mathbf{D} \to \nabla$.

[4] The N-adapted coefficients of $\widehat{\mathbf{D}} = \{\widehat{\Gamma}^\gamma_{\ \alpha\beta} = (\widehat{L}^i_{\ jk}, \widehat{L}^a_{\ bk}, \widehat{C}^i_{\ jc}, \widehat{C}^a_{\ bc})\}$ and $\widehat{\mathbf{Z}}^\gamma_{\ \alpha\beta}$, depending only on $g_{\alpha\beta}$ and N_i^a, can be computed by the following equations:

$$\widehat{L}^i_{\ jk} = \frac{1}{2}g^{ir}\left(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}\right),$$
$$\widehat{C}^a_{\ bc} = \frac{1}{2}g^{ad}\left(e_c g_{bd} + e_b g_{cd} - e_d g_{bc}\right)$$
$$\widehat{C}^i_{\ jc} = \frac{1}{2}g^{ik} e_c g_{jk}, \; \widehat{L}^a_{\ bk} = e_b(N_k^a)$$
$$+ \frac{1}{2}g^{ac}\left(\mathbf{e}_k g_{bc} - g_{dc}\, e_b N_k^d - g_{db}\, e_c N_k^d\right),$$

see proofs, for instance, in [3,4].

$$\widehat{\mathbf{R}}_{\alpha\beta} = \left\{ \widehat{R}_{ij} := \widehat{R}^{k}{}_{ijk},\ \widehat{R}_{ia} := -\widehat{R}^{k}{}_{ika},\ \widehat{R}_{ai} : = \widehat{R}^{b}{}_{aib},\ \widehat{R}_{ab} := \widehat{R}^{c}{}_{abc} \right\}, \quad (10)$$

which allows us to compute an 'alternative' scalar curvature

$$\widehat{R} := \mathbf{g}^{\alpha\beta}\widehat{\mathbf{R}}_{\alpha\beta} = g^{ij}\widehat{R}_{ij} + g^{ab}\widehat{R}_{ab}. \quad (11)$$

We can also introduce the Einstein d-tensor of $\widehat{\mathbf{D}}$,

$$\widehat{\mathbf{E}}_{\alpha\beta} \doteqdot \widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\ \widehat{R}. \quad (12)$$

The values $\widehat{\mathcal{R}}$, $\widehat{\mathcal{R}}ic$ and $\widehat{R}$ for the canonical d-connection $\widehat{\mathbf{D}}$ are different from the similar ones, $\mathcal{R}$, $\mathcal{R}ic$ and R, computed for the LC-connection ∇. Nevertheless, both classes of such fundamental geometric objects are related via unique distorting relations derived from (1) for a N-connection splitting (2). To work with $\widehat{\mathbf{D}}$ is convenient for various purposes in generalized gravity theories with non-trivial torsion. The most surprising property of the Ricci d-tensor $\widehat{\mathcal{R}}ic = \{\widehat{\mathbf{R}}_{\beta\gamma}\}$ is that the corresponding modified Einstein equations of type $\widehat{\mathbf{R}}_{\beta\gamma} = \Upsilon_{\beta\gamma}$ decouple in very general forms with respect to certain classes of N-adapted frames. This property holds for a generic off-diagonal ansatz of type (7) (in principle, depending on all coordinates) and for certain formally diagonalized and N-adapted sources $\Upsilon_{\beta\gamma}$. This allows us to generate various classes of exact solutions in commutative and noncommutative gravity theories with 4-d and higher dimensions spacetimes, see details and examples in Refs. [3,4,9–11]. Such a geometric method of constructing exact solutions in gravity is conventionally called the anholonomic frame deformation method (AFDM).

The AFDM can be used for constructing off-diagonal exact solutions in general relativity (GR) and other theories involving the LC-connection ∇. In such cases, $\widehat{\mathbf{D}} = \{\widehat{\Gamma}^{\gamma}{}_{\alpha\beta}\}$ can be considered as an 'auxiliary' connection which together with certain convenient sets of N-coefficients, N_i^a, are introduced with the aim to decouple certain systems of nonlinear partial differential equations (PDE) and solve them in very general forms. Such solutions are determined by corresponding classes of generating and integration functions and, in principle, by an infinite number of integration/symmetry parameters. On corresponding integral varieties of solutions, we can impose additional nonholonomic constraints when the torsion (9) vanishes and $\widehat{\mathbf{D}} \to \nabla$. Such constraints result in first order PDE equations which can be of type

$$\widehat{L}^{c}_{aj} = e_a(N^c_j),\ \widehat{C}^{i}{}_{jb} = 0,\ \Omega^{a}{}_{ji} = 0. \quad (13)$$

These equations can be solved also in very general forms and this allows us to extract LC-configurations. We note that if we work from the very beginning with ∇, we cannot decouple for general off-diagonal metrics, for instance, the Einstein equations. This is a consequence of the generic nonlinearity of the gravitational field equations. The significance of $\widehat{\mathbf{D}}$ is that we can 'relax' a bit the zero torsion conditions, decouple the corresponding nonlinear PDEs for certain convenient systems of reference determined by 'flexible' N_i^a and find general classes of solutions. At the end (after a class of generalized metrics and connections was defined), we can constrain nonholonomically/parametrically the nonlinear system and find torsionless configurations.

The main goal of this work is to show that the AFDM allows us to generate exact solutions with nonholonomic deformations of wormhole objects in modified and/or massive gravity.

2.2 Nonholonomic massive $f(R, T)$ gravity

We study modified gravity theories derived for the action

$$S = \frac{1}{16\pi}\int \delta u^4 \sqrt{|\mathbf{g}_{\alpha\beta}|} \times \left[f(\widehat{R}, T) - \frac{\mu_g^2}{4}\mathcal{U}(\mathbf{g}_{\mu\nu}, \mathbf{K}_{\alpha\beta}) + {}^{m}L \right]. \quad (14)$$

Such theories generalize the so-called modified $f(R, T)$ gravity, see reviews and original results in [12–14], and the ghost-free massive gravity (by de Rham, Gabadadze and Tolley, dRGT) [15–17]. This evades from certain problems of the bi-metric theory by Hassan and Rosen [18,19] and connects us to a variety of recent research in black hole physics and modern cosmology [20–22]. In this paper, we shall use the units when $\hbar = c = 1$ and the Planck mass M_{Pl} is defined via $M^2_{\mathrm{Pl}} = 1/8\pi G$ with the 4-d Newton constant G. We write δu^4 instead of d^4u because there are used N-elongated differentials (3) and consider the constant μ_g as the mass parameter for gravity. The geometric and physical meaning of the values contained in this formula will be explained below.

There are at least three most important motivations to consider in this work such generalized models of gravity. (1) Using nonholonomic deformations described in previous section, we can transform certain classes of solutions in modified gravity into certain equivalent ones for massive gravity. (2) Via off-diagonal gravitational interactions in Einstein gravity, it is possible to mimic various classes of physical effects in modified, massive, bi-metric, and bi-connection gravity. (3) The AFDM seems to be an effective geometric tool for constructing exact solutions in such 'sophisticate' gravity theories.

In the action (14), the Lagrangian density ${}^{m}L$ is used for computing the stress–energy tensor of matter via variation in N-adapted form, using operators (3) and (4), on inverse metric d-tensor (8), $\mathbf{T}_{\alpha\beta} = -\frac{2}{\sqrt{|\mathbf{g}_{\mu\nu}|}}\frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|}\ {}^{m}L)}{\delta \mathbf{g}^{\alpha\beta}}$, when the trace is computed $T := \mathbf{g}^{\alpha\beta}\mathbf{T}_{\alpha\beta}$. The functional $f(\widehat{R}, T)$ modifies the standard Einstein–Hilbert Lagrangian (with R

for the CL connection ∇) to that for the modified f-gravity but with dependence on ${}^{s}\widehat{R}$ (11) and T. In a large class of generalized cosmological models, we can assume that the stress–energy tensor of the matter is given by

$$\mathbf{T}_{\alpha\beta} = (\rho + p)\mathbf{v}_{\alpha}\mathbf{v}_{\beta} - p\mathbf{g}_{\alpha\beta} \quad (15)$$

for the approximation of perfect fluid matter with the energy density ρ and the pressure p; the four-velocity $\mathbf{v}_{\alpha}$ being subject to the conditions $\mathbf{v}_{\alpha}\mathbf{v}^{\alpha} = 1$ and $\mathbf{v}^{\alpha}\widehat{\mathbf{D}}_{\beta}\mathbf{v}_{\alpha} = 0$, for ${}^{m}L = -p$ in a corresponding local N-adapted frame. For simplicity, we can parametrize

$$f(\widehat{R}, T) = {}^{1}f(\widehat{R}) + {}^{2}f(T) \quad (16)$$

and denote ${}^{1}F(\widehat{R}) := \partial\, {}^{1}f(\widehat{R})/\partial\widehat{R}$ and ${}^{2}F(T) := \partial\, {}^{2}f(T)/\partial T$.

In addition to the usual f-gravity term (in particular, to the Einstein–Hilbert one) in (14), we consider a mass term with 'gravitational mass' μ_g and potential

$$\begin{aligned}\mathcal{U}/4 = &-12 + 6[\sqrt{\mathcal{S}}]+[\mathcal{S}] - [\sqrt{\mathcal{S}}]^2 \\ &+\alpha_3\{18[\sqrt{\mathcal{S}}] - 6[\sqrt{\mathcal{S}}]^2 + [\sqrt{\mathcal{S}}]^3 + 2[\mathcal{S}^{3/2}] \\ &-3[\mathcal{S}]([\sqrt{\mathcal{S}}] - 2) - 24\} \\ &+\alpha_4\{[\sqrt{\mathcal{S}}](24 - 12[\sqrt{\mathcal{S}}] - [\sqrt{\mathcal{S}}]^3) \\ &-12[\sqrt{\mathcal{S}}][\mathcal{S}] + 2[\sqrt{\mathcal{S}}]^2(3[\mathcal{S}] + 2[\sqrt{\mathcal{S}}]) \\ &+3[\mathcal{S}](4-[\mathcal{S}])-8[\mathcal{S}^{3/2}](\sqrt{\mathcal{S}} - 1)+6[\mathcal{S}^2]-24\},\end{aligned} \quad (17)$$

where the trace of a matrix $\mathcal{S} = (S_{\mu\nu})$ is denoted by $[\mathcal{S}] := S^{\nu}_{\ \nu}$; the square root of such a matrix, $\sqrt{\mathcal{S}} = (\sqrt{\mathcal{S}}^{\nu}_{\ \mu})$, is understood to be a matrix for which $\sqrt{\mathcal{S}}^{\nu}_{\ \alpha}\sqrt{\mathcal{S}}^{\alpha}_{\ \mu} = S^{\nu}_{\ \mu}$, and α_3 and α_4 are free parameters. This nonlinearly extended Fierz–Pauli type potential was shown to result in a theory of massive gravity which is free from ghost-like degrees of freedom and takes the special form of a total derivative in the absence of dynamics (see [16,17] and additional arguments in [23]). The potential generating matrix $\mathcal{S}$ is constructed in a special form to result in a d-tensor $\mathbf{K}^{\nu}_{\ \mu} = \delta^{\nu}_{\ \mu} - \sqrt{\mathcal{S}}^{\nu}_{\ \mu}$ characterizing metric fluctuations away from a fiducial (flat) 4-d spacetime. The coefficients

$$\mathbf{S}^{\nu}_{\ \mu} = \mathbf{g}^{\nu\alpha}\eta_{\overline{\nu}\overline{\mu}}\mathbf{e}_{\alpha}s^{\overline{\nu}}\mathbf{e}_{\mu}s^{\overline{\mu}}, \quad (18)$$

with the Minkowski metric $\eta_{\overline{\nu}\overline{\mu}} = \mathrm{diag}(1, 1, 1, -1)$, are generated by introducing four scalar Stükelberg fields $s^{\overline{\nu}}$, which is necessary for restoring the diffeomorphism invariance. Using N-adapted values $\mathbf{g}^{\nu\alpha}$ and $\mathbf{e}_{\alpha}$ we can always transform a tensor $S_{\mu\nu}$ into a d-tensor $\mathbf{S}_{\mu\nu}$ characterizing nonholonomically constrained fluctuations. This is possible for the values $\mathbf{K}^{\nu}_{\ \mu}, \mathbf{S}^{\nu}_{\ \mu}, \sqrt{\mathcal{S}}^{\nu}_{\ \mu}$ etc.; even $s^{\overline{\nu}}$ transforms as a scalar field under coordinate and frame transformations.

Varying the action (14) in N-adapted form for the coefficients of d-metric $\mathbf{g}_{\nu\alpha}$ (8), we obtain certain effective Einstein equations, see (12), for the modified massive gravity,

$$\widehat{\mathbf{E}}_{\alpha\beta} = \Upsilon_{\beta\delta}, \quad (19)$$

with source

$$\Upsilon_{\beta\delta} = {}^{ef}\eta\, G\, \mathbf{T}_{\beta\delta} + {}^{ef}\mathbf{T}_{\beta\delta} + \mu_g^2\, {}^{K}\mathbf{T}_{\beta\delta}. \quad (20)$$

The first component in such a source is determined by the usual matter fields with energy–momentum $\mathbf{T}_{\beta\delta}$ tensor but with effective polarization of the gravitational constant ${}^{ef}\eta = [1 + {}^{2}F/8\pi]/\,{}^{1}F$. The f-modification of the energy–momentum tensor also results in the section term as an additional effective source

$$\begin{aligned}{}^{ef}\mathbf{T}_{\beta\delta} = \Big[&\frac{1}{2}({}^{1}f - {}^{1}F\, \widehat{R} + 2p\, {}^{2}F + {}^{2}f)\mathbf{g}_{\beta\delta} \\ &-(\mathbf{g}_{\beta\delta}\, \widehat{\mathbf{D}}_{\alpha}\widehat{\mathbf{D}}^{\alpha} - \widehat{\mathbf{D}}_{\beta}\widehat{\mathbf{D}}_{\delta})\, {}^{1}F\Big]/\, {}^{1}F\end{aligned} \quad (21)$$

and the 'mass gravity' contribution (the third term) is computed as a dimensionless effective stress–energy tensor

$$\begin{aligned}{}^{K}\mathbf{T}_{\alpha\beta} := &\frac{1}{4\sqrt{|\mathbf{g}_{\mu\nu}|}}\frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|}\,\mathcal{U})}{\delta\mathbf{g}^{\alpha\beta}} \\ = &-\frac{1}{12}\Big\{\mathcal{U}\mathbf{g}_{\alpha\beta}/4 - 2\mathbf{S}_{\alpha\beta} + 2([\sqrt{\mathcal{S}}] - 3)\sqrt{\mathcal{S}}_{\alpha\beta} \\ &+\alpha_3\Big[3(-6 + 4[\sqrt{\mathcal{S}}] + [\sqrt{\mathcal{S}}]^2 - [\mathcal{S}])\sqrt{\mathcal{S}}_{\alpha\beta} \\ &+6([\sqrt{\mathcal{S}}] - 2)\mathbf{S}_{\alpha\beta} - \mathcal{S}^{3/2}_{\alpha\beta}\Big] \\ &-\alpha_4\Big[24\left(\mathcal{S}^2_{\alpha\beta} - ([\sqrt{\mathcal{S}}] - 1)\mathcal{S}^{3/2}_{\alpha\beta}\right)\Big] \\ &+12(2 - 2[\sqrt{\mathcal{S}}] - [\mathcal{S}] + [\sqrt{\mathcal{S}}]^2)\mathbf{S}_{\alpha\beta} \\ &+(24 - 24[\sqrt{\mathcal{S}}] + 12[\sqrt{\mathcal{S}}]^2 - [\sqrt{\mathcal{S}}]^3 - 12[\mathcal{S}] \\ &+12[\mathcal{S}][\sqrt{\mathcal{S}}] - 8[\mathcal{S}^{3/2}])\sqrt{\mathcal{S}}_{\alpha\beta}\Big\}.\end{aligned}$$

In 'hidden' form, ${}^{K}\mathbf{T}_{\alpha\beta}$ encodes a bi-metric configuration with the second (fiducial) d-metric $\mathbf{f}_{\alpha\mu} = \eta_{\overline{\nu}\overline{\mu}}\mathbf{e}_{\alpha}s^{\overline{\nu}}\mathbf{e}_{\mu}s^{\overline{\mu}}$ determined by the St ükelberg fields $s^{\overline{\nu}}$. The potential $\mathcal{U}$ (17) defines interactions between $\mathbf{g}_{\mu\nu}$ and $\mathbf{f}_{\mu\nu}$ via $\sqrt{\mathcal{S}}^{\nu}_{\ \mu} = \sqrt{\mathbf{g}^{\nu\mu}\mathbf{f}_{\alpha\nu}}$ and $\mathcal{S}^{\nu}_{\ \mu} := \mathbf{g}^{\nu\mu}\mathbf{f}_{\alpha\nu}$. For simplicity, we shall study in this paper bi-metric gravity models with ${}^{K}\mathbf{T}_{\alpha\beta} = \lambda(x^k)\,\mathbf{g}_{\alpha\beta}$, which can be generated by such $s^{\overline{\nu}}$ when $\mathbf{g}_{\mu\nu} = \iota^2(x^k)\mathbf{f}_{\mu\nu}$ up to a non-trivial conformal factor ϖ^2. Using (18), we can compute $\mathcal{S}^{\nu}_{\ \mu} := \iota^{-2}\delta^{\nu}_{\ \mu}$ which allows us to express the effective polarized anisotropic constant encoding the contributions of $s^{\overline{\nu}}$ as a functional $\lambda[\iota^2(x^k)]$. In general, the solutions of (19) depend on the type of symmetries of the interactions we prescribe for $\mathbf{f}_{\alpha\mu}$ which, in our model, are N-adapted and subjected to additional nonholonomic constraints.

The gravitational field equations (19) are similar to the Einstein ones in GR but for a different metric compatible linear connection, $\widehat{\mathbf{D}}$, and with nonlinear 'gravitationally polarized' coupling in the effective source $\Upsilon_{\beta\delta}$ (20). Such nonlinear systems of PDE can be integrated in general form for any

N-adapted parameterizations

$$\mathbf{\Upsilon}^{\beta}_{\ \delta} = \mathrm{diag}[\mathbf{\Upsilon}_{\alpha} : \mathbf{\Upsilon}^{1}_{\ 1} = \mathbf{\Upsilon}^{2}_{\ 2} = \Upsilon(x^k, y^3); \quad \mathbf{\Upsilon}^{3}_{\ 3} = \mathbf{\Upsilon}^{4}_{\ 4} = {}^{v}\Upsilon(x^k)], \tag{22}$$

in particular, if

$$\Upsilon = {}^{v}\Upsilon = \Lambda = \text{const}, \tag{23}$$

for an effective cosmological constant Λ, see details in [3,4]. A solution of equations (19) for a source (22) can be modeled effectively by certain classes of solutions generated by N-adapted constant coefficients (23) if the generating and integration functions are redefined to mimic certain classes of solutions. This is equivalent to a procedure of fixing a value for the auxiliary scalar curvature $\widehat{R}$ (11) by frame/coordinate transformations of N_i^a and related N-adapted bases which does not holds true for arbitrary 2+2 splitting but for certain classes of nonholonomic frames resulting in decoupling of the generalized Einstein equations and necessary parameterizations for the sources. Here we note that $\widehat{\mathbf{D}}_{\delta}\ {}^{1}F_{|\Upsilon=\Lambda} = 0$ in (21) if we prescribe a functional dependence $\widehat{R} = \text{const}$. For rather general distributions of matter fields and effective matter, we can prescribe such values for (23) with $\mathbf{T}_{\beta\delta} = \check{T}(x^k)\mathbf{g}_{\beta\delta}$ and ${}^{s}\widehat{R} = \widehat{\Lambda}$ in (22),

$$\begin{aligned}\Upsilon = \Lambda = {}&{}^{ef}\eta\, G\, \check{T}(x^k) + \frac{1}{2}({}^{1}f(\widehat{\Lambda}) - \widehat{\Lambda}\ {}^{1}F(\widehat{\Lambda}) + 2p\ {}^{2}F(\check{T}) + {}^{2}f(\check{T})) + \mu_g^2\, \lambda(x^k),\\ {}^{ef}\eta = {}&[1 + {}^{2}F(\check{T})/8\pi]/\ {}^{1}F(\widehat{\Lambda}).\end{aligned} \tag{24}$$

In general, any term may depend on the coordinates x^i, but via re-definition of generating functions they can be transformed into certain effective constants, see footnote 6. Prescribing values $\widehat{\Lambda}, \check{T}, \lambda, p$ and functionals ${}^{1}f$ and ${}^{2}f$, we describe the nonholonomic matter and effective matter fields dynamics with respect to N-adapted frames.

Finally, we note that the effective source $\Upsilon^{\beta}_{\ \delta} = \Lambda\delta^{\beta}_{\ \delta}$ (via nonholonomic constraints and the canonical d-connection $\widehat{\mathbf{D}}$) encodes all information on modifications of the GR theory to certain classes of f-modified and/or massive gravity theories. Imposing additional constraints when $\widehat{\mathbf{D}}_{\mathcal{T}=0} \to \nabla$, i.e. solving equations (13), we extract LC-configurations for the above-mentioned gravitational models.

3 Decoupling and integrability of MG field equations

In this section, we formulate and analyze possible conditions on the nonholonomic frame structure and matter fields and effective matter distributions when the gravitational field equations for f-modified bi-metric field equations decouple and can be integrated in very general forms. We show that such generic off-diagonal solutions depend on various classes of generating and integration functions and parameters. Such modified spacetimes describe nonholonomic deformations of a prime (fiducial and/or well-defined metric in GR, for simplicity, taken in a diagonal form with two Killing symmetries) into certain 'target' configurations in modified gravity theories.

Three classes of target solutions are analyzed: (1) non-vacuum off-diagonal deformations to LC configurations with effective cosmological constants encoding contributions from massive and f-modified gravity; (2) possible generalizations to non-trivial nonholonomically induced torsion configurations; and (3) nonholonomic deformations on a small parameter.

3.1 Decoupling with respect to N-adapted frames

The local coordinates on a 4-d manifold $\mathbf{V}$ are parameterized in the form $u^{\mu} = (x^i, y^a) = (x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t)$ (or, in brief, $u = (x, y)$), where the indices run over values $i, j, \ldots = 1, 2$ and $a, b, \ldots = 3, 4$ and t is a time-like coordinate. In brief, the partial derivatives $\partial_{\alpha} = \partial/\partial u^{\alpha}$ will be labeled in the forms $s^{\bullet} = \partial s/\partial x^1, s' = \partial s/\partial x^2, s^{*} = \partial s/\partial y^3, s^{\diamond} = \partial s/\partial y^4$.

We shall study nonholonomic deformations of a prime metric[5]

$$\begin{aligned}\mathring{\mathbf{g}} &= \mathring{g}_{\alpha}(u)\mathring{\mathbf{e}}^{\alpha} \otimes \mathring{\mathbf{e}}^{\beta} = \mathring{g}_i(x)\, dx^i \otimes dx^i + \mathring{g}_a(x,y)\mathring{\mathbf{e}}^a \otimes \mathring{\mathbf{e}}^a,\\ &\text{for } \mathring{\mathbf{e}}^{\alpha} = (dx^i, \mathbf{e}^a = dy^a + \mathring{N}_i^a(u)\, dx^i),\\ \mathring{\mathbf{e}}_{\alpha} &= (\mathring{\mathbf{e}}_i = \partial/\partial y^a - \mathring{N}_i^b(u)\partial/\partial y^b,\ e_a = \partial/\partial y^a),\end{aligned}$$

into a target off-diagonal one

$$\begin{aligned}\mathbf{g} &= g_{\alpha}(u)\mathbf{e}^{\alpha} \otimes \mathbf{e}^{\beta} = g_i(x)\, dx^i \otimes dx^i + g_a(x,y)\mathbf{e}^a \otimes \mathbf{e}^a\\ &= \eta_i(x^k)\mathring{g}_i\, dx^i \otimes dx^i + \eta_a(x^k, y^b)\mathring{h}_a\mathbf{e}^a \otimes \mathbf{e}^a,\end{aligned} \tag{25}$$

where $\mathbf{e}^a$ are taken as in (4). Our goal is to generate $\mathbf{g}$ as an exact solution in a (modified) gravity theory even if $\mathring{\mathbf{g}}$ is not necessarily constrained to the condition to be a solution of any gravitational field equations. For certain bi-metric models, the prime metric $\mathring{\mathbf{g}}$ can be considered as a fiducial one which via nonholonomic nonlinear gravitational interactions results in a solution in modified/ massive gravity. In the next sections, we shall take $\mathring{\mathbf{g}}$ as a wormhole solution in GR and study possible off-diagonal deformations induced in generalized gravity theories. We shall study the conditions when modified gravity effects can be explained alternatively by certain effective nonlinear interactions in GR.

The non-trivial components of the Einstein equations (19) with source (22) parameterized with respect to N-adapted

[5] We assume that such a metric comes with two Killing vector symmetries and that in certain systems of coordinates it can be diagonalized.

bases (3) and (4) for a metric ansatz (25) with data (31) for $\omega = 1$ are

$$-\widehat{R}^1_1 = -\widehat{R}^2_2 = \frac{1}{2g_1g_2}\left[g_2^{\bullet\bullet} - \frac{g_1^{\bullet}g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1'' - \frac{g_1'g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}\right] = {}^v\Upsilon, \quad (26)$$

$$-\widehat{R}^3_3 = -\widehat{R}^4_4 = \frac{1}{2h_3h_4}\left[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^*h_4^*}{2h_3}\right] = \Upsilon, \quad (27)$$

$$\widehat{R}_{3k} = \frac{w_k}{2h_4}\left[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^*h_4^*}{2h_3}\right] + \frac{h_4^*}{4h_4}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}\right) - \frac{\partial_k h_4^*}{2h_4} = 0, \quad (28)$$

$$\widehat{R}_{4k} = \frac{h_4}{2h_3}n_k^{**} + \left(\frac{h_4}{h_3}h_3^* - \frac{3}{2}h_4^*\right)\frac{n_k^*}{2h_3} = 0, \quad (29)$$

when the torsionless (LC) conditions (13) transform into

$$w_i^* = (\partial_i - w_i\partial_3)\ln\sqrt{|h_3|}, (\partial_i - w_i\partial_3)\ln\sqrt{|h_4|} = 0,$$
$$\partial_k w_i = \partial_i w_k, n_i^* = 0, \partial_i n_k = \partial_k n_i. \quad (30)$$

Proofs of such equations (but for other types of sources in GR and commutative and noncommutative Finsler-like generalizations) are contained in Refs. [3,4]. The above system of nonlinear PDE possess an important decoupling property which allows us to integrate step by step such equations.

3.2 Generating off-diagonal solutions

We can integrate the Einstein equations (19) for a source (22) if the N-adapted coefficients of a metric (25) are parameterized in the form

$$g_i = e^{\psi(x^k)}, g_a = \omega(x^k, y^b)h_a(x^k, y^3), \ N_i^3 = w_i(x^k, y^3), \ N_i^4 = n_i(x^k), \quad (31)$$

and assuming that frame/coordinate transformations are used we can satisfy the conditions $h_a^* \neq 0$, $\Upsilon_{2,4} \neq 0$. In a more general context, it is possible to consider any class of metrics which via frame and coordinate transformations can be related to such an ansatz. For parameterizations (31), the system (26)–(29) transforms correspondingly into

$$\psi^{\bullet\bullet} + \psi'' = 2\,{}^v\Upsilon \quad (32)$$
$$\phi^* h_4^* = 2h_3h_4\Upsilon \quad (33)$$
$$\beta w_i - \alpha_i = 0, \quad (34)$$
$$n_i^{**} + \gamma n_i^* = 0, \quad (35)$$
$$\partial_i\omega - (\partial_i\phi/\phi^*)\omega^* - n_i\omega^{\diamond} = 0, \quad (36)$$

for

$$\alpha_i = h_4^*\partial_i\phi, \beta = h_4^*\,\phi^*, \gamma = \left(\ln|h_4|^{3/2}/|h_3|\right)^*, \quad (37)$$

where

$$\phi = \ln|h_4^*/\sqrt{|h_3h_4|}| \quad (38)$$

is considered as a generating function. Equation (36) is necessary if we introduce a non-trivial conformal (in the vertical 'subspace') factor depending on all four coordinates. It will be convenient to work also with the value $\Phi := e^\phi$.

The above systems of nonlinear PDE can be integrated step by step in very general forms following the following procedure.

1. The (32) is just a 2-d Laplace equation which allows us to find ψ for any given source ${}^v\Upsilon$.
2. For $h_a := \epsilon_a z_a^2(x^k, y^3)$, when $\epsilon_a = \pm 1$ depending on the signature (we do not consider summation on repeating indices in this formula), equations (33) and (38) are written correspondingly in the form

$$\phi^* z_4^* = \epsilon_3 z_4(z_3)^2\Upsilon \quad \text{and} \quad e^\phi z_3 = 2\epsilon_4 z_4^*. \quad (39)$$

Multiplying both equations for nonzero z_4^*, ϕ^*, z_a and introducing the result instead of the first equation, this system transforms into

$$\Phi^* = 2\epsilon_3\epsilon_4 z_3 z_4\Upsilon \quad \text{and} \quad \Phi z_3 = 2\epsilon_4 z_4^*. \quad (40)$$

Taking z_3 from the second equation and introducing in the first one, we obtain $[(z_4)^2]^* = \frac{\epsilon_3[\Phi^2]^*}{4\Upsilon}$. This allows us to integrate on y^3 and write

$$h_4 = \epsilon_4(z_4)^2 = {}^0h_4(x^k) + \frac{\epsilon_3\epsilon_4}{4}\int dy^3\frac{[\Phi^2]^*}{\Upsilon}, \quad (41)$$

for an integration function ${}^0h_4(x^k)$.[6] Using the first equation in (39), we find

$$h_3 = \epsilon_3(z_3)^2 = \frac{\phi^*}{\Upsilon}\frac{z_4^*z_4}{z_4z_4} = \frac{1}{2\Upsilon}(\ln|\Phi|)^*(\ln|h_4|)^*. \quad (42)$$

For $\Upsilon = \Lambda$, we can redefine the coordinates and Φ, introduce $\epsilon_3\epsilon_4$ in Λ and consider solutions of type

$$h_3[\Phi] = (\Phi^*)^2/\Lambda\Phi^2 \quad \text{and} \quad h_4[\Phi] = \Phi^2/4\Lambda. \quad (43)$$

3. We have to solve algebraic equations for w_i by introducing the coefficients (37) in (34) for the generating function ϕ, or using any equivalent variables ϕ, Φ, and/or $\tilde{\Phi}$,

6 We can always redefine a generating function $\Phi(x^k, y^3) \rightarrow \tilde{\Phi}(x^k, y^3)$ and a source $\Upsilon(x^k, y^3) \rightarrow \Lambda$, reconsidering (40), in a form when $[\Phi^2]^*/4\Upsilon = [\tilde{\Phi}^2]^*/4\Lambda$, which allows us to perform a formal integration in (41) and get $h_4 = {}^0h_4(x^k) + \epsilon_3\epsilon_4[\tilde{\Phi}^2]^*/4\Lambda$.

$$w_i = \partial_i \phi/\phi^* = \partial_i \Phi/\Phi^*. \quad (44)$$

4. The solution of equation (35) can be obtained by integrating twice on y^3,

$$n_k = {}_1n_k + {}_2n_k \int dy^3\, h_3/(\sqrt{|h_4|})^3, \quad (45)$$

where ${}_1n_k(x^i)$, ${}_2n_k(x^i)$ are integration functions.

5. The LC-conditions (30) consist a set of nonholonomic constraints which cannot be solved in explicit form for arbitrary data (Φ, Υ) and all types of integration functions ${}_1n_k$ and ${}_2n_k$. Nevertheless, we can find explicit solutions if we assume that via frame and coordinate transformations we can choose ${}_2n_k = 0$ and ${}_1n_k = \partial_k n$ with a function $n = n(x^k)$. We emphasize that $(\partial_i - w_i\partial_3)\Phi \equiv 0$ for any $\Phi(x^k, y^3)$ if w_i is defined by (44). Introducing instead of Φ a new functional $H(\Phi)$, we obtain $(\partial_i - w_i\partial_3)H = \frac{\partial H}{\partial \Phi}(\partial_i - w_i\partial_3)\Phi = 0$. Using equations (43) for functionals of type $h_4 = H(|\tilde{\Phi}(\Phi)|)$, we solve always the equations $(\partial_i - w_i\partial_3)h_4 = 0$, which is equivalent to the second system of equations in (30) because $(\partial_i - w_i\partial_3)\ln\sqrt{|h_4|} \sim (\partial_i - w_i\partial_3)h_4$. For a subclass of generating functions $\Phi = \check{\Phi}$ for which

$$(\partial_i \check{\Phi})^* = \partial_i \check{\Phi}^*, \quad (46)$$

we compute for the left part of the second equation in (30), $(\partial_i - w_i\partial_3)\ln\sqrt{|h_4|} = 0$. The first system of equations in (30) can be solved in explicit form if the w_i are determined by equations (44), and $h_3[\tilde{\Phi}]$ and $h_4[\tilde{\Phi}, \tilde{\Phi}^*]$ are chosen, respectively, for $\Upsilon = \Lambda$. We can consider $\tilde{\Phi} = \tilde{\Phi}(\ln\sqrt{|h_3|})$ for a functional dependence $h_3[\tilde{\Phi}[\check{\Phi}]]$. This allows us to obtain the equations $w_i = \partial_i|\tilde{\Phi}|/|\tilde{\Phi}|^* = \partial_i|\ln\sqrt{|h_3|}|/|\ln\sqrt{|h_3|}|^*$. Taking the derivative ∂_3 on both sides of this equation, we get

$$w_i^* = \frac{(\partial_i|\ln\sqrt{|h_3|}|)^*}{|\ln\sqrt{|h_3|}|^*} - w_i\frac{|\ln\sqrt{|h_3|}|^{**}}{|\ln\sqrt{|h_3|}|^*}.$$

If the conditions (46) are satisfied, we can construct generic off-diagonal configurations with $w_i^* = (\partial_i - w_i\partial_3)\ln\sqrt{|h_3|}$, which is necessary for the zero torsion conditions. Finally, we note that the conditions $\partial_k w_i = \partial_i w_k$ from the second line in (30) are solved for any

$$\check{w}_i = \partial_i\check{\Phi}/\check{\Phi}^* = \partial_i\widetilde{A}, \quad (47)$$

with a non-trivial function $\widetilde{A}(x^k, y^3)$ depending functionally on the generating function $\check{\Phi}$.

The class of off-diagonal metrics of type (25) constructed following steps 1–5 for $\Upsilon = \check{\Upsilon} = \Lambda$, $\Phi = \check{\Phi} = \tilde{\Phi}$ and ${}_2n_k = 0$ in (45) are determined by quadratic elements of type

$$ds^2 = e^{\psi(x^k)}\left[(dx^1)^2 + (dx^2)^2\right] + \frac{(\check{\Phi}^*)^2}{\Lambda\check{\Phi}^2}\left[dy^3 + (\partial_i\widetilde{A}[\check{\Phi}])\,dx^i\right]^2 - \frac{\check{\Phi}^2}{4|\Lambda|}\left[dt + (\partial_k n)\,dx^k\right]^2. \quad (48)$$

We can consider arbitrary generating functions but take the effective cosmological constant Λ for a model of f-modified massive gravity for a source (24). If $\Upsilon = \Lambda$ (44) is for a source (22), we obtain an effective pseudo-Riemannian metric with N-adapted coefficients determined by effective sources in modified gravity. Via nonlinear off-diagonal interactions in GR, and certain corresponding effective sources encoding the contributions from modified gravity, we mimic both massive gravitational and/or f-functional contributions. Here we emphasize that off-diagonal configurations (of vacuum and non-vacuum type) are possible even if the effective sources from modified bi-metric gravity are constrained to be zero.

For arbitrary ϕ and Υ, and related Φ, or $\widetilde{\Phi}$, and Λ, we can generate off-diagonal solutions of (26)–(29) with a nonholonomically induced torsion,

$$ds^2 = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + (z_3)^2\left[dy^3 + \frac{\partial_i\Phi}{\Phi^*}dx^i\right]^2 - (z_4)^2\left[dt + \left({}_1n_k + {}_2n_k \times \int dy^3\frac{(z_3)^2}{(z_4)^3}\,dx^k\right)\right]^2, \quad (49)$$

for $\epsilon_3 = 1, \epsilon_4 = -1$, where the functions $z_3(x^k, y^3)$ and $z_4(x^k, y^3)$ are defined by equations (42) and (41). In N-adapted frames, the ansatz for such solutions define a nontrivial distorting tensor as in $\widehat{\mathbf{Z}} = \{\widehat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma}\}$ in (1).

3.3 Formal integration via polarization functions

We cannot distinguish the coefficients and multiples in a general off-diagonal solution (48) and (49) which are determined by a prime fiducial, f-modified and/or any diagonal exact solution in GR. Such contributions mix for general coordinate/frame transforms. Our goal is to find certain parameterizations of target metrics when the coefficients of prime metrics can be defined in explicit form together with possible 'gravitational polarizations' of effective constants and nonholonomic deformations of the coefficients of metrics. For certain additional assumptions, such deformations can be parameterized on a small parameter.

3.3.1 Levi–Civita deformations in massive gravity

Metrics of type (25) can be used for constructing nonholonomic deformations $(\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^{v}\mathring{\Upsilon}, \mathring{\Upsilon}) \rightarrow (\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}, {}^{v}\widetilde{\Upsilon}, \widetilde{\Upsilon})$, when the prime metric $\mathring{\mathbf{g}}$ may, or may not be, an exact solution of the Einstein or other modified gravitational equations but the target metric $\mathbf{g}$ positively defines a generic off-diagonal solution of field equations in a model of gravity.

We are interested in deformations of the metrics $\mathring{\mathbf{g}}(x^k)$ possessing two Killing vector symmetries (in particular, such a metric may define a black hole, or wormhole solution). The N-adapted deformations of the coefficients of the metrics, frames, and sources are chosen in the form

$$\left[\mathring{g}_i, \mathring{h}_a, \mathring{w}_i, \mathring{n}_i\right] \rightarrow \left[\widetilde{g}_i = \widetilde{\eta}_i \mathring{g}_i, \widetilde{h}_3 = \widetilde{\eta}_3 \mathring{h}_3, \widetilde{h}_4 = \widetilde{\eta}_4 \mathring{h}_4,\right.$$
$$\left.\widetilde{w}_i = \mathring{w}_i + {}^{\eta}w_i, n_i = \mathring{n}_i + {}^{\eta}n_i\right],$$
$${}^{v}\widetilde{\Upsilon} = {}^{v}\hat{\Upsilon}(x^k)\, {}^{v}\mathring{\Upsilon},\ {}^{v}\hat{\Upsilon}(x^k) = \widetilde{\Upsilon} = \mu_g^2\, \lambda(x^k)$$
$$(\mathring{h}_3)^{-1}, \check{\Phi}^2 = \exp[2\varpi]\, \mathring{h}_3\mathring{h}_4,$$

where the source $\mu_g^2\, \lambda(x^k)$ for massive gravity is taken as in (24) and the values $\widetilde{\eta}_a, \widetilde{w}_i, \widetilde{n}_i$ and ϖ are functions of the three coordinates (x^k, y^3), and $\widetilde{\eta}_i(x^k)$ depends only on the h-coordinates. The prime data $\mathring{g}_i, \mathring{h}_a, \mathring{w}_i, \mathring{n}_i$ (which can be determined by an exact solution in gravity theory, by any fiducial metric) are given by the coefficients depending only on (x^k). The value ${}^{v}\mathring{\Upsilon}$ can be defined from certain physical assumptions on the matter and effective sources if $\mathring{\mathbf{g}}$ is chosen as a solution of certain gravitational field equations in a theory of gravity. Conventionally, we can take ${}^{v}\mathring{\Upsilon} = 1$ if, for instance, a general pseudo-Riemannian metric $\mathring{\mathbf{g}}$ is transformed into a solution of some (generalized) field equations with source $({}^{v}\widetilde{\Upsilon}, \widetilde{\Upsilon})$.

In terms of the η-functions resulting in $h_a^* \neq 0$ and $g_i = c_i e^{\psi(x^k)}$, the solutions (48) can be rewritten in the form

$$\mathrm{d}s^2 = e^{\psi(x^k)}\left[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\right] + \frac{(\varpi^*)^2}{\mu_g^2\, \lambda}\mathring{h}_3\left[\mathrm{d}y^3 + (\partial_i\, {}^{\eta}\widetilde{A})\, \mathrm{d}x^i\right]^2 - \frac{e^{2\varpi}}{4\mu_g^2|\,\lambda|}\mathring{h}_4\left[\mathrm{d}t + (\partial_k\, {}^{\eta}n(x^i))\, \mathrm{d}x^k\right]^2. \quad (50)$$

The gravitational polarizations (η_i, η_a) and N-coefficients (w_i, n_i) are computed by the following equations:

$$e^{\psi(x^k)} = \widetilde{\eta}_1\mathring{g}_1 = \widetilde{\eta}_2\mathring{g}_2,\ \widetilde{\eta}_3 = \frac{(\varpi^*)^2}{\mu_g^2\, \lambda}, \widetilde{\eta}_4 = \frac{e^{2\varpi}}{4\mu_g^2|\lambda|},$$
$$w_i = \mathring{w}_i + {}^{\eta}w_i = \partial_i({}^{\eta}\widetilde{A}[\varpi]),\ n_k = \mathring{n}_k + {}^{\eta}n_k = \partial_k({}^{\eta}n),$$

where ${}^{\eta}\widetilde{A}(x^k, y^3)$ is introduced via equations and assumptions similar to (46)–(47) and $\psi^{\bullet\bullet} + \psi'' = 2\, {}^{v}\hat{\Upsilon}(x^k)\, {}^{v}\mathring{\Upsilon}$. For N-coefficients, the parameterizations $w_i = \mathring{w}_i + {}^{\eta}w_i = \partial_i(e^{\varpi}\sqrt{|\mathring{h}_3\mathring{h}_4|})/\ \varpi^* e^{\varpi}\sqrt{|\mathring{h}_3\mathring{h}_4|} = \partial_i\, {}^{\eta}\widetilde{A}$ are used. We can take any function ${}^{\eta}n(x^k)$ and put $\lambda = \mathrm{const} \neq 0$ for both the prime (if this is an exact solution with non-trivial cosmological constant) and the target metrics.

3.3.2 Induced torsion in massive gravity

This class of solutions with non-trivial d-torsion (9) is determined by the metric (49) when the coefficients (41)–(45) are computed for the source $\Upsilon = \mu_g^2\, \lambda(x^k)$ in massive gravity and for possible effective anisotropic polarizations. The corresponding off-diagonal quadratic element is given by

$$\mathrm{d}s^2 = e^{\psi(x^k)}\left[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\right] + \frac{(\Phi^*)^2}{\mu_g^2\lambda\Phi^2}\left[\mathrm{d}y^3 + \frac{\partial_i\Phi}{\Phi^*}\,\mathrm{d}x^i\right]^2 - \frac{\Phi^2}{4\mu_g^2|\lambda|}\left[\mathrm{d}t + \left({}_1n_k + {}_2n_k\frac{4\mu_g(\Phi^*)^2}{\Phi^5}\right)\mathrm{d}x^k\right]^2. \quad (51)$$

We can see that non-trivial stationary off-diagonal torsion effects may result in additional effective rotation proportional to μ_g if the integration function ${}_2n_k \neq 0$. Such terms do not exist for the LC massive configurations of type (50). Using different classes of off-diagonal metrics (51) and (50) we can study if a massive gravity theory comes with induced torsion or is characterized by additional nonholonomic constraints as GR and zero torsion.

3.3.3 Small f-modifications and massive gravity

Additional modifications of GR are possible by f-functionals with an effective source Λ (24). Using the two nonholonomic deformations $(\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^{v}\mathring{\Upsilon}, \mathring{\Upsilon}) \rightarrow (\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}, {}^{v}\widetilde{\Upsilon}, \widetilde{\Upsilon}) \rightarrow (\mathbf{g}[\varepsilon], \mathbf{N}[\varepsilon], \Lambda)$, we construct off-diagonal solutions type (25) with $\mathbf{g}$ and $\mathbf{N}$ depending on a small parameter $\varepsilon, 0 < \varepsilon \ll 1$, when the source in massive gravity $\mu_g^2|\lambda|$ is generalized to an effective cosmological constant Λ with additional contributions by matter fields and f-modifications of gravity. The corresponding N-adapted transforms are parameterized thus:

$$[\mathring{g}_i, \mathring{h}_a, \mathring{w}_i, \mathring{n}_i] \rightarrow [g_i = (1 + \varepsilon\chi_i)\widetilde{\eta}_i\mathring{g}_i, h_3 = (1 + \varepsilon\chi_3)\widetilde{\eta}_3\mathring{h}_3, h_4 = (1 + \varepsilon\chi_4)\widetilde{\eta}_4\mathring{h}_4,$$
$${}^{\varepsilon}w_i = \mathring{w}_i + \widetilde{w}_i + \varepsilon\overline{w}_i,\ {}^{\varepsilon}n_i = \mathring{n}_i + \widetilde{n}_i + \varepsilon\overline{n}_i], \quad (52)$$
$$\mu_g^2\, \lambda(x^k) = \Lambda[1 - \varepsilon\, {}^{\mu}\chi(x^k)], \varpi^*[\varepsilon] = \varpi^*(1 + \varepsilon\, {}^{\varpi}\chi(x^k, y^3)),$$

where the values $\chi_i(x^k)$, ${}^{\lambda}\chi(x^k), \overline{n}_i(x^k)$, ${}^{\varpi}\chi(x^k, y^3)$, $\chi_a(x^k, y^3)$ and $\overline{w}_i$ (x^k, y^3) can be computed to define LC-configurations as solutions of the system (26)–(30).

The deformations (52) of the off-diagonal solutions (50) result in a new class of ε-deformed solutions if

$$\chi_3 = {}^{\mu}\chi + {}^{\varpi}\chi, \chi_4 = {}^{\mu}\chi + \varpi^{-1}\int dy^3({}^{\varpi}\chi\varpi^*),$$
$$\overline{w}_i = \partial_i\left({}^{\varpi}\chi\sqrt{|\mathring{h}_3\mathring{h}_4|}\right)/\varpi^* e^{\varpi}\sqrt{|\mathring{h}_3\mathring{h}_4|} = \partial_i\overline{A}, \overline{n}_i = \partial_i\overline{n}.$$

The coefficients for the h-metric $g_i = \exp\psi(x^i) = (1 + \varepsilon\chi_i)\widetilde{\eta}_i\mathring{g}_i$ are solutions of (26) with ${}^{v}\Upsilon = \Lambda = \check{\Upsilon}(x^k) + \mu_g^2\lambda$, where $\check{\Upsilon}(x^k)$ is determined by possible contributions of matter fields and f-modifications parametrized in (24).

In the next sections, we shall construct such solutions in explicit form for ellipsoid, toroid, and solitonic deformations. If ε-deformations of type (52) are considered for the metrics (51), we can generate new classes of off-diagonal solutions with nonholonomically induced torsion determined both by massive and f-modifications of GR.

4 Off-diagonal deformations of wormhole metrics

In this section, we construct and analyze two examples when a wormhole solution matching an exterior Schwarzschild–de Sitter spacetime is nonholonomically deformed into new classes of off-diagonal solutions. The target metrics are constructed for modifications of GR with effectively polarized cosmological constants and 'polarization' multiples and additional terms to, respectively, diagonal and non-diagonal coefficients of metrics. The deformations resulting from massive gravity are studied for an effectively polarized cosmological constant proportional to μ_g^2. The modifications determined by the f-terms are computed for a small deformation parameter ε.

4.1 Prime metrics for 4-d wormholes

Let us consider a diagonal prime wormhole metric,

$$\begin{aligned}\mathring{\mathbf{g}} &= \mathring{g}_i(x^k)\,dx^i\otimes dx^i + \mathring{h}_a(x^k)\,dy^a\otimes dy^a\\ &= [1-b(r)/r]^{-1}\,dr\otimes dr + r^2(d\theta\otimes d\theta\\ &\quad + \sin^2\theta d\varphi\otimes d\varphi) - e^{2B(r)}\,dt\otimes dt,\end{aligned}\tag{53}$$

where $B(r)$ and $b(r)$ are called, respectively, the red-shift and form functions; see details in [5–8]. The radial coordinate has a range $r_0 \leq r < a$, where the minimum value r_0 is for the wormhole throat and a is the distance at which the interior spacetime joins to an exterior vacuum solution ($a \to \infty$ for specific asymptotically flat wormhole geometries). Certain conditions have to be imposed on the coefficients of (53) and on the diagonal components of the stress–energy tensor,

$$\mathring{T}^{\mu}_{\nu} = \text{diag}[{}^{r}p = \tau(r),\ {}^{\theta}p = p(r),\ {}^{\varphi}p = p(r),\ {}^{t}p = \rho(r)],\tag{54}$$

in order to generate wormhole solutions of the Einstein equations in GR.

A well-known class of wormhole metrics is constructed so as to possess the conformal symmetry determined by a vector $\mathbf{X} = \{X^{\alpha}(u)\}$, when the Lie derivative $X^{\alpha}\partial_{\alpha}\mathring{g}_{\mu\nu} + \mathring{g}_{\alpha\nu}\partial_{\mu}X^{\alpha} + \mathring{g}_{\alpha\mu}\partial_{\nu}X^{\alpha} = \sigma\mathring{g}_{\mu\nu}$, where $\sigma = \sigma(u)$ is the conformal factor. Such solutions are parameterized by

$$\begin{aligned}B(r) &= \frac{1}{2}\ln(C^2r^2) - \kappa\int r^{-1}(1-b(r)/r)^{-1/2}\,dr,\\ &\quad b(r) = r[1-\sigma^2(r)],\\ \tau(r) &= \frac{1}{\kappa^2r^2}(3\sigma^2 - 2\kappa\sigma - 1),\ \ p(r) = \frac{1}{\kappa^2r^2}\\ &\quad\times(\sigma^2 - 2\kappa\sigma + \kappa^2 + 2r\sigma\sigma^{\bullet}),\\ \rho(r) &= \frac{1}{\kappa^2r^2}(1-\sigma^2 - 2r\sigma\sigma^{\bullet}).\end{aligned}\tag{55}$$

The data (55) generate 'diagonal' wormhole configurations determined by 'exotic' matter because the null energy condition (NEC) $\mathring{T}_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ (k^{ν} is any null vector) is violated.

We shall study configurations which match the interior geometries to an exterior de Sitter one which (in general) can also be determined by an off-diagonal metric. The exotic matter and effective matter configurations are considered to be restricted to spatial distributions in the throat neighborhood which limit the dimension of the locally isotropic and/or anisotropic wormhole not to be arbitrarily large.

4.2 Parametric deformations and exterior de Sitter spacetimes

The Schwarzschild–de Sitter (SdS) metric,

$$ds^2 = q^{-1}(r)(dr^2 + r^2\,d\theta^2) + r^2\sin^2\theta\,d\varphi^2 - q(r)\,dt^2,\tag{56}$$

can be re-parameterized for any $(x^1(r,\theta), x^2(r,\theta), y^3 = \varphi, y^4 = t)$ when

$$q^{-1}(r)(dr^2 + r^2\,d\theta^2) = e^{\mathring{\psi}(x^k)}\left[(dx^1)^2 + (dx^2)^2\right].$$

Such a metric defines two real static solutions of the Einstein equations with cosmological constant Λ if $M < 1/3\sqrt{|\Lambda|}$, for $q(r) = 1 - 2\overline{M}(r)/r$, $\overline{M}(r) = M + \Lambda r^3/6$, where M is a constant mass parameter. For diagonal configurations, we can identify Λ with the effective cosmological constant (24).

In this work, we study conformal, ellipsoid, and/or solitonic/toroidal deformations related in certain limits to the Schwarzschild–de Sitter metric written in the form

$$\begin{aligned}{}_{\Lambda}\mathbf{g} &= d\xi\otimes d\xi + r^2(\xi)\,d\theta\otimes d\theta\\ &\quad + r^2(\xi)\sin^2\theta\,d\varphi\otimes d\varphi - q(\xi)\,dt\otimes dt,\end{aligned}\tag{57}$$

for local coordinates,

$$x^1 = \xi = \int dr/\sqrt{|q(r)|}, x^2 = \vartheta, y^3 = \varphi, y^4 = t,\tag{58}$$

for a system of h-coordinates when $(r,\theta)\rightarrow(\xi,\vartheta)$ with ξ and ϑ of length dimension. The data for this primary metric are written

$$\mathring{g}_i=\mathring{g}_i(x^k)=e^{\mathring{\psi}(x^k)},\ \mathring{h}_3=r^2(x^k)\sin^2\theta(x^k),\\ \mathring{h}_4=-q(r(x^k)),\ \mathring{w}_i=0,\ \mathring{n}_i=0.$$

Let us analyze how such diagonal metrics can be off-diagonally deformed by contributions from massive and f-modified gravity.

4.2.1 Off-diagonal de Sitter deformations in massive gravity

Solutions resulting in the LC configurations can be generated similarly to (50) but using data (57),

$$\begin{aligned}ds^2 &= e^{\tilde{\psi}(\xi,\theta)}(d\xi^2+d\vartheta^2)\\ &+\frac{(\varpi^*)^2}{\mu_g^2\,\lambda(\xi,\vartheta)}r^2(\xi)\sin^2\theta(\xi,\vartheta)[d\varphi+(\partial_\xi\,{}^\eta\widetilde{A})\,d\xi\\ &+(\partial_\vartheta\,{}^\eta\widetilde{A})\,d\vartheta]^2-\frac{e^{2\varpi}}{4\mu_g^2|\,\lambda(\xi,\vartheta)|}q(\xi)\\ &\times[dt+\partial_\xi\,{}^\eta n(\xi,\vartheta)\,d\xi+\partial_\vartheta\,{}^\eta n(\xi,\vartheta)\,d\vartheta]^2,\end{aligned}\qquad(59)$$

where $e^{\tilde{\psi}(\xi,\vartheta)}=\widetilde{\eta}_1\mathring{g}_1=\widetilde{\eta}_2\mathring{g}_2$ are solutions of $\tilde{\psi}^{\bullet\bullet}+\tilde{\psi}''=2\,\mu_g^2\,\lambda(\xi,\vartheta)$. The generating function $\varpi(\xi,\vartheta,\varphi)$, the effective source $\lambda(\xi,\vartheta)$ and the mass parameter μ_g should be fixed from physical assumptions on systems of reference, fixed prime Stükelberg fields [using algebraic conditions of type (18)] and observable effects in modern cosmology. The value $\widetilde{n}_i={}^\eta n_i(\xi,\vartheta)=\partial_i\,{}^\eta n(\xi,\vartheta)$ is an integration function and ${}^\eta\widetilde{A}(\xi,\vartheta,\varphi)$ is determined by $e^{2\varpi}$ following equation (47) and

$$\begin{aligned}\widetilde{w}_i &= {}^\eta w_i=\frac{\partial_i(e^{\varpi}r(\xi)\sin\theta(\xi,\vartheta)\sqrt{|q(\xi)|})}{\varpi^*e^{\varpi}r(\xi)\sin\theta(\xi,\vartheta)\sqrt{|q(\xi)|}}\\ &=\partial_i\,{}^\eta\widetilde{A},\ \text{ for } x^i=(\xi,\vartheta).\end{aligned}\qquad(60)$$

It should be noted here that the N-coefficients in (59) result in nonzero anholonomy coefficients (6) for nonholonomic relations of type (5). This proves that such solutions cannot be diagonalized via frame/coordinate transformations and that, in general, they are characterized by six (from possibly ten) independent coefficients of metrics. We can mimic such configurations by off-diagonal interactions in GR with corresponding effective matter source determined by terms induced by μ_g taken as an integration parameter. It can be related to Killing symmetries of such metrics; see details in Ref. [24].

4.2.2 Ellipsoidal f-modifications

Deformations (52) on a parameter ε, $0\le\varepsilon<1$, are considered for the solutions in massive gravity (59), with

$$\begin{aligned}\chi_3 &= {}^\varpi\chi,\ \chi_4=\varpi^{-1}\int d\varphi({}^\varpi\chi\,\varpi^*),\\ \overline{w}_i &= \frac{\partial_i({}^\varpi\chi\, r(\xi)\sin\theta(\xi,\vartheta)\sqrt{|q(\xi)|})}{\varpi^*e^{\varpi}r(\xi)\sin\theta(\xi,\vartheta)\sqrt{|q(\xi)|}}=\partial_i\overline{A},\overline{n}_i=\partial_i\overline{n},\end{aligned}\qquad(61)$$

for $x^i=(\xi,\vartheta)$, and we fix, for simplicity, ${}^\mu\chi=0$ (a possible physical motivation is to consider models with constant mass gravity parameter and zero related polarization). The coefficients of the h-metric $g_i=\exp\psi(\xi,\vartheta)=(1+\varepsilon\chi_i)\widetilde{\eta}_i\mathring{g}_i$ are solutions of (26) with ${}^\upsilon\Upsilon=\Lambda=\check{\Upsilon}(x^k)+\mu_g^2\lambda$, where $\check{\Upsilon}(\xi,\vartheta)$ is determined by possible contributions of matter fields and f-modifications parameterized in (24). The resulting target off-diagonal quadratic element is parameterized in the form

$$\begin{aligned}\mathbf{ds}^2 &= e^{\tilde{\psi}(\xi,\vartheta)}\left(d\xi^2+d\vartheta^2\right)+\frac{(\varpi^*)^2}{\mu_g^2\,\lambda(\xi,\vartheta)}\\ &\quad\times[1+\varepsilon\chi_3(\xi,\vartheta,\varphi)]r^2(\xi)\sin^2\theta(\xi,\vartheta)(\delta\varphi)^2\\ &\quad-\frac{e^{2\varpi}}{4\mu_g^2|\,\lambda(\xi,\vartheta)|}[1+\varepsilon\chi_4(\xi,\vartheta,\varphi)]q(\xi)(\delta t)^2,\\ \delta\varphi &= d\varphi+[\widetilde{w}_i(\xi,\vartheta,\varphi)+\varepsilon\overline{w}_i(\xi,\vartheta,\varphi)]\,dx^i,\\ \delta t &= dt+\ [\tilde{n}_i(\xi,\vartheta)+\varepsilon\overline{n}_i(\xi,\vartheta)]\,dx^i,\end{aligned}$$

when $\widetilde{w}_i$ are given by equations (60). For such small deformations re-parameterized in (r,θ)-coordinates, the coefficient

$$\begin{aligned}h_4 &= -\frac{e^{2\varpi}}{4\mu_g^2|\,\lambda(\xi,\vartheta)|}[1+\varepsilon\chi_4]q\\ &\simeq-\frac{e^{2\varpi}}{4\mu_g^2|\,\lambda(r,\theta)|}\left[1-\frac{2M(r,\theta,\varphi)}{r}\right]\end{aligned}\qquad(62)$$

is related to small gravitational polarizations of the mass coefficients,

$$M(r,\theta,\varphi)\simeq\overline{M}(r)\left[1+\varepsilon\left(1-\frac{r}{2\overline{M}}\right)\chi_4(r,\theta,\varphi)\right].$$

We generate rotoid f-deformations if

$$\chi_4=\overline{\chi}_4(r,\varphi):=\frac{2\overline{M}(r)}{r}\left(1-\frac{2\overline{M}(r)}{r}\right)^{-1}\underline{\zeta}\sin(\omega_0\varphi+\varphi_0),\qquad(63)$$

for some constants $\underline{\zeta}$, ω_0 and φ_0, taken as a polarization function. With respect to N-adapted frames, there is a smaller "ellipsoidal horizon" (when $h_4=0$ in (62), we get the parametric equation for an ellipse),

$$r_+ \simeq \frac{2\,\overline{M}(r_+)}{1+\varepsilon \underline{\zeta}\,\sin(\omega_0\varphi+\varphi_0)},$$

where ε is the eccentricity parameter. Using equations (61) for a prescribed value $\varpi(r,\theta,\varphi)$, and $\chi_3 = {}^{\varpi}\chi = \partial_\varphi[\overline{\chi}_4\varpi]/\partial_\varphi\varpi$, we compute

$$\overline{w}_i = \frac{\partial_i(r(\xi)\sin\theta(\xi,\vartheta)\sqrt{|q(\xi)|}\partial_\varphi[\overline{\chi}_4\varpi])}{e^{\varpi}r(\xi)\sin\theta(\xi,\vartheta)\sqrt{|q(\xi)|}\partial_\varphi\varpi} = \partial_i\overline{A}.$$

The resulting solutions in massive f-gravity with rotoid symmetry are parameterized

$$\begin{aligned}\mathbf{ds}^2 =& e^{\tilde{\psi}(\xi,\vartheta)}\left(d\xi^2+d\vartheta^2\right)+\frac{(\varpi^*)^2}{\mu_g^2\,\lambda}\left(1+\varepsilon\frac{\partial_\varphi[\overline{\chi}_4\varpi]}{\partial_\varphi\varpi}\right)r^2(\xi)\\ &\times\sin^2\theta(\xi,\vartheta)\,(\delta\varphi)^2\\ &-\frac{e^{2\varpi}}{4\mu_g^2|\,\lambda|}[1+\varepsilon\overline{\chi}_4]q(\xi)\,(\delta t)^2, \qquad (64)\\ \delta\varphi =& d\varphi+[\partial_i\,{}^{\eta}\widetilde{A}+\varepsilon\partial_i\overline{A}]\,dx^i,\ \ \delta t = dt+[\partial_i\,{}^{\eta}n+\varepsilon\partial_i\overline{n}]\,dx^i.\end{aligned}$$

Such stationary configurations are generated by nonlinear off-diagonal interactions in massive gravity with non-trivial $\mu_g^2\,\lambda(\xi,\vartheta)$ terms. We note that, in general, the limit $\mu_g\to 0$ is not smooth for such classes of solutions. There are necessarily additional assumptions on the nonholonomic constraints resulting in diagonal metrics with two Killing symmetries or for selecting black rotoid–de Sitter configurations. It is possible to model such solutions via locally anisotropic effective polarizations of the coefficients of metrics and physical constants (treated as integration functions and constants) in GR. For this class of solutions, the contributions related to 'massive' gravity terms are very different from those generated by f-deformations. In the latter case, there are smooth limits for $\varepsilon\to 0$, when (for instance) rotoid symmetries may transform into spherical ones.

5 Ellipsoid, solitonic, and toroid deformations of wormhole metrics

In this section, we explore rotoid deformations of wormhole configurations determined by off-diagonal effects in massive gravity and f-modifications. The general ansatz for such metrics is taken in the form

$$\begin{aligned}\mathbf{ds}^2 =& e^{\tilde{\psi}(\widetilde{\xi},\theta)}(d\widetilde{\xi}^2+d\vartheta^2)+\frac{[\partial_\varphi\varpi(\widetilde{\xi},\vartheta,\varphi)]^2}{\mu_g^2\lambda(\widetilde{\xi},\vartheta)}\\ &\times\left(1+\varepsilon\frac{\partial_\varphi[\chi_4(\widetilde{\xi},\vartheta,\varphi)\varpi(\widetilde{\xi},\vartheta,\varphi)]}{\partial_\varphi\varpi(\widetilde{\xi},\vartheta,\varphi)}\right)r^2(\widetilde{\xi})\\ &\times\sin^2\theta(\widetilde{\xi},\vartheta)(\delta\varphi)^2\\ &-\frac{e^{2\varpi(\widetilde{\xi},\vartheta,\varphi)]}}{4\mu_g^2|\,\lambda(\widetilde{\xi},\vartheta)|}[1+\varepsilon\chi_4(\widetilde{\xi},\vartheta,\varphi)]e^{2B(\widetilde{\xi})}(\delta t)^2, \quad (65)\end{aligned}$$

$$\begin{aligned}\delta\varphi =& d\varphi+\partial_{\widetilde{\xi}}[\,{}^{\eta}\widetilde{A}(\widetilde{\xi},\vartheta,\varphi)+\varepsilon\overline{A}(\widetilde{\xi},\vartheta,\varphi)]\,d\widetilde{\xi}\\ &+\partial_\vartheta[\,{}^{\eta}\widetilde{A}(\widetilde{\xi},\vartheta,\varphi)+\varepsilon\overline{A}(\widetilde{\xi},\vartheta,\varphi)]\,d\vartheta,\\ \delta t =& dt+\partial_{\widetilde{\xi}}[\,{}^{\eta}n(\widetilde{\xi},\vartheta)+\varepsilon\partial_i\overline{n}(\widetilde{\xi},\vartheta)]\,d\widetilde{\xi}\\ &+\partial_\vartheta[\,{}^{\eta}n(\widetilde{\xi},\vartheta)+\varepsilon\partial_i\overline{n}(\widetilde{\xi},\vartheta)]\,d\vartheta,\end{aligned}$$

where $\widetilde{\xi}=\int dr/\sqrt{|1-b(r)/r|}$, and $B(\widetilde{\xi})$ are determined by the prime metric (53). We can choose such generating and integration functions when the metrics (in corresponding limits) define exterior spacetimes (64), for coordinates (58) and $e^{2B(\widetilde{\xi})}\to q(r)$, see (57).

The class of solutions (65) are for stationary configurations determined by respective general and small ε-parametric μ_g- and f-modifications.

5.1 Ellipsoidal off-diagonal wormhole deformations

Rotoid ε-configurations are 'extracted' from (65) if we take for the f-deformations

$$\chi_4=\overline{\chi}_4(r,\varphi):=\frac{2\overline{M}(r)}{r}\left(1-\frac{2\overline{M}(r)}{r}\right)^{-1}\underline{\zeta}\,\sin(\omega_0\varphi+\varphi_0), \quad (66)$$

for r considered as a function $r(\widetilde{\xi})$. This is different from $r(\xi)$ taken in the previous section but may be parameterized to have $r(\widetilde{\xi})\to r(\xi)$ in exterior spacetimes. Let us define

$$\begin{aligned}h_3 =& \widetilde{\eta}_3(\widetilde{\xi},\vartheta,\varphi)\left[1+\varepsilon\chi_3(\widetilde{\xi},\vartheta,\varphi)\right]\,{}^{0}\overline{h}_3(\widetilde{\xi},\vartheta),\\ h_4 =& \widetilde{\eta}_4(\widetilde{\xi},\vartheta,\varphi)\left[1+\varepsilon\overline{\chi}_4(\widetilde{\xi},\varphi)\right]\,{}^{0}\overline{h}_4(\widetilde{\xi}),\end{aligned}$$

for ${}^{0}\overline{h}_3=r^2(\widetilde{\xi})\sin^2\theta(\widetilde{\xi},\vartheta)$, ${}^{0}\overline{h}_4=q(\widetilde{\xi})$ and

$$\widetilde{\eta}_3=\frac{[\partial_\varphi\varpi(\widetilde{\xi},\vartheta,\varphi)]^2}{\mu_g^2\,\lambda(\widetilde{\xi},\vartheta)},\ \widetilde{\eta}_4=\frac{e^{2\varpi(\widetilde{\xi},\vartheta,\varphi)]}}{4\mu_g^2|\,\lambda(\widetilde{\xi},\vartheta)|q(\widetilde{\xi})}e^{2B(\widetilde{\xi})}, \quad (67)$$

where $e^{2B(\widetilde{\xi})}\to q(\widetilde{\xi})$ if $\widetilde{\xi}\to\xi$. Introducing (66) in the respective equations (61) for any prescribed generating function $\widetilde{\varpi}(\widetilde{\xi},\vartheta,\varphi)$, we can compute

$$\begin{aligned}\widetilde{\chi}_3 =& \chi_3(\widetilde{\xi},\vartheta,\varphi)={}^{\varpi}\chi=\partial_\varphi[\overline{\chi}_4\widetilde{\varpi}]/\partial_\varphi\widetilde{\varpi},\ \text{and}\\ \overline{w}_i =& \frac{\partial_i(r(\widetilde{\xi})\sin\theta(\widetilde{\xi},\vartheta)\sqrt{|q(\widetilde{\xi})|}\partial_\varphi[\overline{\chi}_4\varpi])}{e^{\varpi}r(\widetilde{\xi})\sin\theta(\widetilde{\xi},\vartheta)\sqrt{|q(\widetilde{\xi})|}\partial_\varphi\varpi}=\partial_i\overline{A}(\widetilde{\xi},\vartheta,\varphi),\end{aligned}$$

for $x^i=(\widetilde{\xi},\vartheta)$. With respect to N-adapted frames, we model an ellipsoidal configuration with $r_+(\widetilde{\xi}=\widetilde{\xi}_+)\simeq\frac{2\,\overline{M}(\widetilde{\xi}_+)}{1+\varepsilon\underline{\zeta}\,\sin(\omega_0\varphi+\varphi_0)}$, for a corresponding value of $\widetilde{\xi}_+$, constants $\underline{\zeta}$, ω_0 and φ_0, and eccentricity ε.

Putting together the above equations, we obtain

$$\mathbf{ds}^2 = e^{\tilde{\psi}(\widetilde{\xi},\theta)}(\mathrm{d}\widetilde{\xi}^2 + \mathrm{d}\vartheta^2) + \frac{[\partial_\varphi \widetilde{\varpi}]^2}{\mu_g^2\,\lambda}\left(1+\varepsilon\frac{\partial_\varphi[\overline{\chi}_4\widetilde{\varpi}]}{\partial_\varphi\widetilde{\varpi}}\right)^0 \overline{h}_3[\mathrm{d}\varphi+\partial_{\widetilde{\xi}}({}^\eta\widetilde{A} +\varepsilon\overline{A})\,\mathrm{d}\widetilde{\xi}+\partial_\vartheta({}^\eta\widetilde{A}+\varepsilon\overline{A})\,\mathrm{d}\vartheta]^2 - \frac{e^{2\widetilde{\varpi}}}{4\mu_g^2|\,\lambda|}[1+\varepsilon\overline{\chi}_4(\widetilde{\xi},\varphi)]e^{2B(\widetilde{\xi})}[\mathrm{d}t+\partial_{\widetilde{\xi}}({}^\eta n+\varepsilon\overline{n})\,\mathrm{d}\widetilde{\xi} +\partial_\vartheta({}^\eta n+\varepsilon\overline{n})\,\mathrm{d}\vartheta]^2. \quad (68)$$

If the generating functions $\widetilde{\varpi}$ and effective source λ in massive gravity are chosen in such a way that the polarization functions (67) can be approximated by $\widetilde{\eta}_a \simeq 1$, and ${}^\eta\widetilde{A}$ and ${}^\eta n$ are 'almost constant', with respect to certain systems of radial coordinates, the metric (68) mimics small rotoid wormhole-like configurations with off-diagonal terms and f-modifications of the diagonal coefficients. It is possible to choose such integration functions and constants that this class of stationary solutions define wormhole-like metrics depending generically on three space coordinates with self-consistent 'embedding' in an effective massive gravity background.

For more general classes of nonholonomic deformations, we can preserve certain rotoid type symmetries but the 'wormhole character' of solutions becomes less clear.

5.2 Solitonic waves for wormholes and black ellipsoids

Let us consider two examples of gravitational solitonic deformations in massive f-modified gravity.

5.2.1 *Sine-Gordon two dimensional nonlinear waves*

An interesting class of off-diagonal solutions depending on all spacetime coordinates can be constructed by designing a configuration when a 1-solitonic wave preserving an ellipsoidal wormhole configuration. Such a spacetime metric can be written in the form

$$\mathbf{ds}^2 = e^{\tilde{\psi}(x^i)}(\mathrm{d}\widetilde{\xi}^2 + \mathrm{d}\vartheta^2) + \omega^2(\widetilde{\xi},t) \times\left[\widetilde{\eta}_3\left(1+\varepsilon\frac{\partial_\varphi[\overline{\chi}_4\widetilde{\varpi}]}{\partial_\varphi\widetilde{\varpi}}\right){}^0\overline{h}_3(\delta\varphi)^2 -\widetilde{\eta}_4[1+\varepsilon\overline{\chi}_4(\widetilde{\xi},\varphi)]\,{}^0\overline{h}_4(\delta t)^2\right],$$
$$\delta\varphi = \mathrm{d}\varphi+\partial_i({}^\eta\widetilde{A}+\varepsilon\overline{A})\,\mathrm{d}x^i,\ \delta t=\mathrm{d}t+\,{}_1n_i(\widetilde{\xi},\vartheta)\,\mathrm{d}x^i, \quad (69)$$

for $x^i = (\widetilde{\xi},\vartheta)$ and $y^a = (\varphi, t)$. The 'vertical' conformal factor

$$\omega(\widetilde{\xi},t) = 4\arctan e^{m\gamma(\widetilde{\xi}-vt)+m_0}, \quad (70)$$

where $\gamma^2 = (1-v^2)^{-1}$ and constants m, m_0, v, defines a 1-soliton solution of the sine-Gordon equation $\frac{\partial^2\omega}{\partial t^2} - \frac{\partial^2\omega}{\partial\widetilde{\xi}^2} + \sin\omega = 0$.

For $\omega = 1$, the metrics (69) are of type (68). A non-trivial value ω depends on the time-like coordinate t and has to be constrained to conditions of type (36), which can be written for ${}_1n_2 = 0$ and ${}_1n_1 = $ const in the form $\frac{\partial\omega}{\partial\widetilde{\xi}} - {}_1n_1\frac{\partial\omega}{\partial t} = 0$. A gravitational solitonic wave (70) will propagate self-consistently in a rotoid wormhole background with ${}_1n_1 = v$, which solves both the sine-Gordon and the constraint equations. Re-defining the system of coordinates with $x^1 = \widetilde{\xi}$ and $x^2 = \theta$, we can transform any non-trivial ${}_1n_i(\widetilde{\xi},\theta)$ into the necessary ${}_1n_1 = v$ and ${}_1n_2 = 0$.

5.2.2 *Three dimensional solitonic waves*

In general, we can construct various types of vacuum gravitational 2-d and 3-d configurations characterized by solitonic hierarchies and related bi-Hamilton structures, for instance, of the Kadomtsev–Petivashvili, KdP, equations [25,26] with possible mixtures with solutions for 2-d and 3-d sine-Gordon equations etc., see details in Ref. [27].

Let us consider a solution of the KdP equation for the v-conformal factor $\omega = \check{\omega}(\widetilde{\xi},\varphi,t)$, when $y^4 = t$ is taken as a time-like coordinate, thus

$$\pm\,\check{\omega}^{**} + (\partial_t\check{\omega} + \check{\omega}\,\check{\omega}^\bullet + \epsilon\check{\omega}^{\bullet\bullet\bullet})^\bullet = 0, \quad (71)$$

with dispersion ϵ. In the dispersionless limit $\epsilon\to 0$, we can assume that the solutions are independent on φ and determined by Burgers' equation $\partial_t\check{\omega} + \check{\omega}\,\check{\omega}^\bullet = 0$. For 3-d solitonic configurations, the conditions (36) are written in the form $\mathbf{e}_1\check{\omega} = \check{\omega}^\bullet + w_1(\widetilde{\xi},\vartheta,\varphi)\check{\omega}^* + n_1(\widetilde{\xi},\vartheta)\partial_t\check{\omega} = 0$. If $\check{\omega}' = 0$, we can fix $w_2 = 0$ and $n_2 = 0$.

Such solitonic deformations of the wormhole metrics and their massive gravity and f-modifications can be parameterized in the form

$$\mathbf{g} = e^{\psi(\widetilde{\xi},\vartheta)}(\mathrm{d}\widetilde{\xi}\otimes\mathrm{d}\widetilde{\xi} + \mathrm{d}\vartheta\otimes\mathrm{d}\vartheta) + \left[\check{\omega}(\widetilde{\xi},\varphi,t)\right]^2 \times h_a(\widetilde{\xi},\varphi)\,\mathbf{e}^a\otimes\mathbf{e}^a,$$
$$\mathbf{e}^3 = \mathrm{d}\varphi + w_1(\widetilde{\xi},\vartheta,\varphi)d\widetilde{\xi},\ \mathbf{e}^4 = \mathrm{d}t + n_1(\widetilde{\xi},\vartheta)d\widetilde{\xi}.$$

This class of metrics does not have (in general) Killing symmetries but may possess symmetries determined by solitonic solutions of (71).

In a similar form, we can construct solutions for any $\check{\omega}$ defined by any 3-d solitonic and/or other nonlinear wave equations, or we can generate solitonic deformations for $\omega = \check{\omega}(\vartheta,\varphi,t)$.

5.3 Ringed wormholes

Using the AFDM, we can generate an ansatz for a rotoid wormhole plus a torus (ring) configuration,

$$\begin{aligned} \mathbf{ds}^2 &= e^{\tilde{\psi}(x^i)}(\mathrm{d}\widetilde{\xi}^2 + \mathrm{d}\vartheta^2) + \widetilde{\eta}_3 \left(1 + \varepsilon \frac{\partial_\varphi [\overline{\chi}_4 \widetilde{\varpi}]}{\partial_\varphi \widetilde{\varpi}}\right) {}^0\overline{h}_3(\delta\varphi)^2 \\ &\quad - F(\widetilde{\xi}, \vartheta, \varphi)\widetilde{\eta}_4[1 + \varepsilon \overline{\chi}_4(\widetilde{\xi}, \varphi)] \, {}^0\overline{h}_4(\delta t)^2 \\ \delta\varphi &= \mathrm{d}\varphi + \partial_i({}^\eta\widetilde{A} + \varepsilon \overline{A})\, \mathrm{d}x^i, \delta t = \mathrm{d}t + {}_1n_i(\widetilde{\xi}, \vartheta)\, \mathrm{d}x^i, \end{aligned} \tag{72}$$

for $x^i = (\widetilde{\xi}, \vartheta)$ and $y^a = (\varphi, t)$, where the function $F(\widetilde{\xi}, \vartheta, \varphi)$ in conventional spherical coordinates can be rewritten equivalently in conventional Cartesian coordinates as $F(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (R_0 - \sqrt{\widetilde{x}^2 + \widetilde{y}^2})^2 + \widetilde{z}^2 - a_0$, for $a_0 < a$, $R_0 < r_0$. We get a ring around the wormhole throat[7]. The ring configuration is defined by the condition $F = 0$. For $F = 1$, we get a metric of type (69) with $\omega = 1$. In the above equations, R_0 is the distance from the center of the tube to the center of the torus/ring and a_0 is the radius of the tube.

If the wormhole objects exist, the variants ringed by a torus may be stable for certain nonholonomic geometry and exotic matter configurations. We omit in this work a rigorous stability analysis as well as a study of issues related to cosmic censorship criteria etc.

5.4 Modified wormholes with induced torsion

The examples for wormhole nonholonomic deformations considered above are for effective LC-configurations which can be effectively modeled by nonlinear off-diagonal interactions in GR. Here, we provide an example of a class of stationary off-diagonal solutions with non-trivial torsion effects resulting in additional effective rotation proportional to μ_g, see the similar configuration (51). The corresponding off-diagonal quadratic element is given by

$$\begin{aligned} \mathrm{d}s^2 &= e^{\tilde{\psi}(\widetilde{\xi},\vartheta)}(\mathrm{d}\widetilde{\xi}^2 + \mathrm{d}\vartheta^2) \\ &\quad + \frac{(\partial_\varphi \Phi)^2}{\mu_g^2 \lambda(\widetilde{\xi}, \vartheta)\Phi^2} \left[\mathrm{d}y^3 + \frac{\partial_i \Phi}{\partial_\varphi \Phi}\, \mathrm{d}x^i\right]^2 \\ &\quad - \frac{\Phi^2}{4\mu_g^2 |\lambda(\widetilde{\xi}, \vartheta)|} \left[\mathrm{d}t + \left({}_1n_k(\widetilde{\xi}, \vartheta) \right.\right. \\ &\quad \left.\left. + {}_2n_k(\widetilde{\xi}, \vartheta) \frac{4\mu_g (\partial_\varphi \Phi)^2}{\Phi^5}\right) \mathrm{d}x^k\right]^2, \end{aligned} \tag{73}$$

for $x^i = (\widetilde{\xi}, \vartheta)$ and generating function $\Phi = \exp[2\widetilde{\varpi}(\widetilde{\xi}, \vartheta, \varphi)]$. The d-torsion coefficients (9) for this metric are not trivial if ${}_2n_k \neq 0$. This and other settings for more general sources $\Upsilon(\widetilde{\xi}, \vartheta, \varphi)$, see (22), and different classes of N-coefficients lead to characteristic geometric and physical properties, which are very different from LC-configurations.

We can parameterize (73) in any form (59), (65), (68), and (69) in order to generate off-diagonal solutions with μ_g- and/of f-modifications possessing rotoid and/or solitonic symmetries characterized by nonholonomic torsion. If a vertical conformal factor ω similar to (69) is considered, the metric and induced torsion fields might depend on all four spacetime coordinates. Toroidal configurations of type (72) can be constructed if a toroidal function of type $F(\widetilde{x}, \widetilde{y}, \widetilde{z})$ is introduced before the v-components of metrics in (73).

6 Concluding remarks and discussion

Modified gravity theories with functional dependence on curvature and other traces of energy–momentum tensors for matter fields, torsion sources etc. and/or with contributions by massive/bi-metric and generalized connection terms for Lagrangians belong to the most active research area oriented to a solution of important problems in modern cosmology and particle physics. As we can see in the recent literature, many interesting and original classical and quantum scenarios can be elaborated for naive additions of mass terms, non-trivial geometric backgrounds, and nonlinear interactions via polarized constants and quantum corrections. Such constructions are grounded on geometric models and solutions for certain (generalized) effective Einstein equations with high degrees of symmetries (for Killing vectors) and diagonalizable metrics.

In our research, we focus on exact and approximate generic off-diagonal solutions in gravity theories with generalized symmetries and dependencies via generating and integration functions on as many as possible spacetime coordinates (for instance, on three and four ones on 4-d manifolds). It is a difficult mathematical task to construct such solutions in analytic form and to provide and study certain physical important examples and interesting effects related to outstanding issues, for instance, in cosmology and astrophysics. Nevertheless, all candidates for gravity theories are characterized by complex off-diagonal systems of nonlinear partial equations, and the fundamental classical and quantum properties of gravitational and matter fields interactions should be studied with regard to the most general classes of solutions and nonlinear nonholonomically constrained configurations found. Here we note that although the equations of modified gravity theories are rather involved, they became very simple in certain adapted reference systems and certain types of nonholonomic constraints on certain classes for generic off-diagonal solutions. The surprising thing is that, in many cases, under well-defined geometric conditions, we can model certain classes of nonlinear solutions both in an effective Einstein-like theory (with off-diagonal metrics and gen-

[7] We can consider wormholes in the limit $\varepsilon \to 0$ and for corresponding approximations $\widetilde{\eta}_a \simeq 1$ and ${}^\eta\widetilde{A}$ and ${}^\eta n$ to be almost constant.

eralized, or the LC connection) and in modified bi-metric/-connection gravity models, in general, with non-trivial mass terms. Hence, a generic off-diagonal solution with arbitrary generating and integration functions and constants in GR can be regarded as a possible analog of various types of similar solutions in modified gravity theories. In many cases, we can argue for a quite conservative opinion: maybe it is not necessary to modify the 'canonical' Einstein gravity if in the framework of this theory we are able to explain many fundamental issues and observable cosmological effects via certain generalized off-diagonal solutions with generic off-diagonal interactions and nonholonomic constraints?

In order to investigate certain physical implications of off-diagonal solutions and the possibility to mimic physical effects in one theory by effective analogs of such solutions in another class of theories, a general geometric/analytic method of constructing exact solutions should be applied. For such purposes, we developed the anholonomic frame deformation method, AFDM; see [3,4,9–11] and references therein. Following such a geometric method, various classes of gravitational and matter field equations in modified gravity (MG) and Einstein gravity theories can be decoupled and integrated in very general forms if the necessary types of adapted frame and connection structures are considered. We can impose constraints, at the end, for extracting Levi Civita configurations. In this way, a wide variety of generalized locally anisotropic wormhole and matched exterior black holes can be constructed. They can be derived for certain exotic matter and/or for off-diagonal configurations of metrics describing nonlinear gravitational and matter field interactions which may limit certain de Sitter spacetimes with effective 'anisotropically' polarized cosmological and matter fields constants. The assumption on the stationary properties of such locally anisotropic spacetimes is introduced from the very beginning; even solitonic waves may be involved. Note that the method allows us to find a wide variety of non-stationary exact solutions.

In this paper, we focused on generic off-diagonal solutions which are constructed as nonholonomic deformations of pseudo-Riemannian metrics with two Killing vectors (in particular, they can be solutions of the Einstein equations) into certain classes of generalized exact solutions in massive gravity with possible small parametric deformations related to f-modified gravity. We proved that there exists a formal integration procedure via effective polarization functions, which allows us to construct various classes of exact solutions depending generically on three and four coordinates on (generalized) four dimensional spacetimes. In explicit form, we constructed and studied off-diagonal deformations of wormhole solutions matching exterior (in general, nonholonomically deformed) de Sitter spacetimes with contributions by non-trivial massive gravitational terms and ellipsoidal f-modifications of de Sitter metrics. We also analyzed soliton waves, possible 'ringed wormhole'-like configurations, modified wormholes 'distorted' in nonholonomically induced torsion, etc.

There is still much to be learned about the possibilities of the AFDM and possible relations of off-diagonal solutions constructed in such a way with massive, f-modified, Finsler-like gravity theories, etc. Here, it should be noted that such nonholonomic structures were originally considered in Finsler-like theories, fractional generalizations etc. as applied to modern cosmological scenarios [28–31]. This paper and the discussion provide just a glimpse to potential applications and future work.

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