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Bound states of massive fermions in Aharonov–Bohm-like fields

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Abstract Bound states of massive fermions in Aharonov–Bohm (AB)-like fields have analytically been studied. The Hamiltonians with the (AB)-like potentials are essentially singular and therefore require specification of a one-parameter self-adjoint extension. We construct self-adjoint Dirac Hamiltonians with the AB potential in 2+1 dimensions that are specified by boundary conditions at the origin. It is of interest that for some range of the extension parameter the AB potential can bind relativistic charged massive fermions. The bound-state energy is determined by the AB magnetic flux and depends upon the fermion spin and extension parameter; it is a periodical function of the magnetic flux. We also construct self-adjoint Hamiltonians for the so-called Aharonov–Casher (AC) problem, show that nonrelativistic neutral massive fermions can be bound by the (AC) background, determine the range of the extension parameter in which fermion bound states exist, and find their energies as well as wave functions.

1 Introduction

The quantum Aharonov–Bohm (AB) effect [1] is an important phenomenon analyzed in various physical situations in numerous works (see, e.g., Ref. [2]). Considering an electron traveling in a region with the magnetic flux restricted to a thin solenoid, the electron wave function may develop a quantum (geometric) phase, which describes the real behavior of the electrons propagation. Thus, the AB vector potential can produce observable effects because the relative (gauge-invariant) phase of the electron wave function, correlated with a nonvanishing gauge vector potential in the domain where the magnetic field vanishes, depends on the magnetic flux [3].

It has been observed that the AB problem is governed by Hamiltonians that are essentially singular and therefore require specification of a one-parameter self-adjoint extension in order for them to be treated as self-adjoint quantum-mechanical operators [4–7]. Self-adjoint Hamiltonians are specified by boundary conditions at the singular point.

One-parameter self-adjoint extensions of the Dirac Hamiltonian for the AB problem in 2+1 dimensions were constructed in [5,6,8]. In [5] a formal solution was constructed, which describes a bound fermion state in the field of a cosmic string. Recently great interest in different effects in the two-dimensional systems has been shown after successful fabrication of graphene [9–11]. We note that while descriptions of electron states in the graphene in [12–14] were based on the Dirac equation for massless fermions, [15] has shown that the massive case can also be created.

It seems that the physical reason for the additional specification of the above Dirac Hamiltonians is also related to the interaction between the fermion spin magnetic moment and the source field [16]. Since the interaction potential is repulsive or attractive for different signs of the spin projection, this feature must be taken into account in the behavior of the wave functions at the origin. The existence of weakly bound electron states, which can emerge due to the interaction between the electron spin magnetic moment and the AB magnetic field in 3+1 dimensions, was shown in [17].

Fermion bound states can emerge in the Aharonov–Casher (AC) problem [18] of the motion of a neutral fermion with an anomalous magnetic moment (AMM) in the electric field of an electrically charged conducting long straight thin thread oriented perpendicularly to the plane of fermion motion resulting from the interaction between the AMM of the moving fermion and the electric field [19]. The authors of Ref. [19] argue that such a kind of point interaction also appears in several AB-like problems [20–24].

In this paper, we analyze the AB problem taking into account the fermion spin term in the Dirac Hamiltonian. We find all self-adjoint Dirac Hamiltonians as well as their spectra in the AB potential in 2+1 dimensions using the so-called form asymmetry method developed in Refs. [25,26]. In particular, expressions for the wave functions and bound-

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state energies are obtained as functions of the magnetic flux, spin, and extension parameters. By constructing self-adjoint Hamiltonians for the AC problem we show that fermion bound states exist, and we find their energies as well as wave functions. We note that the AB and AC scattering problems were studied in [16,27] using corresponding self-adjoint Hamiltonians.

We shall adopt units where $c = \hbar = 1$.

2 Self-adjoint radial Dirac Hamiltonians in an Aharonov–Bohm potential in 2+1 dimensions

In two spatial dimensions, the Dirac γ^μ-matrix algebra can be represented in terms of the two-dimensional Pauli matrices σ_j, and the parameter $s = \pm 1$ can be introduced to label two types of fermions [28]; and this is applied to characterizing the two states of the fermion spin (spin 'up and 'down') [29,30]. Then, the Dirac Hamiltonian for a fermion of the mass m and charge $e = -e_0 < 0$ in an AB potential, $A_0 = 0$, $A_r = 0$, $A_\varphi = B/r$, $r = \sqrt{x^2 + y^2}$, $\varphi = \arctan(y/x)$, is

$$H_D = \sigma_1 P_2 - s\sigma_2 P_1 + \sigma_3 m, \tag{1}$$

where $P_\mu = -i\partial_\mu - eA_\mu$ is the generalized fermion momentum operator (a three-vector). The Hamiltonian (1) should be defined as a self-adjoint operator in the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^2)$ of square-integrable two-spinors $\Psi(\mathbf{r})$, $\mathbf{r} = (x, y)$ with the scalar product

$$(\Psi_1, \Psi_2) = \int \Psi_1^\dagger(\mathbf{r})\Psi_2(\mathbf{r})d\mathbf{r}, \quad \mathrm{d}\mathbf{r} = \mathrm{d}x\mathrm{d}y. \tag{2}$$

The total angular momentum $J \equiv L_z + s\sigma_3/2$, where $L_z \equiv -i\partial/\partial\varphi$, commutes with H_D; therefore, we can consider separately each eigenspace of the operator J and the total Hilbert space is the direct orthogonal sum of the subspaces of J.

In the real (three-dimensional) space, the quantity B characterizes the flux of the magnetic field $\mathbf{H} = (0, 0, H) = \nabla \times \mathbf{A} = B\delta(x)\delta(y)$ through the surface of an infinitely thin solenoid (the radius $R \to 0$). Thus, there appears an interaction potential of the electron spin magnetic moment with the magnetic field in the form $-seB\delta(r)/r$, which is singular and must influence the behavior of the solutions at the origin. The 'spin' potential is invariant under the changes $e \to -e$, $s \to -s$, and it hence suffices to consider only the case $e = -e_0 < 0$ and $e_0 B \equiv \mu > 0$; μ is the magnetic flux Φ in units of the elementary magnetic flux $\Phi_0 \equiv 2\pi/e_0$. Then the potential is attractive for $s = -1$ and repulsive for $s = 1$. For cosmic strings $\Phi = e/Q$, where Q is the Higgs charge [5–7].

The eigenfunctions of the Hamiltonian (1) are [31]

$$\Psi(t, \mathbf{r}) = \frac{1}{\sqrt{2\pi r}} \begin{pmatrix} f_1(r) \\ f_2(r)\mathrm{e}^{is\varphi} \end{pmatrix} \exp(-iEt + il\varphi), \tag{3}$$

where E is the fermion energy, and l is an integer. The wave function Ψ is an eigenfunction of the operator J with eigenvalue $j = l + s/2$ and

$$\check{h}F = EF, \quad F = \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix}, \tag{4}$$

where

$$\check{h} = is\sigma_2 \frac{\mathrm{d}}{\mathrm{d}r} + \sigma_1 \frac{l + \mu + s/2}{r} + \sigma_3 m, \quad \mu \equiv e_0 B. \tag{5}$$

Thus, the problem is reduced to that for the radial Hamiltonian $\check{h}$ in the Hilbert space of doublets $F(r)$ square-integrable on the half-line.

As was shown in [30,31] any doublets $F(r)$, $G(r)$ of the Hilbert space $\mathfrak{H} = \mathfrak{L}^2(0, \infty)$ must satisfy

$$\lim_{r\to 0} G^\dagger(r) i\sigma_2 F(r) = 0. \tag{6}$$

Then, for $\nu = |l + \mu + s/2| \neq n/2, n = 1, 2, \ldots$ the linear independent solutions of (4) needed are [30]

$$U_1(r; E) = A(kr)^{1/2} \left(\frac{2m}{k}\right)^\nu \Gamma(1/2 + \nu)\mathrm{e}^{-i\frac{\pi}{4}(1-s)} \times \begin{pmatrix} \sqrt{E + m} J_{\nu - s/2}(kr) \\ \sqrt{E - m} J_{\nu + s/2}(kr) \end{pmatrix} \tag{7}$$

and

$$U_2(r; E) = B(kr)^{1/2} \left(\frac{2m}{k}\right)^{-\nu} \Gamma(1/2 - \nu)\mathrm{e}^{i\frac{\pi}{4}(1+s)} \times \begin{pmatrix} \sqrt{E + m} J_{-\nu + s/2}(kr) \\ -\sqrt{E - m} J_{-\nu - s/2}(kr) \end{pmatrix}, \tag{8}$$

with the asymptotic behavior at $r \to 0$:

$$U_1(r; E) = (mr)^\nu \begin{pmatrix} 1 + s \\ 1 - s \end{pmatrix} + O(r^{\nu+1}), \quad r \to 0,$$

$$U_2(r; E) = (mr)^{-\nu} \begin{pmatrix} 1 - s \\ 1 + s \end{pmatrix} + O(r^{-\nu+1}), \quad r \to 0,$$

where A, B are complex constants, $k = \sqrt{E^2 - m^2}$, and $J_\mu(z)$ are the Bessel functions. Also we have

$$V_1(r; E) = U_1(r; E) + \frac{1}{4s\lambda}\omega(E)U_2(r; E), \tag{9}$$

where $\omega(E) = \mathrm{Wr}(U_1, V_1)$ is the Wronskian:

$$\omega(E) = \mathrm{Wr}(U_1, V_1) = \frac{\Gamma(2\nu)\Gamma[-\nu + (1-s)/2]}{\Gamma(-2\nu)\Gamma[\nu + (1-s)/2]} \frac{(2\lambda)^{-2\nu}}{m^{-2\nu}} 4s\lambda \equiv \frac{\tilde{w}(E)}{\Gamma(-2\nu)}, \tag{10}$$

where $\lambda = \sqrt{m^2 - E^2}$. The doublet V_1 also can be represented via the MacDonald functions:

$$V_1(r; E) = C(mr)^{1/2}\left(\frac{m}{\lambda}\right)^{\nu-1/2}\frac{2}{\Gamma(1/2-\nu)} \times \begin{pmatrix} K_{\nu-s/2}(\lambda r) \\ sK_{\nu+s/2}(\lambda r) \end{pmatrix}, \tag{11}$$

where C is a complex constant. We note that

$$\nu(\pm l, s = 1, \mu) = \nu(\pm l + 1, s = -1, \mu). \tag{12}$$

Any doublet of the domain $D(h)$ must satisfy

$$(F^{\dagger}(r)i\sigma_2 F(r))|_{r=0} = (\bar{f}_1 f_2 - \bar{f}_2 f_1)|_{r=0} = 0. \tag{13}$$

$D(h)$ is the space of absolutely continuous doublets $F(r)$ regular at $r = 0$ with $hF(r)$ belonging to $\mathfrak{L}^2(0, \infty)$.

If $\nu > 1/2$ there exist only solutions belonging to the continuous spectrum (7). If $0 < \nu < 1/2$ Eq. (13) is not satisfied and its left-hand side

$$(\bar{f}_1 f_2 - \bar{f}_2 f_1)|_{r=0} = 4s\lambda(\bar{c}_1 c_2 - \bar{c}_2 c_1). \tag{14}$$

Therefore the adjoint operator h^* is not symmetric and we need to construct the nontrivial self-adjoint extensions of the initial symmetric operator h^0. By means of the linear transformation

$$c_{1,2} \to c_{\pm} = c_1 \pm ic_2 \tag{15}$$

Equation (14) is reduced to the quadratic diagonal form

$$(\bar{f}_1 f_2 - \bar{f}_2 f_1)|_{r=0} = -i4s\lambda(|c_+|^2 - |c_-|^2) \tag{16}$$

with the inertia indices $(1, 1)$, which means that the deficiency indices of the symmetric operator h^0 for $0 < \nu < 1/2$ are $(1, 1)$. Equation (13) will be satisfied for any c_- related to c_+ by

$$c_- = e^{i\theta}c_+, \quad 0 \le \theta \le 2\pi, \quad 0 \sim 2\pi. \tag{17}$$

The angle θ parameterizes the self-adjoint extensions h_θ of the symmetric operator h^0. These self-adjoint extensions are different for various θ except for two equivalent cases $\theta = 0$ and $\theta = 2\pi$. If we denote $\xi = \tan(\theta/2)$, then the relation (17) is equivalent to

$$c_2 = -\xi c_1, \quad -\infty \le \xi = \tan\frac{\theta}{2} \le +\infty, \quad -\infty \sim +\infty. \tag{18}$$

The values of $\xi = \pm\infty$ are equivalent; they imply $c_1 = 0$ so we can consider only $\xi = \infty$. Hence, in the range $0 < \nu < 1/2$ there is a one-parameter $U(1)$-family of the operators $h_\theta \equiv h_\xi$ with the domain D_ξ,

$$h_\xi : \begin{cases} D_\xi = \begin{cases} F(r):\ F(r) \text{ are absolutely continuous in } (0,\infty),\ F, hF \in \mathfrak{L}^2(0,\infty), \\ F(r) = C\left[(mr)^{\nu}\begin{pmatrix}1+s\\1-s\end{pmatrix} - \xi(mr)^{-\nu}\begin{pmatrix}1-s\\1+s\end{pmatrix}\right],\ r \to 0,\ -\infty < \xi < +\infty, \\ F(r) = C(mr)^{-\nu}\begin{pmatrix}1-s\\1+s\end{pmatrix} + O(r^{1/2}),\ r \to 0,\ \xi = \infty \end{cases} \\ h_\xi F = \check{h}F, \end{cases} \tag{19}$$

where C is a complex constant. Then

$$U_\xi(r; E) = U_1(r; E) - \xi U_2(r; E) \tag{20}$$

and

$$V_1(r; E) \equiv V_\xi = U_\xi(r; E) + \frac{1}{4s\lambda}\omega_\xi(E)U_2(r; E) \tag{21}$$

with

$$\omega_\xi(E) = \mathrm{Wr}(U_\xi, V_\xi) = \omega(E) + 4s\lambda\xi, \tag{22}$$

where $\omega(E)$ is determined by (10). For $-\infty < \xi < \infty$, the energy eigenstates (doublets) in the range $|E| \ge m$ are

$$F(r) = U_1(r; E) - \xi U_2(r; E),$$

where $U_1(r; E)$ and $U_2(r; E)$ are determined by (7) and (8) with $0 < \nu < 1$. The operator h^0 is not determined as a unique self-adjoint operator and so an additional specification of its domain, given with the real parameter ξ (the self-adjoint extension parameter), is required in terms of the self-adjoint boundary conditions. It is well to note that the self-adjoint boundary conditions permit an integrable singularity in the wave functions at the origin. Physically, they show that the probability current density is equal to zero at the origin.

The spectrum of the radial Hamiltonian is determined by [25,31]

$$\frac{\mathrm{d}\sigma(E)}{\mathrm{d}E} = \frac{1}{\pi}\lim_{\epsilon\to 0}\mathrm{Im}\frac{1}{\omega_\xi(E + i\epsilon)}, \tag{23}$$

where the generalized function $\omega_\xi(E + i\epsilon)$ is obtained by analytic continuation of the corresponding Wronskian in the complex plane of E. It coincides with the corresponding function $\omega(E)$ on the real axis of E. It can be verified that in the range $|E| > m$ the functions $\omega(E)$ and $\omega_\xi(E)$ are continuous, complex-valued, and not equal to zero for real

E; the spectral function $\sigma(E)$ exists and is absolutely continuous. Thus, the energy spectrum in the range $|E| \geq m$ is continuous. In the range $|E| < m(-m < E < m)$ the functions $\omega(E)$ and $\omega_\xi(E)$ are real and $\lim_{\epsilon\to 0} \omega_\xi^{-1}(E + i\epsilon)$ can be complex only at the points where $\omega_\xi(E) = 0$ and the energy spectrum of bound states is determined by roots of this equation. The Wronskians as functions of the complex E have two cuts $(-\infty, -m]$ and $[m, \infty)$ in the complex plane of E, so we determine the first (second) sheet with $\mathrm{Re}\lambda > 0$ ($\mathrm{Re}\lambda < 0$) on the real axis of E. The real bound states are situated on the first (physical) sheet.

3 Relativistic bound fermion states in 2+1 dimensions

For negative ξ there exists a bound state. The bound-state energy $E_\xi(\nu, s)$ is implicitly determined by the equation $\omega_\xi(E) = 0$, i.e.

$$\frac{\Gamma(2\nu)\Gamma(-\nu + (1-s)/2)}{\Gamma(-2\nu)\Gamma(\nu + (1-s)/2)} \frac{(\lambda)^{-2\nu}}{m^{-2\nu}} = \xi. \tag{24}$$

Let us write

$$\mu = [\mu] + \beta \equiv n + \beta, \tag{25}$$

where $[\mu] \equiv n$ denotes the largest integer $\leq \mu$, and $1 > \beta \geq 0$. Hence $n = 0, 1, 2, \ldots$ for $\mu > 0$ and $n = -1, -2, -3, \ldots$ for $\mu < 0$. Since the signs of e and B are fixed it is enough to consider only the case $\mu > 0$. One can assume that a bound state exists due to the interaction of the fermion spin magnetic moment with the AB magnetic field.

We define particle bound states as the states that tend to the boundary of the continuous spectrum $E = m$ upon adiabatically slow switching of the external field (see, for instance [32,33]). For $l + n = 0, \mu = \beta > 0$ the only (particle) bound state $s = -1$ satisfies the self-adjoint condition (19). We rewrite (24) for this case as follows:

$$\frac{\Gamma(1-2\beta)\Gamma(1/2+\beta)}{\Gamma(2\beta-1)\Gamma(3/2-\beta)} \left(\frac{m}{\lambda}\right)^{2\beta-1} = \xi, \quad 1/2 > \beta > 0 \tag{26}$$

and

$$\frac{\Gamma(2\beta-1)\Gamma(3/2-\beta)}{\Gamma(1-2\beta)\Gamma(1/2+\beta)} \left(\frac{m}{\lambda}\right)^{1-2\beta} = \xi, \quad 1 > \beta > 1/2. \tag{27}$$

It is easy to see that these equations hold for $l + n = -1, s = 1$. Since $K_{-\gamma}(z) = K_\gamma(z)$ it is seen from Eq. (11) that the bound fermion states with $l + n = 0, s = -1$ or $l + n = -1, s = 1$ are doublets represented via the two MacDonald functions $K_{1-\beta}(\lambda r)$ and $K_\beta(\lambda r)$.

It follows from Eqs. (26) and (27) that an adiabatic increase of the magnetic flux μ between the integers $n \to n + 1$ lifts the energy level $E = m \to E = -m$ [5] on the physical sheet $\mathrm{Re}\lambda > 0$) and $E = -m \to E = m$ on the second (unphysical) sheet $\mathrm{Re}\lambda < 0$). The second sheet is below the first one. The given bound-state energy is decreased (increased), $E = m \to E = -m$ for $\mathrm{Re}\lambda > 0$ ($E = -m \to E = m$ for $\mathrm{Re}\lambda < 0$), upon adiabatic increase of the flux Φ between the integers $n \to n+1$ and is increased (decreased), $E = -m \to E = m$ ($E = m \to E = -m$), upon adiabatic increase of Φ between $n+1 \to n+2$. Therefore, any bound-state energy is a periodic function of the magnetic flux similar to the case of the fermion motion in the AB potential along a closed circle [34]; it is repeated every time we change μ by an integer. It is interesting that the induced current due to vacuum polarization in the AB field is a finite periodical function of the magnetic flux [35].

For $\xi = -1$ any curve $E(\beta)$ is symmetric upon reflection with respect to the point $\beta = 1/2$, $E = 0$. One also can see that there exists at $\beta = 1/2$ a normalizable state with $E = 0$; for ξ it lies in the middle of the gap $2m$. The wave function of this (particle) state is

$$F(r) = D(mr)^{1/2} \begin{pmatrix} 1 \\ s \end{pmatrix} K_{1/2}(mr). \tag{28}$$

We give a few comments.

1. In the range of parameters $0 > \xi > -\infty$ the constructed self-adjoint Hamiltonians h_ξ have real localized solutions (fermionic bound state); physically they exist if the additional potential (in our case, $s\mu\delta(\mathbf{r})$ type) is attractive.
2. We define antiparticle bound states as the states that tend to the boundary of the lower continuum, $E = -m$, upon adiabatically slow switching of the external field. Then, we can treat an antiparticle as a particle with opposite signs for e, s, E, and we see that the Dirac Hamiltonian (5) possesses a conjugation symmetry.
Jackiw and Rebbi [36] observed that, in a time-inversion, charge-conjugation symmetric theory of one-dimensional Dirac fermions interacting with a solitonic background field (a kink), the effective Hamiltonian possesses conjugation symmetry. Because of this symmetry an isolated nondegenerate, charge-self-conjugate, zero-energy state (zero mode) lying in the middle of the gap $2m$ exists [36–38] and the vacuum of the model must acquire a half-integer fermionic charge [36]. In the presence of a vector potential, the Dirac Hamiltonian does not exhibit a charge-conjugation symmetry since a charge coupling treats particles and antiparticles differently. So the existence of fermion states with zero energy does not necessarily imply a fractional fermion number [39]. The presence of a magnetic field breaks time-inversion invariance. In the case considered, the wave function (a doublet) of the antiparticle F^a is related to that of the particle F by means of the charge-conjugation operator given by

the Pauli matrix $C = \sigma_3$, i.e. if F is a solution of the Dirac equation (4) with $(l+\mu), s$ and energy E, then $F^a = \sigma_3 F^*$ is also a solution of the same equation, but with $-(l+\mu), -s, -E$. For $\xi = -1$ the antiparticle energy as a function of β is equal to zero at $\beta = 1/2$, and the wave function of the antiparticle state with $E^a = 0$ is $F^a(r) = \sigma_3 F^*(r)$, where $F(r)$ is determined by (28). Therefore, the AB vector potential can yield bound states and localized spin-polarized charged zero modes [39,40]. Since $F^a(r)$ does not coincide with $F(r)$ the fermionic charge stays integer.

3. The behavior of the lowest particle energy level near the upper boundary $E = -m$ of the lower continuum in the relativistic AB problem differs from the one in the cut-off Coulomb problem. In the (cutoff) Coulomb problem, the lowest electron energy level can dive into the lower continuum $[-m, -\infty)$, then turn into a resonance that can be described as a quasistationary state with 'complex energy' (directly associated with the creation of an electron–positron pair) [41] (see also [42]); when the bound-state pole disappears from the physical sheet the quasistationary state pole resides on the second (unphysical) sheet.

We see that there are no particle bound states diving into the lower continuum, no quasistationary states with 'complex energy' in the relativistic AB problem (there is not particle creation); also only fermionic bound states with real E can appear on the second sheet.

4 Bound fermion states in the Aharonov–Casher problem

The Dirac–Pauli equation for a neutral fermion with mass m, an AMM M in the form of the Schrödinger equation for the case of fermion motion in an electric field reads

$$i\frac{\partial\Psi}{\partial t} = H_{DP}\Psi \tag{29}$$

with the Hamiltonian

$$H_{DP} = \boldsymbol{\alpha}\cdot\mathbf{P} + iM\boldsymbol{\gamma}\mathbf{E} + \beta m. \tag{30}$$

Here $\mathbf{P} = -i\nabla$ is the canonical momentum operator, Ψ is a bispinor, $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$, $\boldsymbol{\alpha}$ are the Dirac matrices, and $\mathbf{E}$ is the electric field strength.

Introducing the function

$$\Psi = \Psi_n e^{-imt} \tag{31}$$

and representing Ψ_n in the form

$$\Psi_n = \begin{pmatrix}\phi\\ \chi\end{pmatrix}, \tag{32}$$

where ϕ and χ are spinors, we obtain an equation for the neutral fermion in the electric field of an electrically charged homogeneous long straight thin thread directed along the z axis in the nonrelativistic approximation in the form

$$i\frac{\partial\phi}{\partial t} = \frac{(\mathbf{P}-\mathbf{E}\times\mathbf{M})^2 - M^2\mathbf{E}^2 + M\nabla\cdot\mathbf{E}}{2m}\phi, \tag{33}$$

where $\mathbf{M} = M\boldsymbol{\sigma}$, $\boldsymbol{\sigma}$ are the Pauli matrices and the term $\nabla\cdot\mathbf{E}$ is equal to 4π times the electric field charge density.

In the AC field configuration

$$E_x = \frac{ax}{r^2},\quad E_y = \frac{ay}{r^2},\quad E_z = 0,\quad E_r = \frac{a}{r},\quad E_\varphi = 0, \tag{34}$$

is the electric field for an electrically charged homogeneous long straight thin (a zero radius) thread and $a/2$ is the total surface charge density. We also assume that the projection of the fermion momentum on the z axis is equal to zero. The radial component of the (macroscopic) electric field is determined by the mean surface charge density as $\nabla\cdot\mathbf{E} = 4\pi\rho$, and the expression $\rho = a\delta(r)/4\pi r$, therefore, well approximates ρ. We seek the solutions of (33) in the polar coordinates in the form

$$\phi(t,r,\varphi) = \exp(-iE_n t)\sum_{l=-\infty}^{\infty} F_l(r)\exp(il\varphi)\psi, \tag{35}$$

where E_n is the particle energy, l is an integer, and ψ is a constant spinor. The Hamiltonian of a neutral fermion in the AC background contains only the matrix σ_3, and the wave function ϕ therefore depends only on the number ζ characterizing the conserved spin projection on the z axis, and its eigenvalue $\zeta = \pm 1$ can be substituted for the operator σ_3 in (33). After this substitution, the spin part of the wave function ψ becomes inessential, and we need to consider only the scalar coordinate function ϕ depending on ζ (see, e.g., [43]). Thus, the radial Dirac–Pauli equation for the neutral fermion with AMM in the electric field of a thread oriented perpendicular to the plane of fermion motion in 3+1 dimensions in the nonrelativistic approximation coincides with the nonrelativistic equation in the AB problem and reads [18,16]

$$h^n F_l(r) = E_n F_l(r),\quad h^n = -\frac{1}{2m}\times\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(l+Ma\zeta)^2}{r^2} - Ma\frac{\delta(r)}{r}\right). \tag{36}$$

Here E_n is related to E by $E = m + E_n$, $|E_n| \ll m$. We also note that the analogous singular term ($\sim \delta(r)/r$) also appears in the quadratic Dirac equation in the AB problem; there it includes the spin parameter in the form of an additional delta-function interaction of the spin with the magnetic field. The additional term must influence the behavior of the solutions at the origin and it can be taken into account by means of boundary conditions at the point $r = 0$. In the nonrelativistic AC problem the boundary condition (13) can be given by [16] (see also [44,45])

$$(\bar{f}' f - \bar{f} f')|_{r=0} = 0, \tag{37}$$

where $f(r) \equiv F_l(r)/\sqrt{r}$ and $\bar{f}$ is the complex conjugate function f. Here we restrict ourself to considering the case $\gamma = |l + \zeta Ma| < 1$ when bound states can exist. Then, for each l in the range $0 < \gamma < 1$, there is a one-parameter $U(1)$-family of self-adjoint Hamiltonians h_ξ^n parameterized by (18) with the domain D_ξ^n

$$h_\xi^n : \begin{cases} D_\xi^n = \begin{cases} f(r),\ f'(r) \text{ are absolutely continuous in } (0,\infty);\ f, h_\xi^n f \in \mathfrak{L}^2(0,\infty), \\ f(r) = A[(mr)^\gamma - \xi (mr)^{-\gamma}] + O(r),\ r \to 0, \quad -\infty < \xi < +\infty, \\ f(r) = A(mr)^{-\gamma},\ r \to 0, \quad \xi = \infty \end{cases} \\ h_\xi^n f = \check{h}^n f, \end{cases} \tag{38}$$

where A is a complex constant. It is obvious that the functions $f(r)$ are Bessel functions of the order $\pm\gamma$. Then, calculating the corresponding Wronskian we obtain

$$\omega(E_n) = \frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)} \left(\frac{2m}{\lambda}\right)^{2\gamma}, \tag{39}$$

where $\lambda = \sqrt{-2mE_n}$. By analytic continuation of (39) in the complex plane of E_n we obtain the function $\omega_\xi(E_n + i\epsilon)$. Now the Wronskian as a function of the complex E_n has a cut $(0, \infty)$ in the complex plane of E_n and the first (second) sheet is determined, $\mathrm{Re}\sqrt{-2mE_n} > 0$ ($\mathrm{Re}\sqrt{-2mE_n} < 0$). Real bound states are situated on the first (physical) sheet.

It can be verified that in the range $E_n > 0$ the functions $\omega(E_n)$ and $\omega_\xi(E_n)$ are continuous, complex-valued, and not equal to zero for real E_n; the function $\sigma(E_n)$ exists and is absolutely continuous. Thus, the energy spectrum in this range is continuous. One can show there also exists a bound state (with $E_n < 0$) in the range of parameters $-\infty < \xi < 0$) for $0 < \gamma < 1$ and its energy is determined by

$$\frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)} \left(\sqrt{\frac{-E_n}{2m}}\right)^{-2\gamma} = -\xi. \tag{40}$$

The bound-state energy is the same on the first and second sheets; it is given by (compare with formula (90) in [19])

$$E_n = -2m \left(-\xi \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)}\right)^{-1/\gamma}. \tag{41}$$

The wave function of the bound state is $N\sqrt{mr}K_\gamma(\sqrt{-2mE_n}r)$ where N is a normalization factor. Since signs of M and a are fixed it is enough to consider only the (attractive) case $Ma < 0$ and because of the bound states existing for $\gamma < 1$ we must have $Ma < -1$. It is seen that there are bound states with $\zeta = \pm 1$ for $l = 0$ and with $\zeta = 1(-1)$ for $l = 1(-1)$. We denote $-Ma \equiv c > 0$; we rewrite (41) for these cases as follows:

$$E_n^0 = -2m \left(-\xi \frac{\Gamma(1-c)}{\Gamma(1+c)}\right)^{-1/c}, \quad l = 0, 0 < c < 1, \tag{42}$$

$$E_n^{\pm 1} = -2m \left(-\xi \frac{\Gamma(c)}{\Gamma(2-c)}\right)^{1/(c-1)}, \quad l = \pm 1, 0 < c < 1. \tag{43}$$

It is evident that $E_n^0(c) = E_n^{\pm 1}(c = 1 - b), 1 > b > 0$. This means that an adiabatic increase of c in the interval (0, 1) lifts the levels $E_n^0(c)$ on the first (physical) sheet and $E_n^{\pm 1}(c)$ on the second (unphysical) sheet in the opposite direction. The second sheet is below the first one.

The special case $\gamma = 0$ can be of some interest (the analogous case was considered in [17,44] for the nonrelativistic AB problem in 2+1 dimensions). One can show that for $|\xi| = \infty$ the energy spectrum is continuous and nonnegative and also that for $-\infty < \xi < 0$ there exists (in addition to the continuous part of the spectrum) one negative level

$$E_0 = -4me^{2(\xi - \mathcal{C})}, \tag{44}$$

where $\mathcal{C} = 0.57721$ is the Euler constant [46]. The wave function of the bound state for $\gamma = 0$ is $N\sqrt{mr}K_0(\sqrt{-2mE_0}r)$.

5 Summary

By constructing a one-parameter self-adjoint extension of the Dirac Hamiltonian with the AB potential in 2+1 dimensions, we have studied the bound states of fermions in this background. It has been shown that for negative values of the extension parameter ξ, the spectrum of self-adjoint Dirac Hamiltonians, in addition to its continuous part, has one bound level. Therefore, the AB vector potential can bind relativistic charged massive fermions in 2+1 dimensions. The bound-state energy depends upon the extension parameter and is a periodical function of the AB magnetic flux. It is of interest that the AB vector potential can yield localized spin-polarized charged zero modes.

We also have studied the AC problem in the context of the nonrelativistic limit of the Dirac–Pauli equation in 3+1 dimensions. We show that the AC background can bind nonrelativistic neutral massive fermions, we determine the range of extension parameter in which fermion bound states exist, and we find their energies as well as wave functions.

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