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T-duality in a weakly curved background

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Abstract We consider the closed string propagating in a weakly curved background which consists of a constant metric and a Kalb–Ramond field with an infinitesimally small coordinate-dependent part. We propose a procedure for constructing the T-dual theory, performing T-duality transformations along the coordinates on which the Kalb–Ramond field depends. The theory obtained is defined in the non-geometric double space, described by the Lagrange multiplier y_μ and its T-dual in the flat space $\tilde{y}_\mu$. We apply the proposed T-duality procedure to the T-dual theory and obtain the initial one. We discuss the standard relations between the T-dual theories that imply that the equations of motion and momentum modes of one theory are the Bianchi identities and the winding modes of the other.

1 Introduction

In string theory, duality symmetry was for the first time described in the context of toroidal compactification in [1–3] (thoroughly explained in [4,5]). If only one dimension is compactified on a circle with radius R, then under the transformation

$$R \to \frac{\alpha'}{R}, \quad \Phi \to \Phi - \log\left(\frac{R}{\sqrt{\alpha'}}\right), \tag{1}$$

where α' is the Regge slope parameter, the physical features of the interacting theory remain the same. This kind of symmetry can be generalized to the arbitrary toroidal compactification [6], and extended to the non-flat conformal backgrounds [7,8]. In the case of the open string, there exists a relation between the T-dual background fields and the coordinate non-commutativity parameters [9], as well as the relation between fermionic T-dual fields and the momenta non-commutativity parameters [10].

In Buscher's construction of the T-dual theory [7,8,11], one starts with the manifold containing the metric $G_{\mu\nu}$, the antisymmetric field $B_{\mu\nu}$, and the dilaton field Φ. It is required that the metric admits at least one continuous abelian isometry which leaves the action for the σ-model invariant. One can choose the target space coordinates $x^\mu = (x^i, x^a)$, such that the isometry acts by translation of the periodic coordinates x^a. The T-duality along these directions changes x^a-independent background fields G, B, Φ into the corresponding T-dual fields $\tilde{G}$, $\tilde{B}$, $\tilde{\Phi}$. In this way, one connects different geometries and two seemingly different σ-models. This method was originally obtained in a non-covariant way (because of the choice of coordinates), but it was soon slightly modified, which led to a covariant construction [12].

In the covariant construction, the isometry is gauged by introducing the gauge fields v_α^μ. To preserve the physical meaning of the original theory, one requires that the new fields v_α^μ do not carry the additional degrees of freedom. This means that these fields are pure gauge with vanishing field strength,

$$F_{\alpha\beta}^\mu = \partial_\alpha v_\beta^\mu - \partial_\beta v_\alpha^\mu. \tag{2}$$

This requirement is included in the theory by adding the term $y_\mu F_{01}^\mu$ into the Lagrangian, with y_μ being the Lagrange multiplier. This guarantees that at the classical level the dual theory will be equivalent to the original one. Integrating over the Lagrange multipliers y_μ, one simply recovers the original theory. The integration over the gauge fields v_α^μ, produces the T-dual theory. The non-abelian extension to T-duality has been considered in [13–16].

In the construction above, the background fields were constant along the x^a directions. In the present article, we consider a weakly curved background. We allow the background fields to depend on the coordinates along which we perform duality transformations. Note that the constant shift of coordinates remains a global symmetry in this background.

To gauge the global isometry, we introduce the gauge fields v_α^μ, as usual. The replacement of the derivatives $\partial_\alpha x^\mu$

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with the covariant ones $D_\alpha x^\mu$, does not make the whole action invariant. The obstacle is the background field $B_{\mu\nu}$, depending on x^μ, which is not locally gauge invariant. Therefore, in addition we should covariantize x^μ as well. At this point, we will depart from the conventional approach. We take the invariant coordinates as the line integral of the covariant derivatives of the original coordinates,

$$\begin{aligned}\Delta x^\mu_{\text{inv}} &= \int_P (\mathrm{d}\xi^+ D_+ x^\mu + \mathrm{d}\xi^- D_- x^\mu) \\ &= x^\mu - x^\mu(\xi_0) + \Delta V^\mu \left[v_+, v_-\right],\end{aligned} \tag{3}$$

where ΔV^μ is a line integral of the gauge fields v^μ_α. As before, to obtain a theory physically equivalent to the original one, all degrees of freedom carried by the gauge fields v^μ_α should be eliminated. Therefore, we add a Lagrange multiplier term $y_\mu F^\mu_{01}$ into the Lagrangian. This allows us to consider x^μ_{inv} and V^μ as primitives up to a constant of $D_\alpha x^\mu$ and v^μ_α, respectively. Using the local gauge freedom, we fix the gauge, taking $x^\mu(\xi) = x^\mu(\xi_0)$.

Thus we succeed in generalizing the gauged action. Substituting the solution of the equations of motion for the Lagrange multiplier into the gauge fixed action one obtains the original action. The T-dual theory is obtained for the equations of motion for the gauge fields v^μ_α. These equations must be resolved iteratively, because

1. the action is not bilinear in v^μ_α as the background fields depend on $V^\mu[v_\alpha]$;
2. V^μ is the line integral of v^μ_α.

The fact that this background is characterized by an infinitesimally small parameter enables one to solve the problem and find the T-dual action. There are two essential differences in the T-dual action in comparison to the flat background case. The first one is that the target space of the T-dual theory in the weakly curved background turns out to be a non-geometrical one [15–28]. This is a doubled space with two coordinates, one of them being the Lagrange multiplier as in the case of the flat background. The second one is the T-dual of the first in the flat space. The second difference is the coordinate dependence of both dual background fields as a consequence of the coordinate-dependent initial Kalb–Ramond field.

The theory defined above has one additional difficulty: the invariant coordinates $\Delta x^\mu_{\text{inv}}$ and ΔV^μ are defined as line integrals along arbitrary path P. We will show that the equation of motion for the Lagrangian multiplier y_μ, forces the field strength F^μ_{01} to vanish, which guarantees that $\Delta x^\mu_{\text{inv}}$ and ΔV^μ are independent on the choice of the path P.

Because T-duality leads to an equivalent theory, we expected that the T-dual of the T-dual theory is the initial one. The T-dual theory is defined in doubled space, but is still globally invariant under the shift of the T-dual coordinate y_μ. Gauging this symmetry, we show that the T-dual of the T-dual is indeed the original theory.

2 Bosonic string in the weakly curved background

Let us consider the action [29,30]

$$S = \kappa \int_\Sigma \mathrm{d}^2\xi \left[\frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu}[x] + \varepsilon^{\alpha\beta} B_{\mu\nu}[x]\right] \partial_\alpha x^\mu \partial_\beta x^\nu, \tag{4}$$

describing the propagation of the bosonic string in the nontrivial background, defined by the space-time metric $G_{\mu\nu}$ and the Kalb–Ramond field $B_{\mu\nu}$. The integration goes over a two-dimensional world-sheet Σ parametrized by ξ^α ($\xi^0 = \tau,\ \xi^1 = \sigma$), where the intrinsic world-sheet metric $g_{\alpha\beta}$ is taken in the conformal gauge, $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$. Here, $x^\mu(\xi),\ \mu = 0, 1, \ldots, D-1$ are the coordinates of the D-dimensional space-time, $\kappa = \frac{1}{2\pi\alpha'}$ and $\varepsilon^{01} = -1$.

Introducing the light-cone coordinates and their derivatives

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma), \quad \partial_\pm = \partial_\tau \pm \partial_\sigma, \tag{5}$$

and defining

$$\Pi_{\pm\mu\nu}[x] = B_{\mu\nu}[x] \pm \frac{1}{2} G_{\mu\nu}[x], \tag{6}$$

the action (4) can be written in the form

$$S[x] = \kappa \int_\Sigma \mathrm{d}^2\xi\ \partial_+ x^\mu \Pi_{+\mu\nu}[x]\, \partial_- x^\nu. \tag{7}$$

The consistency of the theory requires the conformal invariance of the world-sheet on the quantum level. This requirement results in the space-time equations of motion for the background fields. To the lowest order in the slope parameter α', these equations have the form

$$\begin{aligned}&R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu^{\ \rho\sigma} + 2D_\mu \partial_\nu \Phi = 0, \\ &D_\rho B^\rho_{\ \mu\nu} - 2\partial_\rho \Phi B^\rho_{\ \mu\nu} = 0, \\ &4(\partial\Phi)^2 - 4D_\mu \partial^\mu \Phi + \frac{1}{12} B_{\mu\nu\rho} B^{\mu\nu\rho} - R = 0,\end{aligned} \tag{8}$$

where $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are the Ricci tensor and the covariant derivative with respect to the space-time metric. Φ is the dilaton field and D is the dimension of the space-time. We consider one of the simplest coordinate-dependent solutions of (8). This is the weakly curved background, defined by the following expressions:

$$G_{\mu\nu} = \text{const},$$
$$B_{\mu\nu}[x] = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho \equiv b_{\mu\nu} + h_{\mu\nu}[x], \tag{9}$$
$$\Phi = \text{const}.$$

This background is the solution of the space-time equations of motion, if the constant $B_{\mu\nu\rho}$ is taken to be infinitesimally small and all the calculations are done in the first order in $B_{\mu\nu\rho}$.

A weakly curved background was considered in [31–33], where the influence of the boundary conditions on the non-commutativity of the open bosonic string has been investigated. The same approximation (not referred to as a weakly curved background) was considered in [34], where non-commutativity of the closed string was investigated. In the present paper, we will investigate the closed bosonic string moving in the weakly curved background, with the goal to find the generalization of the Buscher construction of the T-dual theory.

3 Generalized Buscher construction

In the standard Buscher construction of the T-dual theory, the premise is that the target space has isometries. It is possible to choose adapted coordinates $x^\mu = (x^i, x^a)$, so that the isometries act as translations of the x^a components; the background fields are taken to be x^a-independent, and the action is invariant under the global shift symmetry. The weakly curved background preserves this symmetry, despite the x^a-dependence of the background fields. As this is not obvious, let us first demonstrate that the constant shift

$$\delta x^\mu = \lambda^\mu = \text{const}, \tag{10}$$

leaves the action (7) for the closed string invariant. For simplicity, we assume that all the coordinates are compact.

As $B_{\mu\nu}$ is linear in the coordinates, one has

$$\begin{aligned}\delta S &= \frac{\kappa}{3}B_{\mu\nu\rho}\lambda^\rho \int d^2\xi\, \partial_+ x^\mu \partial_- x^\nu \\ &= \frac{\kappa}{3}B_{\mu\nu\rho}\lambda^\rho \varepsilon^{\alpha\beta} \int d^2\xi\, \partial_\alpha x^\mu \partial_\beta x^\nu.\end{aligned} \tag{11}$$

This is proportional to the total divergence

$$\delta S = \frac{\kappa}{3}B_{\mu\nu\rho}\lambda^\rho \varepsilon^{\alpha\beta} \int d^2\xi\, \partial_\alpha (x^\mu \partial_\beta x^\nu) = 0, \tag{12}$$

which vanishes in the case of a closed string and a topologically trivial mapping of the world-sheet into the space-time.

3.1 Gauging shift symmetry

In the present paper, the procedure for gauging the global shift symmetries is different from the conventional one [7,8,15,16,35]. The coordinate dependence of the Kalb–Ramond field separates us from the conventional approach.

To localize the global symmetry, we introduce the independent world-sheet gauge fields v^μ_α and substitute the ordinary derivatives with the covariant ones

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v^\mu_\alpha. \tag{13}$$

We want the covariant derivatives to be gauge invariant, so we impose the transformation law for the gauge fields

$$\delta v^\mu_\alpha = -\partial_\alpha \lambda^\mu, \quad (\lambda^\mu = \lambda^\mu(\tau,\sigma)). \tag{14}$$

This replacement is, however, not sufficient to make the action locally invariant because the background field $B_{\mu\nu}$ in a weakly curved background depends on the coordinate x^μ, which is not gauge invariant. Thus, there is one more important step to be done. We should replace the coordinate x^μ, with some extension of it, where only the gauge fields v^μ_α already introduced will appear. Let us define the invariant coordinate by

$$\begin{aligned}\Delta x^\mu_{\text{inv}} &\equiv \int_P d\xi^\alpha D_\alpha x^\mu = \int_P (d\xi^+ D_+ x^\mu + d\xi^- D_- x^\nu) \\ &= x^\mu - x^\mu(\xi_0) + \Delta V^\mu,\end{aligned} \tag{15}$$

where

$$\Delta V^\mu \equiv \int_P d\xi^\alpha v^\mu_\alpha = \int_P \left(d\xi^+ v^\mu_+ + d\xi^- v^\mu_-\right). \tag{16}$$

The line integral is taken along the path P, from the initial point $\xi^\alpha_0(\tau_0, \sigma_0)$ to the final point $\xi^\alpha(\tau, \sigma)$.

We require the T-dual theory to be equivalent to the initial one. Therefore, we do not want to introduce new degrees of freedom, originating from the gauge fields. Therefore, we will require the corresponding field strength,

$$F^\mu_{\alpha\beta} \equiv \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha, \tag{17}$$

to vanish. We can achieve this by introducing the Lagrange multiplier y_μ, and the appropriate term in the Lagrangian which will force $F^\mu_{+-} \equiv \partial_+ v^\mu_- - \partial_- v^\mu_+ = -2F^\mu_{01}$ to vanish. Thus, the gauge invariant action is

$$\begin{aligned}S_{\text{inv}} = \kappa \int d^2\xi \Big[&D_+ x^\mu \Pi_{+\mu\nu}[\Delta x_{\text{inv}}] D_- x^\nu \\ &+ \frac{1}{2}\left(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu\right)\Big],\end{aligned} \tag{18}$$

where the last term is equal to $\frac{1}{2} y_\mu F^\mu_{+-}$ up to a total divergence. Now, we can use the gauge freedom to fix the gauge. It is easy to see that $x^\mu(\xi) = x^\mu(\xi_0)$ is a good gauge fixing. So, the gauge-fixed action equals

$$\begin{aligned}S_{\text{fix}}[y, v_\pm] = \kappa \int d^2\xi \Big[&v^\mu_+ \Pi_{+\mu\nu}[\Delta V] v^\nu_- \\ &+ \frac{1}{2}\left(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu\right)\Big],\end{aligned} \tag{19}$$

where y_μ and $v^\mu_\pm$ are independent variables and ΔV^μ is defined in (16).

Note that we can also define x^μ_{inv} and V^μ as the solutions of the equations $\partial_\alpha x^\mu_{\text{inv}} = D_\alpha x^\mu$ and $\partial_\alpha V^\mu = v^\mu_\alpha$. Thus, x^μ_{inv} and V^μ are in fact primitives and due to the presence of the term $\frac{1}{2} y_\mu F^\mu_{+-}$ in the Lagrangian they are well defined up to a constant.

4 From gauge-fixed action to the original and T-dual action

From the gauge-fixed action (19), we can obtain equations of motion, varying over the Lagrange multiplier y_μ and the gauge fields $v^\mu_\pm$. Substituting the solution of the equation of motion for the Lagrange multiplier into (19) we will obtain the original action, while substituting the solution of the equations of motion for the gauge fields we will obtain the T-dual theory.

4.1 Eliminating the Lagrange multiplier

Let us show that for the equation of motion for the Lagrange multiplier

$$\partial_+ v^\mu_- - \partial_- v^\mu_+ = 0 \tag{20}$$

the gauge-fixed action (19) reduces to the original action (7). This equation of motion enforces the field strength of the gauge fields F^μ_{+-} to vanish, and therefore makes the variable ΔV^μ defined in (16) path independent. To prove this, let us show that ΔV^μ is equal to zero for a closed path. If P is a closed path, then, using Stokes' theorem, the defining integral along P can be rewritten as the integral over the surface S which spans the path $P = \partial S$ of the field strength of the gauge fields

$$\oint_{P=\partial S} d\xi^\alpha v^\mu_\alpha = \int_S d^2\xi \; \left(\partial_+ v^\mu_- - \partial_- v^\mu_+\right), \tag{21}$$

which is obviously zero on (20).

The solution of (20)

$$v^\mu_\pm = \partial_\pm x^\mu, \tag{22}$$

substituted into (16) gives

$$\Delta V^\mu(\xi) = x^\mu(\xi) - x^\mu(\xi_0). \tag{23}$$

Taking into account that the action does not depend on the constant shift of the coordinate, we can omit $x^\mu(\xi_0)$ and thus the action (19) becomes

$$S_{\text{fix}}\left[v_\pm = \partial_\pm x\right] = \kappa \int d^2\xi \; \partial_+ x^\mu \Pi_{+\mu\nu}[x]\,\partial_- x^\nu, \tag{24}$$

which is just the initial action (7).

4.2 Eliminating the gauge fields

The T-dual action will be obtained by integrating out the gauge fields from (19). The equations of motion with respect to the gauge fields $v^\mu_\pm$ are

$$\Pi_{\mp\mu\nu}[\Delta V]\, v^\nu_\pm + \frac{1}{2}\partial_\pm y_\mu = \mp\beta^\mp_\mu[V], \tag{25}$$

with the terms $\beta^\mp_\mu[V]$ defined by

$$\beta^\pm_\mu[x] = \mp\frac{1}{2} h_{\mu\nu}[x]\,\partial_\mp x^\nu. \tag{26}$$

Notice that the $\beta^\mp_\mu$ come from the variation with respect to $\Delta V^\mu(\xi)$, the argument of the background fields. To show this, let us find the variation with respect to V^μ (which depends on $v^\mu_\pm$):

$$\begin{aligned}\delta_V S_{\text{fix}} &= \kappa \int d^2\xi \; v^\nu_+ \partial_\mu B_{\nu\rho} v^\rho_- \,\delta V^\mu \\ &\equiv \kappa \int d^2\xi \; \eta_\mu \delta V^\mu,\end{aligned} \tag{27}$$

where, with the help of the relation

$$\partial_\alpha V^\mu = v^\mu_\alpha, \tag{28}$$

we have

$$\eta_\mu = \partial_\mu B_{\nu\rho}\varepsilon^{\alpha\beta}\partial_\alpha V^\nu \partial_\beta V^\rho. \tag{29}$$

So, we can write

$$\eta_\mu = \partial_\alpha \beta^\alpha_\mu[V], \quad \beta^\alpha_\mu[x] \equiv -\varepsilon^{\alpha\beta} h_{\mu\nu}[x]\,\partial_\beta x^\nu, \tag{30}$$

with $h_{\mu\nu}$ defined in (9) and consequently

$$\begin{aligned}\delta_V S_{\text{fix}} &= -\kappa \int d^2\xi \beta^\alpha_\mu[V]\,\delta v^\mu_\alpha \\ &= -\kappa \int d^2\xi \left[\beta^+_\mu[V]\,\delta v^\mu_+ + \beta^-_\mu[V]\,\delta v^\mu_-\right],\end{aligned} \tag{31}$$

where $\beta^\pm_\mu[x] = \frac{1}{2}\left(\beta^0_\mu[x] \pm \beta^1_\mu[x]\right)$ is defined in (26).

Because V^μ is function of v^μ_+ and v^μ_-, there are two equations in (25) with two unknown variables v^μ_+ and v^μ_-. We can rewrite (25) in the form

$$v^\mu_\pm(y) = -\kappa\, \Theta^{\mu\nu}_\pm[\Delta V(y)]\left[\partial_\pm y_\nu \pm 2\beta^\mp_\nu[V(y)]\right], \tag{32}$$

where

$$\begin{aligned}\Theta^{\mu\nu}_\pm[\Delta V] &= -\frac{2}{\kappa}\left(G_E^{-1}[\Delta V]\,\Pi_\pm[\Delta V]\,G^{-1}\right)^{\mu\nu} \\ &= \theta^{\mu\nu}[\Delta V] \mp \frac{1}{\kappa}\left(G_E^{-1}\right)^{\mu\nu}[\Delta V],\end{aligned} \tag{33}$$

and $G^E_{\mu\nu} \equiv \left[G - 4BG^{-1}B\right]_{\mu\nu}$, $\theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$ are the open string background fields: the effective metric and the non-commutativity parameter, respectively. Let us note that the variables $G^{\mu\nu}_E$ and $\theta^{\mu\nu}$ correspond to the new fields $\tilde{g}$ and β introduced by the field redefinition in [28,36]. The tensors $\Pi_{\mp\mu\nu}$ and $\Theta_{\pm}^{\mu\nu}$ are connected by the relation

$$\Theta_{\pm}^{\mu\nu}\Pi_{\mp\nu\rho} = \frac{1}{2\kappa}\delta^{\mu}_{\rho}. \tag{34}$$

We will solve (32) iteratively. Let us separate the variables into two parts, as in [34]:

$$v^{\mu}_{\pm} = v^{(0)\mu}_{\pm} + v^{(1)\mu}_{\pm}, \quad y_\mu = y^{(0)}_\mu + y^{(1)}_\mu, \tag{35}$$

where the index (0) denotes the finite part and the index (1) the infinitesimal part (proportional to $B_{\mu\nu\rho}$). In the zeroth order, (32) reduce to

$$v^{(0)\mu}_{\pm}(y) = -\kappa\, \Theta^{\mu\nu}_{0\pm}\partial_\pm y^{(0)}_\nu, \tag{36}$$

where

$$\Theta^{\mu\nu}_{0\pm} = -\frac{2}{\kappa}(g^{-1}\Pi_{0\pm}G^{-1})^{\mu\nu} = \theta^{\mu\nu}_0 \mp \frac{1}{\kappa}(g^{-1})^{\mu\nu}, \tag{37}$$

with $g_{\mu\nu} = G_{\mu\nu} - 4b^2_{\mu\nu}$ and $\theta^{\mu\nu}_0 = -\frac{2}{\kappa}(g^{-1}bG^{-1})^{\mu\nu}$. The tensors $\Pi_{0\mp\mu\nu}$ and $\Theta^{\mu\nu}_{0\pm}$ are connected by the relation

$$\Pi_{0\mp\mu\nu}\Theta^{\nu\rho}_{0\pm} = \frac{1}{2\kappa}\delta^{\rho}_{\mu}, \tag{38}$$

which is an analogue of (34) in the constant background case.

To explicitly express the background fields argument in zeroth order $V^{(0)\mu}$, we introduce the new (double) variable $\tilde{y}_\mu$ in the zeroth order by

$$\Delta\tilde{y}^{(0)}_\mu = \tilde{y}^{(0)}_\mu(\xi) - \tilde{y}^{(0)}_\mu(\xi_0) = \int_P \mathrm{d}\tau\, y^{(0)\prime}_\mu + \mathrm{d}\sigma\, \dot{y}^{(0)}_\mu, \tag{39}$$

where the line integral is independent of the choice of the path P on the equation of motion. The double variable satisfies

$$\dot{\tilde{y}}^{(0)}_\mu = y^{(0)\prime}_\mu, \quad \tilde{y}^{(0)\prime}_\mu = \dot{y}^{(0)}_\mu. \tag{40}$$

Using the last relation and (36), we obtain

$$V^{(0)\mu} = -\kappa\, \theta^{\mu\nu}_0 y^{(0)}_\nu + (g^{-1})^{\mu\nu}\tilde{y}^{(0)}_\nu. \tag{41}$$

Comparing (22) with (32) where the background field argument is taken in the zeroth order (41), we obtain the T-dual transformation law of the variables

$$\partial_\pm x^\mu \cong -\kappa\, \Theta^{\mu\nu}_{\pm}\left[\Delta V^{(0)}\right]\partial_\pm y_\nu \mp 2\kappa\Theta^{\mu\nu}_{0\pm}\beta^{\mp}_\nu\left[V^{(0)}\right]. \tag{42}$$

In a flat background for $b_{\mu\nu} = 0$, we have $\tilde{y}_\mu \cong G_{\mu\nu}x^\nu$. Therefore, the double variable $\tilde{y}_\mu$ in this particular case turns out to be related to x^μ, the T-dual variable of y_μ.

Substituting (32) and (41) into the action (19), we obtain the T-dual action

$$^\star S[y] \equiv S_{\mathrm{fix}}[y] = \frac{\kappa^2}{2}\int \mathrm{d}^2\xi\; \partial_+ y_\mu \Theta^{\mu\nu}_{-}\left[\Delta V^{(0)}(y)\right]\partial_- y_\nu, \tag{43}$$

where we neglected the term $\beta^-_\mu\beta^+_\nu$ as the infinitesimal of the second order.

Comparing the initial action (7) with the T-dual one (43), we see that they are equal under the following transformations:

$$\partial_\pm x^\mu \to \partial_\pm y_\mu, \quad \Pi_{+\mu\nu}[x] \to \frac{\kappa}{2}\Theta^{\mu\nu}_{-}\left[\Delta V^{(0)}\right], \tag{44}$$

which implies

$$\begin{aligned} G_{\mu\nu} &\to {}^\star G^{\mu\nu} = \left(G^{-1}_E\right)^{\mu\nu}\left[\Delta V^{(0)}\right], \\ B_{\mu\nu}[x] &\to {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}\left[\Delta V^{(0)}\right], \end{aligned} \tag{45}$$

where $(G^{-1}_E)^{\mu\nu}$ and $\theta^{\mu\nu}$ are introduced in (33) and

$$\Delta V^{(0)\mu}(y) = -\kappa\theta^{\mu\nu}_0\Delta y^{(0)}_\nu + (g^{-1})^{\mu\nu}\Delta\tilde{y}^{(0)}_\nu. \tag{46}$$

Let us underline that in the initial theory the metric tensor is constant and the Kalb–Ramond field is linear in the coordinates x^μ. In the T-dual theory, both background fields depend on ΔV^μ, which is a linear combination of y_μ and its dual $\tilde{y}_\mu$. Note that the variable V^μ and consequently the T-dual action are not defined on the geometrical space (defined by the coordinate y_μ), but on the so-called doubled target space [17–28] composed of both y_μ and $\tilde{y}_\mu$. A similar procedure, using the first-order Lagrangian, was applied to the flat space-time long ago [37]. The result has the same form as (45), but it is V-independent.

In Appendix A, the equation of motion for the T-dual theory will be given explicitly. It will be shown that this equation is equal to the equation of motion of the gauge-fixed action after the elimination of the gauge fields as auxiliary fields on their equation of motion.

5 The *T*-dual of the *T*-dual theory

Because the T-dual theory (43) is by construction physically equivalent to the initial one (7), we should expect that the T-dual of the T-dual theory is just the initial theory. To demonstrate this, we should first find the global symmetry of the T-dual action. As can be seen from (19), the gauge-fixed action, S_{fix}, is invariant under the global shift

$$\delta y_\mu = \lambda_\mu = \mathrm{const}. \tag{47}$$

As the T-dual theory is equivalent to the gauge-fixed one, it must have the same global symmetry. One can check that this is indeed the symmetry of (43). Note that the action is not invariant under the constant shift of the argument of $\Theta^{\mu\nu}_{-}$, because, in contrast to the original action, the metric of the T-dual theory $^\star G^{\mu\nu}$ is not constant. Therefore, in comparison with the original action, one cannot omit the constant part of the argument $V^\mu(\xi_0)$, as we did at the end of the Sect. 4.1. But the transformation (47) leaves the argument itself, $\Delta V^\mu = V^\mu(\xi) - V^\mu(\xi_0)$, unchanged and consequently the action (43) is invariant too.

5.1 Gauging the symmetry

Let us localize this symmetry and find the corresponding locally invariant action. We covariantize the derivatives introducing the gauge fields $u_{\pm\mu}$

$$D_\pm y_\mu = \partial_\pm y_\mu + u_{\pm\mu}. \tag{48}$$

Demanding $\delta D_\pm y_\mu = 0$, we require that $u_{\pm\mu}$ transform as

$$\delta u_{\pm\mu} = -\partial_\pm \lambda_\mu(\tau, \sigma). \tag{49}$$

The argument of the dual background fields ΔV^μ is not locally invariant. Thus, first we construct the invariant expressions for the two variables y_μ and $\tilde{y}_\mu$,

$$\begin{aligned} \Delta y_\mu^{\text{inv}} &\equiv \int_P d\xi^\alpha D_\alpha y_\mu = \Delta y_\mu + \Delta U_\mu, \\ \Delta \tilde{y}_\mu^{\text{inv}} &\equiv \int_P d\xi^\alpha \varepsilon^\beta{}_\alpha D_\beta y_\mu = \Delta \tilde{y}_\mu + \Delta \tilde{U}_\mu, \end{aligned} \tag{50}$$

where

$$\begin{aligned} \Delta y_\mu &\equiv \int_P d\xi^\alpha \partial_\alpha y_\mu = y_\mu(\xi) - y_\mu(\xi_0), \\ \Delta \tilde{y}_\mu &\equiv \int_P d\xi^\alpha \varepsilon^\beta{}_\alpha \partial_\beta y^\mu, \end{aligned} \tag{51}$$

and

$$\Delta U_\mu \equiv \int_P d\xi^\alpha u_{\alpha\mu}, \quad \Delta \tilde{U}_\mu \equiv \int_P d\xi^\alpha \varepsilon^\beta{}_\alpha u_{\beta\mu}. \tag{52}$$

Now, it is easy to find the generalization of the background fields argument (45):

$$\begin{aligned} \Delta V^\mu_{\text{inv}} &\equiv -\kappa \theta_0^{\mu\nu} \Delta y_\nu^{\text{inv}} + (g^{-1})^{\mu\nu} \Delta \tilde{y}_\nu^{\text{inv}} \\ &= \Delta V^\mu[y] + \Delta V^\mu[U], \end{aligned} \tag{53}$$

which is invariant by construction and will be considered only in the zeroth order.

Finally, we can construct the dual invariant action

$$\begin{aligned} {}^\star S_{\text{inv}} = \frac{\kappa}{2} \int d^2\xi \big[&\kappa D_+ y_\mu \Theta_-^{\mu\nu} [\Delta V_{\text{inv}}] D_- y_\nu \\ &+ u_{+\mu} \partial_- z^\mu - u_{-\mu} \partial_+ z^\mu\big], \end{aligned} \tag{54}$$

where the second term makes the gauge fields $u_{\pm\mu}$ nonphysical. The gauge fixing $y_\mu(\xi) = y_\mu(\xi_0)$ produces $D_\pm y_\mu = u_{\pm\mu}$ and $\Delta V^\mu[y] = 0$, so the action becomes

$$\begin{aligned} {}^\star S_{\text{fix}}[z, u_\pm] = \frac{\kappa}{2} \int d^2\xi \big[&\kappa u_{+\mu} \Theta_-^{\mu\nu} [\Delta V[U]] u_{-\nu} \\ &+ u_{+\mu} \partial_- z^\mu - u_{-\mu} \partial_+ z^\mu\big]. \end{aligned} \tag{55}$$

5.2 Eliminating the Lagrange multiplier

The equation of motion with respect to the Lagrange multiplier z^μ,

$$\partial_+ u_{-\mu} - \partial_- u_{+\mu} = 0, \tag{56}$$

has the solution

$$u_{\pm\mu} = \partial_\pm y_\mu, \tag{57}$$

which substituted into (52) gives $\Delta U_\mu = \Delta y_\mu$ and therefore $\Delta V^\mu[U] = \Delta V^\mu[y]$. Thus, substituting (57) into the action (55), it becomes

$${}^\star S_{\text{fix}}[u_\pm = \partial_\pm y] = \frac{\kappa^2}{2} \int d^2\xi \, \partial_+ y_\mu \Theta_-^{\mu\nu} [\Delta V[y]] \partial_- y_\nu \tag{58}$$

and coincides with the T-dual action (43).

Let us stress that we cannot omit $V(\xi_0)$, because, as we have discussed at the beginning of this section, the T-dual action is invariant under a constant shift in the coordinate y_μ, but it is not invariant under a constant shift of the argument of the background fields.

5.3 Eliminating the gauge fields

Using the fact that

$$\Theta_\pm^{\mu\nu}(x) = \Theta_{0\pm}^{\mu\nu} - 2\kappa \left[\Theta_{0\pm} h(x) \Theta_{0\pm}\right]^{\mu\nu}, \tag{59}$$

we find that the equations of motion for the gauge fields, obtained by varying the action (55) with respect to the gauge fields $u_{\pm\mu}$, are

$$\partial_\pm z^\mu = -\kappa \Theta_\pm^{\mu\nu} [\Delta V(U)] \left[u_{\pm\nu} \pm 2\beta_\nu^\mp [V(U)]\right]. \tag{60}$$

Note that $\Theta_\pm^{\mu\nu}$ depends on $\Delta V^\mu(U)$, while β_ν depends on $V^\mu(U)$. Using the relation (34), we can extract $u_{\pm\mu}$:

$$u_{\pm\mu} = -2\Pi_{\mp\mu\nu} [\Delta V(U)] \partial_\pm z^\nu \mp 2\beta_\mu^\mp [V(U)]. \tag{61}$$

As in Sect. 4.2, we can separate the variables $u_{\pm\mu}$ into finite and infinitesimal parts. In the zeroth order, (61) reduce to

$$u_{\pm\mu}^{(0)} = -2\Pi_{0\mp\mu\nu} \partial_\pm z^{(0)\nu}. \tag{62}$$

Therefore, the zeroth order values of U_μ and $\tilde{U}_\mu$ are

$$\begin{aligned} U_\mu^{(0)} &= -2b_{\mu\nu} z^{(0)\nu} + G_{\mu\nu} \tilde{z}^{(0)\nu}, \\ \tilde{U}_\mu^{(0)} &= -2b_{\mu\nu} \tilde{z}^{(0)\nu} + G_{\mu\nu} z^{(0)\nu}, \end{aligned} \tag{63}$$

and this yields

$$V^{(0)\mu}(U^{(0)}) = (g^{-1})^{\mu\nu} \left[2b_\nu{}^\rho U_\rho^{(0)} + \tilde{U}_\nu^{(0)}\right] = z^{(0)\mu}, \tag{64}$$

and consequently $\beta_\mu^\pm\left[V^{(0)}(U)\right] = \beta_\mu^\pm\left[z^{(0)}\right]$. Substituting (64) into (61), we obtain its solution

$$u_{\pm\mu} = -2\Pi_{\mp\mu\nu}\left[\Delta z^{(0)}\right] \partial_\pm z^\nu \mp 2\beta_\mu^\mp\left[z^{(0)}\right], \tag{65}$$

with $\Delta z^{(0)\mu} = z^{(0)\mu}(\xi) - z^{(0)\mu}(\xi_0)$. Comparing it with (57), we find the T-duality transformation law of the variables

$$\partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}\big[\Delta z^{(0)}\big]\partial_\pm z^\nu \mp 2\beta^\mp_\mu\big[z^{(0)}\big]. \tag{66}$$

Note that this is the inverse transformation of (42). More precisely, substituting $\partial_\pm y_\mu$ from (66) into (42) and using (64), which is a consequence of the zeroth order of (66), one obtains $\partial_\pm x^\mu = \partial_\pm z^\mu$.

Substituting (65) into the action (55), we obtain the action

$${}^\star S_{\text{fix}}[z] = \kappa \int d^2\xi\, \partial_+ z^\mu \Pi_{+\mu\nu}[z(\xi) - z(\xi_0)]\, \partial_- z^\nu, \tag{67}$$

which is invariant under the global shift in the coordinate (as seen in Sect. 3). Thus, we can omit the term $z(\xi_0)$ and obtain the T-dual of the T-dual action:

$${}^{\star\star}S[z] \equiv {}^\star S_{\text{fix}}[z] = \kappa \int d^2\xi\, \partial_+ z^\mu \Pi_{+\mu\nu}[z]\, \partial_- z^\nu, \tag{68}$$

which is in fact the initial action. So, the second T-duality turns the doubled target space $(y_\mu, \tilde{y}_\mu)$ back to the conventional space z^μ.

Similarly to the derivation in Appendix A, one can show that the equation of motion of the T-dual of the T-dual action (original action) (68) is the same as the equation of motion of the gauge-fixed action (55) after elimination of the gauge fields using their equations of motion

$$\partial_+\partial_- z^\mu - B^\mu_{\ \nu\rho}\partial_+ z^\nu \partial_- z^\rho = 0. \tag{69}$$

6 The features of the T-duality

There are two important features of the T-duality which we will consider here. First, the momentum and the winding numbers of the original theory are equal to the winding and the momentum numbers of the T-dual theory, respectively. Second, the equation of motion and the Bianchi identity of the original theory are equal to the Bianchi identity and the equation of motion of the T-dual theory [34,37,38]. Thus, T-duality interchanges momentum and winding numbers, as well as the equations of motion and the Bianchi identities. Because in our case, the action is invariant up to total divergences, the conserved charges may in general differ from the corresponding momenta.

6.1 T-dualities in terms of the conserved currents and charges

We will discuss the above-mentioned features by investigating the Noether and the topological currents and their charges. As a consequence of the global shift invariance of the action (7), there exist conserved Noether currents,

$$\partial_\alpha j^\alpha_\mu = 0, \tag{70}$$

of the form

$$j^\alpha_\mu = \kappa\left[\left(\eta^{\alpha\beta}G_{\mu\nu} + 2\varepsilon^{\alpha\beta}B_{\mu\nu}[x]\right)\partial_\beta x^\nu - \beta^\alpha_\mu[x]\right], \tag{71}$$

where β^α_μ is defined in (30). In the light-cone coordinates, they have the form $j^\pm_\mu = \frac{1}{2}(j^0_\mu \pm j^1_\mu)$

$$j^\pm_\mu = \pm\kappa\Pi_{\pm\mu\nu}[x]\,\partial_\mp x^\nu - \kappa\beta^\pm_\mu[x]. \tag{72}$$

The current conservation equation $\partial_+ j^+_\mu + \partial_- j^-_\mu = 0$ is in fact the equation of motion of the original theory,

$$\partial_+\partial_- x^\mu - B^\mu_{\ \nu\rho}\partial_+ x^\nu \partial_- x^\rho = 0. \tag{73}$$

Let us now turn to the T-dual description. As the consequence of (66), one has

$$j^\alpha_\mu \cong {}^\star i^\alpha_\mu = -\kappa\varepsilon^{\alpha\beta}\partial_\beta y_\mu, \tag{74}$$

where ${}^\star i^\alpha_\mu$ is the topological current, because

$$\partial_\alpha {}^\star i^\alpha_\mu = 0 \tag{75}$$

is the Bianchi identity. Thus, T-duality relates the conservation of the Noether and the topological current laws, which are in fact the equations of motion and the Bianchi identities.

Note that, from (71) and (74), for $\alpha = 0$ one has

$$\pi_\mu - \kappa\beta^0_\mu[x] \cong \kappa y'_\mu, \tag{76}$$

where π_μ is the canonical momentum corresponding to the variable x^μ,

$$\pi_\mu = \kappa\left(G_{\mu\nu}\dot{x}^\nu - 2B_{\mu\nu}[x]\,x'^\nu\right). \tag{77}$$

The charges associated with the conserved currents (71),

$$Q_\mu = \int_{-\pi}^{\pi} d\sigma\, j^0_\mu = \int_{-\pi}^{\pi} d\sigma\left[\pi_\mu - \kappa\beta^0_\mu\right], \tag{78}$$

in general could differ from the momentum quantum numbers (Kaluza–Klein modes). The difference is the infinitesimal part of the momenta. Its mode expansion corresponds to the expressions (3.21) and (3.22) of [34]. In the particular case when the string is curled up around only one compactified dimension, i.e. $x^i = c\sigma$, $x^j = 0$, $j \neq i$, one has $\beta^0_\mu = 0$, because of the antisymmetry of $B_{\mu\nu\rho}$. Then the conserved charges turn to momentum quantum numbers.

The charge corresponding to the conserved topological current,

$${}^\star q_\mu = \int_{-\pi}^{\pi} d\sigma\, {}^\star i^0_\mu = \kappa\int_{-\pi}^{\pi} d\sigma\, y'_\mu, \tag{79}$$

is just the winding number of the T-dual theory. As a consequence of (76), the Noether charges transform under T-duality into the topological charges

$$Q_\mu \cong {}^\star q_\mu. \tag{80}$$

For $\beta^0_\mu = 0$, this just describes the fact that T-duality transforms the momenta numbers of the initial theory into the winding numbers of the T-dual theory.

Because the T-dual of the T-dual theory is the original theory, we can apply the same procedure in the other direction. From (43), we obtain the T-dual Noether currents

$$
\begin{aligned}
{}^\star j^{\alpha\mu} = \kappa \Big[\Big(\eta^{\alpha\beta} \left(G_E^{-1}\right)^{\mu\nu} [\Delta V] + \kappa \varepsilon^{\alpha\beta} \theta^{\mu\nu} [\Delta V] \Big) \partial_\beta y_\nu \\
- (g^{-1})^{\mu\nu} \varepsilon^\alpha{}_\beta \beta^\beta_\nu [V] - \kappa \theta_0^{\mu\nu} \beta^\alpha_\nu [V] \Big].
\end{aligned} \tag{81}
$$

In light-cone coordinates, one has

$$
{}^\star j^{\pm\mu} = \pm \frac{\kappa^2}{2} \Theta^{\mu\nu}_\mp [\Delta V] \left[\partial_\mp y_\nu \mp 2\beta^\pm_\nu [V] \right]. \tag{82}
$$

The conservation law for the T-dual current

$$
\partial_+ {}^\star j^{+\mu} + \partial_- {}^\star j^{-\mu} = 0 \tag{83}
$$

is just the equation of motion in the T-dual theory

$$
\begin{aligned}
&\partial_+ \left[\Theta^{\mu\nu}_- [\Delta V] \partial_- y_\nu - 2\Theta^{\mu\nu}_{0-} \beta^+_\nu [V] \right] \\
&-\partial_- \left[\Theta^{\mu\nu}_+ [\Delta V] \partial_+ y_\nu + 2\Theta^{\mu\nu}_{0+} \beta^-_\nu [V] \right] = 0.
\end{aligned} \tag{84}
$$

T-duality according to (42) transforms Noether currents of the T-dual theory to the topological currents of the original theory (formally to the topological currents of the T-dual of the T-dual theory):

$$
{}^\star j^{\alpha\mu} \cong i^{\alpha\mu} = -\kappa \varepsilon^{\alpha\beta} \partial_\beta x^\mu. \tag{85}
$$

The conservation of the topological currents $\partial_\alpha i^{\alpha\mu} = 0$ are just the Bianchi identities. From (81) and (85) for $\alpha = 0$, it follows that

$$
{}^\star \pi^\mu - \kappa^2 \theta_0^{\mu\nu} \beta^0_\nu [V] \cong \kappa x'^\mu, \tag{86}
$$

where ${}^\star\pi^\mu$ is the canonical momentum in the T-dual theory:

$$
\begin{aligned}
{}^\star \pi^\mu = \kappa \left(G_E^{-1}\right)^{\mu\nu} [\Delta V[y]] \dot{y}_\nu - \kappa^2 \theta^{\mu\nu} [\Delta V[y]] y'_\nu \\
-\kappa (g^{-1})^{\mu\nu} \beta^1_\nu [V[y]].
\end{aligned} \tag{87}
$$

The conserved charges of the dual Noether and the original topological currents for $\beta^\alpha_\mu = 0$

$$
\begin{aligned}
{}^\star Q^\mu &= \int_{-\pi}^{\pi} \mathrm{d}\sigma\, {}^\star j^{0\mu} = \int_{-\pi}^{\pi} \mathrm{d}\sigma\, {}^\star \pi^\mu, \\
q^\mu &= \int_{-\pi}^{\pi} \mathrm{d}\sigma\, i^{0\mu} = \kappa \int_{-\pi}^{\pi} \mathrm{d}\sigma\, x'^\mu,
\end{aligned} \tag{88}
$$

are momentum modes of the dual theory and the winding modes of the original theory. They are also, according to (86), connected by the T-duality transformation

$$
{}^\star Q^\mu \cong q^\mu. \tag{89}
$$

In the following tables, we summarize the relations obtained: a T-duality transformation relates the Noether currents with the topological ones; the corresponding conservation laws relate the equations of motion with Bianchi identities, while the corresponding Noether charges relate momenta and winding modes.

Original theory S *T-dual theory* ${}^\star S$
Noether current j^α_μ *Topological current* ${}^\star i^\alpha_\mu = -\kappa\epsilon^{\alpha\beta}\partial_\beta y_\mu$
Conservation law = *Equation of motion* $\partial_\alpha j^\alpha_\mu = 0$ *Conservation law* = *Bianchi identity* $\partial_\alpha {}^\star i^\alpha_\mu = 0$
Noether conserved charge $Q_\mu = \int_{-\pi}^{\pi} d\sigma j^0_\mu = \int_{-\pi}^{\pi} d\sigma \pi_\mu = P_\mu$ *Topological conserved charge* ${}^\star q_\mu = \int_{-\pi}^{\pi} d\sigma\, {}^\star i^0_\mu = \kappa \int_{-\pi}^{\pi} d\sigma y'_\mu = {}^\star W_\mu$
T-dual theory ${}^\star S$ *T-dual of T-dual theory* ${}^{\star\star}S = S$
Noether current ${}^\star j^{\alpha\mu}$ *Topological current* $i^{\alpha\mu} = -\kappa\epsilon^{\alpha\beta}\partial_\beta x^\mu$
Conservation law = *Equation of motion* $\partial_\alpha {}^\star j^{\alpha\mu} = 0$ *Conservation law* = *Bianchi identity* $\partial_\alpha i^{\alpha\mu} = 0$
Noether conserved charge ${}^\star Q^\mu = \int_{-\pi}^{\pi} d\sigma\, {}^\star j^{0\mu} = \int_{-\pi}^{\pi} d\sigma\, {}^\star\pi^\mu = {}^\star P^\mu$ *Topological conserved charge* $q^\mu = \int_{-\pi}^{\pi} d\sigma\, i^{0\mu} = \kappa \int_{-\pi}^{\pi} d\sigma x'^\mu = W^\mu$

7 Conclusion

In this paper, we consider the closed bosonic string moving in a weakly curved background. This background is defined by a constant space-time metric and a Kalb–Ramond field linear in the coordinates, where the coordinate dependence is infinitesimally small. With such a choice, the space-time equations of motion are satisfied. The aim of the paper was to investigate the T-dual theory in a curved background.

Earlier, in a number of papers, similar topics, restricted to the string in a flat background, were discussed. In these papers, the prescriptions for the construction of the T-dual theories were established. Here, we present the generalization of the covariant Buscher construction.

In Buscher's construction, one starts with the sigma model constructed from background fields G, B, Φ which do not depend on some coordinates x^a. Thus, the corresponding abelian isometries leave the action invariant. We started with the sigma model in a weakly curved background. We found that the action still has the global symmetry $\delta x^\mu = \lambda^\mu =$ const, even though the background fields depend on these coordinates. Therefore, we gauged it in the usual way by introducing the gauge fields v_α^μ, replacing the derivatives $\partial_\alpha x^\mu$ with the covariant ones. In our case, this was not sufficient to construct the invariant action, because the background field $B_{\mu\nu}$ depends on x^μ, which is not gauge invariant. The essential new step in our gauging prescription is the introduction of the invariant coordinate, as the line integral (primitive) of its covariant derivatives. This kind of coordinate enables local invariance. It remains to find to which class of backgrounds this generalized procedure can be applied.

As usual, for the T-dual theory to be physically equivalent to the original theory, one had to eliminate all the degrees of freedom carried by the gauge fields. This was achieved by adding the Lagrange multiplier term $y_\mu F_{01}^\mu$ to the Lagrangian. At this point, we fixed the gauge. The action obtained in this way reduced to the initial one on the equations of motion for the Lagrange multiplier y_μ.

The Lagrange multiplier term $y_\mu F_{01}^\mu$ guarantees that the gauge field is closed ($dv = 0$), but one should consider the topological contribution as well. Because of this, an additional investigation of the holonomies of v should be performed. To solve these problems, connected with the global structure of the theory, following [11,12,15,38,39] we will consider the quantum theory in some of our further papers.

For the solution of the equations of motion of the gauge fields, one obtains the T-dual action. In the case of a flat background, the T-dual action was given in terms of the T-dual variable, which turned out to be the Lagrange multiplier itself. In the weakly curved case, the T-dual action is defined in the doubled space given in terms of the Lagrange multiplier and its T-dual in the flat space. The dual background fields depend on ΔV^μ, a linear combination of these variables.

Starting from the T-dual action and following the T-dual prescription that we proposed, we obtained the initial one. The fact that we succeeded in finding the rules for obtaining the T-dual theory from the initial one in a weakly curved background and the inverse rules allowed us to treat the two main features of T-duality. These are: the T-duality relates the original and the T-dual theory, by mapping the momentum numbers of one theory with the winding numbers of the other, and the equations of motion of one with the Bianchi identities of the other.

Starting with the initial theory and its global shift invariance, we found the conserved Noether currents j_μ^α. The current conservation laws $\partial_\alpha j_\mu^\alpha = 0$ are in fact the equations of motion, and the associated charges $Q_\mu = \int_{-\pi}^{\pi} d\sigma j_\mu^0 = P_\mu$ are momentum numbers. The T-duals of the Noether currents are the topological currents ${}^\star i_\mu^\alpha$ of the T-dual theory. Their conservation laws $\partial_\alpha {}^\star i_\mu^\alpha = 0$ are in fact Bianchi identities and the associated charges ${}^\star q_\mu = \int_{-\pi}^{\pi} {}^\star i_\mu^0 = {}^\star W_\mu$ are the winding numbers. Thus, T-duality relates the equations of motion with the Bianchi identities and the momentum with the winding modes. Analogously, starting with the T-dual action and its global symmetry, we found that the Noether currents and their associated charges (dual momentum numbers) correspond to the topological currents of the initial theory and theirs charges (winding numbers).

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Appendix A: *T*-dual equation of motion

Let us demonstrate that the T-dual equation of motion can be obtained from both the T-dual action (43) and the gauge fixed action (19). The T-dual action depends on y_μ, so there is one equation of motion which depends on y_μ. On the other hand, the gauge-fixed action beside y_μ depends on the gauge fields $v_\pm^\mu$ as well. Treating $v_\pm^\mu$ as auxiliary fields, we will use their equations of motion to eliminate them from the third equation and obtain an y_μ dependent equation.

Appendix A.1: The equation of motion for the gauge-fixed action

Let us find the equations of motion for the gauge-fixed action (19). This will be done iteratively. The equations of motion, obtained varying the action over the finite parts $(v_\pm^{(0)\mu}, y_\mu^{(0)})$ introduced in (35), contain both finite and infinitesimal parts. The finite part of these equations is

$$\Pi_{0+\mu\nu} v_-^{(0)\nu} + \frac{1}{2}\partial_- y_\mu^{(0)} = 0, \tag{90a}$$

$$\Pi_{0-\mu\nu} v_+^{(0)\nu} + \frac{1}{2}\partial_+ y_\mu^{(0)} = 0, \tag{90b}$$

$$\partial_+ v_-^{(0)\mu} - \partial_- v_+^{(0)\mu} = 0. \tag{90c}$$

The equations of motion obtained varying over the infinitesimal parts, $(v_\pm^{(1)\mu}, y_\mu^{(1)})$, are identical to (90a)–(90c). Equation (90c) guarantees that (16) in the zeroth order does not depend on the choice of the path P. Therefore, we can write

$$\Delta V^{(0)\mu} = V^{(0)\mu}(\xi) - V^{(0)\mu}(\xi_0) \tag{91}$$

and

$$\partial_\alpha V^{(0)\mu} = v_\alpha^{(0)\mu}. \tag{92}$$

The solution of (90a) and (90b) is

$$v_\pm^{(0)\mu}(y) = -\kappa\, \Theta_{0\pm}^{\mu\nu} \partial_\pm y_\nu^{(0)}, \tag{93}$$

where $\Theta_{0\pm}^{\mu\nu}$ is defined in (37). Using the zeroth order value of V^μ, we can rewrite (93) as

$$\begin{aligned} &v_\pm^{(0)\mu}(y) = \partial_\pm V^{(0)\mu}(y), \\ &V^{(0)\mu} = -\kappa\, \theta_0^{\mu\nu} y_\nu^{(0)} + (g^{-1})^{\mu\nu} \tilde{y}_\nu^{(0)}. \end{aligned} \tag{94}$$

Let us now turn to the infinitesimal part of the equations of motion obtained varying S_{fix} by the finite parts $(v_\pm^{(0)\mu}, y_\mu^{(0)})$. They are equal to

$$\begin{aligned} &h_{\mu\nu}\big[\Delta V^{(0)}\big] v_-^{(0)\nu} + \Pi_{0+\mu\nu} v_-^{(1)\nu} + \frac{1}{2}\partial_- y_\mu^{(1)} \\ &\quad = \beta_\mu^+\big[V^{(0)}\big], \end{aligned} \tag{95a}$$

$$\begin{aligned} &h_{\mu\nu}\big[\Delta V^{(0)}\big] v_+^{(0)\nu} + \Pi_{0-\mu\nu} v_+^{(1)\nu} + \frac{1}{2}\partial_+ y_\mu^{(1)} \\ &\quad = -\beta_\mu^-\big[V^{(0)}\big], \end{aligned} \tag{95b}$$

$$\partial_+ v_-^{(1)\mu} - \partial_- v_+^{(1)\mu} = 0. \tag{95c}$$

Solving the first two equations, with the help of (36) and (38), we get

$$\begin{aligned} v_\pm^{(1)\mu} &= -\kappa \Theta_{0\pm}^{\mu\nu} \partial_\pm y_\nu^{(1)} - \kappa \Theta_{1\pm}^{\mu\nu}\big[\Delta V^{(0)}\big] \partial_\pm y_\nu^{(0)} \\ &\quad \mp 2\kappa \Theta_{0\pm}^{\mu\nu} \beta_\nu^\mp\big[V^{(0)}\big], \end{aligned} \tag{96}$$

where

$$\Theta_{1\pm}^{\mu\nu}[x] = -2\kappa \Theta_{0\pm}^{\mu\rho} h_{\rho\sigma}[x]\, \Theta_{0\pm}^{\sigma\nu}. \tag{97}$$

So, the solution for $v_\pm^\mu$ is just (32)

$$v_\pm^\mu[y] = -\kappa\, \Theta_\pm^{\mu\nu}\big[\Delta V^{(0)}\big] \partial_\pm y_\nu \mp 2\kappa \Theta_{0\pm}^{\mu\nu} \beta_\nu^\mp\big[V^{(0)}\big], \tag{98}$$

where $\Theta_\pm^{\mu\nu}$, defined in (33), is the inverse of $\Pi_{\pm\mu\nu}$ (see the relation (34)).

Finally, (90c) and (95c) produce $\partial_+ v_-^\mu - \partial_- v_+^\mu = 0$. On the solution (98), this equation becomes

$$\begin{aligned} &\partial_+\big[\Theta_-^{\mu\nu}\big[\Delta V^{(0)}\big]\partial_- y_\nu - 2\Theta_{0-}^{\mu\nu}\beta_\nu^+\big[V^{(0)}\big]\big] \\ &\quad -\partial_-\big[\Theta_+^{\mu\nu}\big[\Delta V^{(0)}\big]\partial_+ y_\nu + 2\Theta_{0+}^{\mu\nu}\beta_\nu^-\big[V^{(0)}\big]\big] = 0. \end{aligned} \tag{99}$$

This is the equation of motion of the T-dual theory.

Appendix A.2: The equation of motion for the T-dual action

Again, we will find the equations of motion iteratively. Variation of the T-dual action (43) with respect to y_μ produces in the zeroth order

$$\partial_+ \partial_- y_\mu^{(0)} = 0. \tag{100}$$

By this equation, $\Delta \tilde{y}_\mu$ becomes path independent and we have $\partial_\pm V^{(0)\mu} = -\kappa \Theta_{0\pm}^{\mu\nu} \partial_\pm y_\nu$. Variation by V^μ, with the help of (97), gives

$$\begin{aligned} &\delta_{V^\star} S \\ &\quad = \kappa^2 \int d\xi^2 \big[\Theta_{0-}^{\mu\nu} \partial_+ \beta_\nu^+\big[V^{(0)}\big] + \Theta_{0+}^{\mu\nu} \partial_- \beta_\nu^-\big[V^{(0)}\big]\big] \delta y_\mu, \end{aligned} \tag{101}$$

where the $\beta_\mu^\pm$ are defined in (26) and so the equation of motion for y_μ is indeed (99).

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