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Ferromagnetic and spin-glass-like transition in the majority vote model on complete and random graphs

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Abstract. Ferromagnetic and spin-glass-like transitions in nonequilibrium spin models in contact with two thermal baths with different temperatures are investigated. The models comprise the Sherrington-Kirkpatrick model and the dilute spin glass model which are the Ising models on complete and random graphs, respectively, with edges corresponding, with certain probability, to positive and negative exchange integrals. The spin flip rates are combinations of two Glauber rates at the two temperatures, and by varying the coefficients of this combination probabilities of contact of the model with each thermal bath and thus the level of thermal noise in the model are changed. Particular attention is devoted to the majority vote model in which one of the two above-mentioned temperatures is zero and the other one tends to infinity. Only in rare cases such nonequilibrium models can be mapped onto equilibrium ones at certain effective temperature. Nevertheless, Monte Carlo simulations show that transitions from the paramagnetic to the ferromagnetic and spin-glass-like phases occur in all cases under study as the level of thermal noise is varied, and the phase diagrams resemble qualitatively those for the corresponding equilibrium models obtained with varying temperature. Theoretical investigation of the model on complete and random graphs is performed using the TAP equations as well as mean-field and pair approximations, respectively. In all cases theoretical calculations yield reasonably correct predictions concerning location of the phase border between the paramagnetic and ferromagnetic phases. In the case of the spin-glass-like transition only qualitative agreement between theoretical and numerical results is achieved using the TAP equations, and the mean-field and pair approximations are not suitable for the study of this transition. The obtained results can be interesting for modeling opinion formation by means of the majority-vote and related models and suggest that in the presence of negative interactions between agents, apart from the ferromagnetic phase corresponding to consensus formation, spin-glass-like phase can occur in the society characterized by local rather than long-range ordering.

1 Introduction

The majority-vote (MV) model is a stochastic model for the opinion formation in which agents with two possible opposite opinions are represented by two-state spins $s_i = \pm 1$ located in the nodes of a network $i = 1, 2 \dots N$ [1,2]. The agents can update their opinions (flip their orientations) under the influence of the opinions of their neighbors connected to them by the edges of the network according to a probabilistic rule in which the spin flip rate depends on a parameter p, $0 \leq p \leq 1/2$, controlling the degree of stochasticity (social temperature) in the model. In Monte-Carlo (MC) simulation random sequential updating of the agents' opinions is assumed. Usually, the following probabilistic rule is used to define the spin flip rate: in each elementary MC simulation step a node is randomly chosen and the corresponding spin is flipped with probability $1 - p$ if majority of the neighboring spins has opposite opinion, with probability p if the majority has the same opinion and with probability $1/2$ if the numbers of neighbors with opposite and the same opinion are equal (this is possible in the case of an even number of neighbors only). From a physical point of view such MV model is equivalent to a nonequilibrium version of the Ising model with ferromagnetic (FM) interactions in which each spin with probabilities $2p$ and $1 - 2p$ is in contact with two thermal baths with infinite and zero temperature, respectively [1,2], and the above-mentioned spin flip rate is a result of competition between two corresponding equilibrium spin flip rates. The FM interactions are described by exchange integrals, e.g., $J_{ij} = J > 0$ associated with edges connecting pairs of nodes i, j. In this interpretation the agents make decisions to flip their opinions under the influence of the local fields acting on spins which are proportional to the resultant opinions of the agents' neighbors, but the nonequilibrium spin flip rate depends only on the sign of these fields. In fact, FM-like continuous ordering phase transition with decreasing p was observed in such MV model on regular networks [1–4], random graphs [5–13] and complete graphs [14].

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Modifications of the above-mentioned probabilistic rule have also been considered, consisting in, e.g., inclusion of agents with independence [15], heterogeneous agents [16], agents with more than two opinions [17], agents with inertia, which leads to the occurrence of a discontinuous FM transition [18], and anticonformist agents, which leads to antiferromagnetic (AFM) or spin-glass-like (SG) rather than FM transition [19,20]. The latter modification amounts to associating AFM-like interactions (not necessarily symmetric) with edges attached to the nodes with anticonformist agents.

In this paper the MV model on more general weighted complete and random graphs is considered. The role of weights is played by exchange integrals J_{ij} which are drawn from given probability distribution $P(J_{ij})$ and associated randomly to the edges. It is assumed that the interactions can have FM or AFM character with $J_{ij} > 0$ or $J_{ij} < 0$, respectively, and, possibly, different strength. Thus, in general, each agent with certain probability can exhibit FM or AFM interactions with different neighbors, corresponding to the tendency to share the same opinion with friends and colleagues or to differ in opinion with disliked persons or enemies. Such assumption concerning the exchange integrals amounts to introducing certain degree of randomness in social interactions which is controlled by the parameters of the probability distribution $P(J_{ij})$. Again, the agents make decisions under the influence of the local fields which are sums of opinions of their neighbors weighted by the appropriate exchange integrals, but the nonequilibrium spin flip rate depends only on the sign of the local fields. From a physical point of view, for certain choices of the distribution of exchange integrals the MV model under study is a kind of a nonequilibrium counterpart to the generic models based on the Ising model and exhibiting SG transition [21–23], e.g., the Sherrington-Kirkpatrick (SK) model on complete graphs [24,25] and dilute spin glass (DSG) model on random graphs [26]. Hence, the model under study can exhibit a richer phase diagram than the usual MV model with only FM-like interactions. In fact, in this paper is shown that nonequilibrium SG-like transition occurs in the model apart from the usual FM transition which is also affected by the presence of the AFM interactions, and that the phase diagrams qualitatively resemble those for the corresponding equilibrium Ising models.

SG-like transition in nonequilibrium models, in particular in those for the opinion formation, is not a widely studied subject, mainly due to lack of analytic methods. Similarly, in this paper it is shown that a possible candidate for the theoretical description of the SG-like transition in the MV model on complete graphs, the (modified) Thouless–Anderson–Palmer (TAP) equation [27], in general yields qualitatively incorrect results. Thus, the following investigation of the SG-like transition in the MV model is based mainly on MC simulations. In contrast, FM and possibly AFM transitions in nonequilibrium models with competing spin flip mechanisms and different kinds of disorder have been broadly studied in spin models, mainly on regular lattices [28–37], analytically using the concept of the effective Hamiltonian [28–30,32], a sort of pair approximation (PA) [31] and numerically via MC simulations [33–37]; the latter studies comprised also models with MV kind of dynamics [34,35]. It should be, however, mentioned that the concept of the effective Hamiltonian in certain special cases can be also useful in the analytic study of other nonequilibrium systems, e.g., neural networks with fast time variation of synapses in which both the FM and SG phases can occur [38]. In this paper the above-mentioned analytic methods as well as simple mean-field approximation (MFA) are applied to investigate the FM and, to small extent, also SG-like transitions in the MV model on random graphs, and the obtained predictions show satisfactory agreement with results of MC simulations. The main outcome of the present study is that nonequilibrium models for the opinion formation in the presence of disorder in social interactions can exhibit more diverse critical behavior than the usual FM ordering transition, which can be seen in numerical simulations and described to some extent using analytic approach based mainly on the MFA, PA and, in specific cases, also on the concept of the effective Hamiltonian.

2 The models

A starting point for the definition of the MV model is the usual Ising model with two-state spins $s_i = \pm 1$ located in nodes $i = 1, 2, \ldots N$ and with non-zero exchange integrals J_{ij} associated with edges of a network of interactions,

$$H = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij} s_i s_j, \tag{1}$$

where $J_{ij} > 0$ ($J_{ij} < 0$) if there is an edge connecting nodes i, j and the interaction between the corresponding agents has FM (AFM) character, respectively. Thus, the exchange integrals play a role of weights associated with edges of the network. Each node (spin) of a network is assumed to be in contact with two thermal baths with temperatures T_1 and T_2 ($T_2 < T_1$), with probability $2p$ and $1-2p$, respectively, where $0 \le p \le 1/2$ is a model parameter. Usually such model is a nonequilibrium one (apart from few exceptional cases, see Ref. [14] and Sect. 4.1) since there is no effective temperature associated with the model; in contrast, there is non-zero heat flux. Consequently, the spin flip rate is a result of competition between two equilibrium spin flip rates for the Ising model with temperature T_1 or T_2 and has a form of an appropriate combination of them. Assuming that the equilibrium rates are Glauber ones the probability of the flip of the spin s_i per unit time (rate) provided that the model is in the spin configuration $\mathbf{s} = \{s_1, s_2, \ldots s_N\}$ is

$$\begin{aligned} w_i(\mathbf{s}) &= 2p\frac{1}{2}\left[1 - s_i \tanh(\beta_1 I_i)\right] \\ &\quad + (1-2p)\frac{1}{2}\left[1 - s_i \tanh(\beta_2 I_i)\right] \\ &= \frac{1}{2}\left\{1 - s_i\left[2p\tanh(\beta_1 I_i) + (1-2p)\tanh(\beta_2 I_i)\right]\right\}, \end{aligned} \tag{2}$$

where $\beta_1 = 1/T_1$, $\beta_2 = 1/T_2$ and $I_i = \sum_{j=1}^{N} J_{ij} s_j$ is a local field acting on the spin in node i. The rate for the MV model is obtained by taking the limits $T_1 \to \infty$ ($\beta_1 \to 0$), $T_2 \to 0$ ($\beta_2 \to \infty$) which yields [1]

$$w_i(\mathbf{s}) = \frac{1}{2}\left[1-(1-2p)s_i \text{sign}\left(I_i\right)\right] = \frac{1}{2}\left[1-(1-2p)s_i \text{sign}\left(\sum_{j=1}^{N} J_{ij}s_j\right)\right], \quad (3)$$

where

$$\text{sign}(x) = \begin{cases} -1 & \text{for} \quad x<0 \\ 0 & \text{for} \quad x=0 \\ +1 & \text{for} \quad x>0 \end{cases} \quad (4)$$

is the signum function.

In the MV model on a complete graph, each spin can interact with all other spins. The exchange integrals in the Hamiltonian (1) are drawn from the normal distribution $P\left(J_{ij}\right) = N\left(\frac{J_0}{N}, \frac{J_1}{\sqrt{N}}\right)$ with

$$J_{ij} = \frac{J_0}{N} + \Delta J_{ij}, \quad i,j = 1,2\ldots N,$$
$$P\left(\Delta J_{ij}\right) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\left(\Delta J_{ij}\right)^2}{2\sigma^2}\right),$$
$$\sigma^2 = \overline{\left(\Delta J_{ij}\right)^2} = \frac{J_1^2}{N}, \quad (5)$$

as in the well-known SK model in the equilibrium case [21,22,24].

In the MV model on random graphs, the structure of interactions is determined by the underlying network, with non-zero exchange integrals associated with the edges. In this paper for simplicity only models on weakly heterogeneous random graphs are considered in which the probability distribution for the degrees of nodes (numbers of edges attached to the nodes) $P(k)$ is narrow and centered around the mean degree $\langle k \rangle$ [39,40]. In particular, the MV model on random regular graphs (RRGs) with $P(k) = \delta_{k,K}$ and on Erdös-Rényi graphs (ERGs) with binomial degree distribution $P(k) = \binom{N-1}{k}\tilde{p}^k(1-\tilde{p})^{N-1-k}$ with $\langle k \rangle = N\tilde{p}$ [41] is studied. An efficient method to generate RRGs is to apply the Configuration Model [42], while ERGs can be efficiently constructed by randomly connecting N nodes with $N\langle k \rangle/2$ edges. Then, an exchange integral is randomly associated with each edge, which is either $J_{ij} = -J < 0$ with probability r (corresponding to AFM-like interaction) or $J_{ij} = J > 0$ with probability $1-r$ (corresponding to FM-like interaction), i.e., the exchange integrals are drawn from the probability distribution

$$P\left(J_{ij}\right) = r\delta\left(J_{ij}+J\right) + (1-r)\,\delta\left(J_{ij}-J\right). \quad (6)$$

If the nodes i, j are not connected by an edge, by definition $J_{ij} = 0$, thus the effective field acting at spin i can be written as $I_i = \sum_{j \in \text{nn}_i} J_{ij} s_j$, where nn_i denotes a set of neighbors of the node i connected to it by an edge. The Hamiltonian (1) with the distribution of exchange integrals (6) in the equilibrium case is that for the DSG model [26].

3 The model on complete graphs

3.1 Theory

It is known that the equilibrium SK model described by the Hamiltonian (1) with the distribution of the exchange integrals (5) with decreasing temperature exhibits transition from the disordered paramagnetic (PM) state to the FM or SG state, depending on the parameters J_0, J_1. Critical temperatures for these transitions can be evaluated analytically using, e.g., the replica approach which is believed to yield quantitatively correct results [21–25]. Besides, in some special cases also, approximate methods lead to correct values of the critical temperatures. In the case of purely FM interactions in the model ($J_0 > 0$, $J_1 = 0$ in equation (5)) simple MFA is enough to predict the critical temperature for the FM transition, while in the case of random unbiased interactions ($J_0 = 0$, $J_1 > 0$ in equation (5)) the TAP equation [21–23,27] correctly predicts the critical temperature for the SG transition. In contrast, in the case of the nonequilibrium MV model on a complete graph the replica approach cannot be applied since the model in general does not obey Gibbs distribution. However, below it is shown that in the two above-mentioned special cases the MFA and the TAP equations can be heuristically modified to describe the FM and SG-like transitions in the MV model, respectively, with decreasing parameter p. In the case of the FM transition quantitative agreement between the critical value of p evaluated in the MFA and that from MC simulations is obtained. Unfortunately, the modified TAP equations only qualitatively confirm the occurrence of the SG-like transition in the MV model, and the evaluated critical value of the parameter p is definitely wrong.

3.1.1 Mean field approximation

In order to avoid singularities in the modified TAP equations it is convenient to consider the model on a complete graph with the distribution of the exchange integrals as in equation (5) and with the spin flip rate (2) and only finally take the limits $\beta_1 \to 0$, $\beta_2 \to \infty$ to obtain results for the MV model. Let us start with the MFA. For the spin system with dynamics governed by the transition rates $w\left(\mathbf{s}'|\,\mathbf{s}\right)$ from the spin configuration $\mathbf{s}$ to $\mathbf{s}'$ the Master equation for the probability $P\left(\mathbf{s},t\right)$ that at time t the spin configuration is $\mathbf{s}$ has a general form

$$\frac{dP\left(\mathbf{s},t\right)}{dt} = \sum_{\mathbf{s}'}\left[w\left(\mathbf{s}|\,\mathbf{s}'\right)P\left(\mathbf{s}',t\right) - w\left(\mathbf{s}'|\,\mathbf{s}\right)P\left(\mathbf{s},t\right)\right]. \quad (7)$$

Taking into account that at each time step transition occurs between spin configurations $\mathbf{s} \to \mathbf{s}'$ differing just by one spin flipped, say s_i, thus the transition rate $w\left(\mathbf{s}'|\,\mathbf{s}\right) =$

$w_i(\mathbf{s})$ is given by equation (2), and performing ensemble average of the Master equation (7) the following system of equations is obtained for the local magnetizations, i.e., mean values $m_i \equiv \langle s_i \rangle$ of spins at sites $i = 1, 2, \ldots N$,

$$\begin{aligned}\frac{dm_i}{dt} &= -2\langle s_i w_i(\mathbf{s})\rangle \\ &= -m_i + 2p\langle\tanh(\beta_1 I_i)\rangle + (1-2p)\langle\tanh(\beta_2 I_i)\rangle. \quad (8)\end{aligned}$$

The MFA consists in replacing

$$\langle\tanh(\beta_\nu I_i)\rangle \to \tanh(\beta_\nu h_i), \quad \nu = 1, 2, \quad (9)$$

where

$$h_i \equiv \langle I_i \rangle = \sum_{i=1}^{N} J_{ij}\langle s_j\rangle = \sum_{i=1}^{N} J_{ij} m_j \quad (10)$$

is the local mean field acting at spin at node i. Fixed points of the system of equations (8) are obtained by setting all $dm_i/dt = 0$,

$$m_i = 2p\tanh(\beta_1 h_i) + (1-2p)\tanh(\beta_2 h_i). \quad (11)$$

Different fixed points correspond to different thermodynamic phases of the model. In particular, there is a PM fixed point with $m_i = 0$ for $i = 1, 2, \ldots N$. As the parameter p is decreased it is expected that the PM fixed point loses stability, and new stable fixed point or points appear corresponding to the transition to FM or SG phases. Just below the transition point the local magnetizations m_i are expected to be small; on the basis of equation (11) the local mean fields h_i are also expected to be small. Thus, by linearizing equation (11) in the vicinity of the PM fixed point and using equation (10) the following system of linear equations for the local magnetizations m_i, $i = 1, 2, \ldots N$ is obtained,

$$\sum_{j=1}^{N}\left\{\delta_{ij} - \left[2p(\beta_1 - \beta_2) + \beta_2\right] J_{ij}\right\} m_j = 0. \quad (12)$$

Non-zero solutions of equation (12) are allowed provided that the determinant of the system of equations is zero. After diagonalizing the matrix $\{J_{ij}\}$, equation (5), it can be seen that the latter condition is fulfilled for different values of the parameter p, connected with different eigenvalues of the matrix of the exchange integrals. The critical value $p_{c,MFA}$ at which the PM fixed point loses stability is the largest of the above-mentioned values of p connected with the maximum eigenvalue $J_{\max}$,

$$p_{c,MFA} = \frac{1}{2(\beta_2 - \beta_1)}\left(\beta_2 - \frac{1}{J_{\max}}\right). \quad (13)$$

Let us put $\beta_1 = 0$ ($T_1 \to \infty$) and consider the two above-mentioned cases, one with purely FM interactions ($J_0 > 0$, $J_1 = 0$ in equation (5)) and the other with random unbiased interactions ($J_0 = 0$, $J_1 > 0$ in Eq. (5)). In the former case the only non-zero eigenvalue of the matrix $\{J_{ij}\}$ is J_0, corresponding to the eigenvector $m_i = 1$, $i = 1, 2, \ldots N$ in equation (12) characteristic of the FM phase, which leads to the critical value of the parameter p for the FM transition,

$$p_{c,MFA}^{(FM)} = \frac{1}{2}\left(1 - \frac{1}{\beta_2 J_0}\right) \overset{\beta_2\to\infty}{\to} \frac{1}{2}, \quad (14)$$

where the latter result is for the MV model on a complete graph. This leads to a reasonable prediction that the FM transition at $p_{c,MFA}^{(FM)} > 0$ can occur only if T_2, the lower of the temperatures of the two thermal baths, fulfills a condition $T_2 < J_0$, i.e., it is below the critical temperature for the FM transition in the corresponding equilibrium model. Besides, $p_{c,MFA}^{(FM)}$ is an increasing function of β_2 and rises to the maximum possible value $p_{c,MFA}^{(FM)} = 1/2$ for $T_2 \to 0$ which means that the MV model exhibits FM transition even if the probability that the spins are in contact with the thermal bath with zero temperature is negligibly small. These predictions are in agreement with results of MC simulations (Sect. 3.2).

In the case with $J_0 = 0$, $J_1 > 0$ in equation (5) transition to the SG-like phase is expected, in analogy with the SK model. The maximum eigenvalue of the random matrix $\{J_{ij}\}$ is $J_{\max} = 2J_1$. Putting again $\beta_1 = 0$ the critical value of the parameter p for the SG-like transition is

$$p_{c,MFA}^{(SG)} = \frac{1}{2}\left(1 - \frac{1}{2\beta_2 J_1}\right) \overset{\beta_2\to\infty}{\to} \frac{1}{2}, \quad (15)$$

where, again, the latter result is for the MV model. This means that the SG-like transition at $p_{c,MFA}^{(SG)} > 0$ can occur only if $T_2 < 2J_1$. This prediction is clearly wrong since for $p = 0$ in equation (2) the nonequilibrium model under study is equivalent to the equilibrium SK model at temperature T_2 for which the critical temperature for the SG transition is J_1 [21–24]. Similar disagreement with predictions of the MFA occurs in the equilibrium SK model. Hence, more rigorous approximation based on the TAP equation is needed to describe the possible SG-like transition in the MV model.

3.1.2 The modified Thouless–Anderson–Palmer equation

The crucial idea in the derivation of the TAP equation is to take into account that the component of the local mean field h_j at node j coming from a neighboring node i modifies local magnetization m_j which in turn modifies the local field acting at spin in node i. This is achieved by introducing Onsager reaction fields $h_{i,R}$ and replacing $h_i \to h_i - h_{i,R}$ in equations derived in the MFA [27], in the present case in equation (11). The Onsager reaction field is

$$h_{i,R} = m_i \sum_{j=1}^{N} J_{ij}^2 \chi_j, \quad (16)$$

where the local susceptibility χ_j of node j is

$$\chi_j = \frac{\partial m_j}{\partial h_j} = 2p\beta_1 \left[1 - \tanh^2(\beta_1 h_j)\right] + (1-2p)\beta_2 \left[1 - \tanh^2(\beta_2 h_j)\right]. \quad (17)$$

The system of equations (11) with all h_i replaced by $h_i - h_{i,R}$ has still a PM fixed point $m_i = 0$, $i = 1, 2, \ldots N$, which can lose stability with decreasing parameter p. In the vicinity of the PM point all $m_i \approx 0$ and since h_i is of the same order of magnitude as m_i (cf. Eq. (11)) there is also $h_i \approx 0$. It can be easily seen that $\chi_j|_{h_j=0} = 2p(\beta_1 - \beta_2) + \beta_2$, $(\partial\chi_j/\partial h_j)|_{h_j=0} = 0$, thus

$$\chi_j(h_j) \approx 2p(\beta_1 - \beta_2) + \beta_2 + O\left(m_j^2\right) \quad (18)$$

and thus in linear approximation with respect to small local magnetizations m_j the reaction field (16) is

$$h_{i,R} \approx \left[2p(\beta_1 - \beta_2) + \beta_2\right] m_i \sum_{j=1}^{N} J_{ij}^2. \quad (19)$$

Let us focus on the case of the model on a complete graph with random unbiased interactions ($J_0 = 0$, $J_1 > 0$ in Eq. (5)) in which SG-like transition can be expected. Taking into account that $\sum_{j=1}^{N} J_{ij}^2 \overset{N\to\infty}{\longrightarrow} J_1^2$ and linearizing in the vicinity of the PM fixed point equation (11) with h_i replaced by $h_i - h_{i,R}$, where the reaction field is given by equation (19), for small m_j, $j = 1, 2, \ldots N$, the following modified TAP equation for the model under study is obtained,

$$\sum_{j=1}^{N} \left\{ \left(1 + \left[2p(\beta_1 - \beta_2) + \beta_2\right]^2 J_1^2\right)\delta_{ij} - \left[2p(\beta_1 - \beta_2) + \beta_2\right] J_{ij} \right\} m_j = 0. \quad (20)$$

As mentioned in Section 3.1.1 the maximum eigenvalue of the matrix $\{J_{ij}\}$ is $J_{\max} = 2J_1$. The corresponding value of p for which the determinant of the system of equation (20) is equal to zero is the critical value for the expected SG-like transition,

$$p_{c,TAP}^{(SG)} = \frac{1}{2(\beta_2 - \beta_1)}\left(\beta_2 - \frac{1}{J_1}\right). \quad (21)$$

Putting $\beta_1 = 0$ yields finally

$$p_{c,TAP}^{(SG)} = \frac{1}{2}\left(1 - \frac{1}{\beta_2 J_1}\right) \overset{\beta_2\to\infty}{\longrightarrow} \frac{1}{2}, \quad (22)$$

where the latter result is for the MV model on a complete graph. This leads to a reasonable prediction that the SG-like transition at $p_{c,TAP}^{(SG)} > 0$ can occur only if $T_2 < J_1$, i.e., it is below the critical temperature for the SG transition in the corresponding equilibrium SK model. In contrast with equation (15) this prediction is in agreement with results of MC simulations (Sect. 3.2). Unfortunately, the predicted value $p_{c,TAP}^{(SG)} = 1/2$ for the MV model with $T_2 \to 0$ is still much overestimated: MC simulations show that the critical value of the parameter p does not increase so much with $\beta_2 \to \infty$ and the MV model exhibits SG-like transition at significantly smaller, though non-zero value of p (Sect. 3.2).

3.2 Numerical results

In this section, theoretical predictions from Section 3.1 are compared with results of numerical simulations. In order to verify the occurrence of the FM or SG-like phase transition predicted theoretically in Section 3.1 MC simulations are performed of the MV model on a complete graph, with the spin flip rate given by equation (3), and also of a more general nonequilibrium Ising model in contact with two thermal baths, with the spin flip rate given by equation (2), in particular with $\beta_1 = 0$ and $\beta_2 \to \infty$. Simulations are performed for graphs with different numbers of nodes, $10^2 \le N \le 10^3$, and for each graph results are averaged over 100−500 (depending on N) different realizations of the distribution of the exchange integrals (5); different sets of parameters J_0, J_1 are considered, leading eventually to the appearance of the FM or SG-like transition. MC simulations are performed using simulated annealing algorithm with random sequential updating of the agents' opinions. For each N, J_0, J_1 and realization of the distribution $P(J_{ij})$ simulation is started in the disordered phase at high p with random initial orientations of spins s_i, $i = 1, 2, \ldots N$. Then the level of internal noise p is decreased in small steps toward zero, and at each intermediate value of p, after a sufficiently long transient, the order parameters for the FM and SG-like transitions are calculated as averages over the time series of the spin configurations.

The possible FM and SG-like transitions in the MV model under study are investigated in the same way as in the Ising model. The order parameter for the FM transition is the absolute value of the magnetization

$$M = \left|\left[\left\langle \frac{1}{N}\sum_{i=1}^{N} s_i \right\rangle_t\right]_{av}\right| \equiv \left|\left[\langle \tilde{m} \rangle_t\right]_{av}\right|, \quad (23)$$

where $\tilde{m}$ denotes a momentary value of the magnetization at a given MCSS, $\langle\cdot\rangle_t$ denotes the time average for a model with given realization of the distribution $P(J_{ij})$ and $[\cdot]_{av}$ denotes average over different realizations of the latter distribution. The order parameter for the SG-like transition (henceforth called the SG order parameter) is the absolute value of the overlap parameter [21–23],

$$Q = \left|\left[\left\langle \frac{1}{N}\sum_{i=1}^{N} s_i^{\alpha} s_i^{\beta} \right\rangle_t\right]_{av}\right| \equiv \left|\left[\langle \tilde{q} \rangle_t\right]_{av}\right|, \quad (24)$$

where α, β denote two copies (replicas) of the system simulated independently with different random initial orientations of spins. In the PM phase both M and Q are close to zero. In the case of the FM transition both M and Q increase as p is decreased. In the case of the SG-like transition the SG order parameter Q increases as p is decreased while the magnetization M remains close to zero.

The critical values $p_{c,MC}^{(FM)}$ and $p_{c,MC}^{(SG)}$ of the parameter p for the FM and SG transitions, respectively, can be determined from the intersection point of the respective Binder cumulants $U^{(M)}$ vs. p and $U^{(Q)}$ vs. p for systems with different numbers of agents N [43], where

$$U^{(M)} = \frac{1}{2}\left[3 - \frac{\langle \tilde{m}^4 \rangle_t}{\langle \tilde{m}^2 \rangle_t^2}\right]_{av}, \tag{25}$$

$$U^{(Q)} = \frac{1}{2}\left[3 - \frac{\langle \tilde{q}^4 \rangle_t}{\langle \tilde{q}^2 \rangle_t^2}\right]_{av}. \tag{26}$$

Besides, in the case of the second-order transition the respective cumulants for all N should decrease monotonically with p, while in the case of the first-order transition they should exhibit minima as functions of p.

Exemplary curves M, Q and Binder cumulants vs. p are shown in Figure 1a for the particular case of the MV model on a complete graph. In the case of purely FM, uniform exchange integrals ($J_0 = 1,\ J_1 = 0$) crossing of the cumulants $U^{(M)}$ for different numbers of nodes N at $p = 1/2$ as well as increase of the magnetization for decreasing p confirm the occurrence of the continuous FM transition at a critical value $p = p_{c,MC}^{(FM)} = p_{c,MFA}^{(FM)} = 1/2$, in agreement with prediction of equation (14). This agreement may be related to the fact that, as an exception, the MV model on a complete graph is an equilibrium model [14]. In the case of purely AFM exchange integrals ($J_0 = 0, J_1 = 1$) crossing of the cumulants $U^{(Q)}$ for different N at $p = p_{c,MC}^{(SG)} \approx 0.08$ and increase of the SG order parameter Q as well as decrease of the magnetization M with increasing number of nodes confirm the occurrence of the continuous SG-like transition. However, this transition occurs at a critical value of $p_{c,MC}^{(SG)}$ much below $p_{c,TAP}^{(SG)} = 1/2$ predicted by equation (22).

MC simulations show that equation (14) correctly predicts the critical value of p for the FM transition in a more general model with purely FM exchange integrals in contact with two thermal baths. For example, for fixed $\beta_1 = 0$ the minimum value of T_2 at which the FM transition is observed is $T_2 = 1/\beta_2 = 1$, and the expected linear dependence of $p_{c,MFA}^{(FM)}$ on $1/\beta_2$ is confirmed by numerical data (Fig. 2a). In contrast, for the above-mentioned model the only correct prediction of equation (22) resulting from even the more accurate TAP equations is that the minimum value of T_2 at which the SG-like transition is observed is again $T_2 = 1/\beta_2 = 1$. For decreasing T_2 the predicted increase of the critical value $p_{c,TAP}^{(SG)}$ for the SG-like transition much exceeds that of $p_{c,MC}^{(SG)}$ observed in MC simulations (Fig. 2a), thus for $T_2 \to 0$ the SG-like transition in the MV model occurs at $p \approx 0.08 < 1/2$.

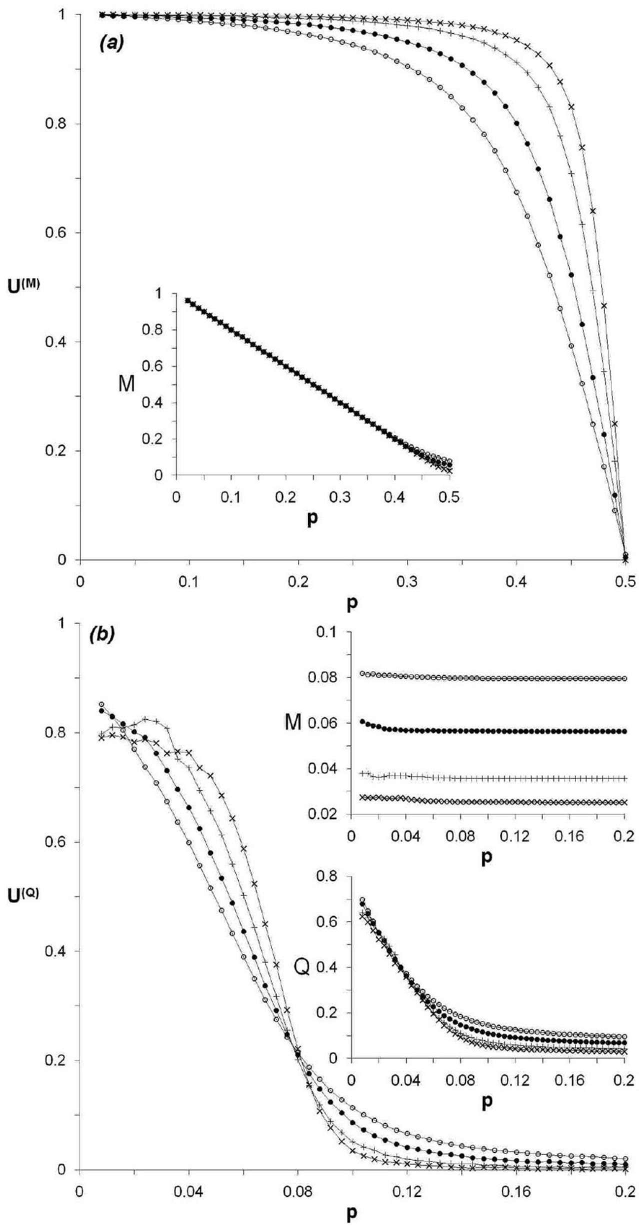

Fig. 1. Binder cumulants and order parameters for the FM and SG-like transitions vs. the parameter p from MC simulations of the MV model on complete graphs with the number of spins $N = 100$ (○), $N = 200$ (●), $N = 500$ (+), $N = 1000$ (×). (a) $U^{(M)}$ vs. p for the model with $J_0 = 1$, $J_1 = 0$ (only FM interactions), inset: magnetization M vs. p; intersection of cumulants indicates second-order phase transition from the PM to the FM phase at $p_{c,MC}^{(FM)} = 0.5$. (b) $U^{(Q)}$ vs. p for the model with $J_0 = 0$, $J_1 = 1$ (only AFM interactions), insets: magnetization M and SG order parameter Q vs. p; intersection of cumulants at $p_{c,MC}^{(SG)} \approx 0.08$ as well as increase of Q with decreasing p and decrease of M with increasing N indicate second-order phase transition from the PM to the SG-like phase.

The phase diagram obtained from MC simulations of the MV model on a complete graph, with exchange integrals drawn from the normal distribution (5) with different parameters J_0, J_1 is shown in Figure 2b. Is shows obvious qualitative similarity with the phase diagram for the equilibrium SK model [21–24], with the parameter p playing the role of temperature. This suggests that the mechanism beyond the occurrence of the SG transition in both cases may be similar; in particular, it may be speculated that with decreasing p and thus increasing probability of contact with thermal bath with $T_2 = 0$ the spin configuration of the MV model approaches that corresponding to a (possibly local) minimum of the energetic landscape given by the Hamiltonian (1), although probability that a model has a given energy is not given by the Gibbs distribution.

4 The model on random graphs

4.1 Theory

The equilibrium DSG model described by the Hamiltonian (1) with the distribution of the exchange integrals (6) with decreasing temperature exhibits transition from the disordered PM state to the FM or SG state, depending on the mean degree of nodes $\langle k \rangle$ and fraction of the AFM exchange integrals r. In the case of the model on ERGs critical temperatures for these transitions can be accurately evaluated analytically using the replica approach [26]; in a similar way, in certain cases critical temperature for the SG transition was obtained for the model on RRGs [44]. Of course, critical temperature for the FM transition can be obtained also from the MFA which should yield more and more correct results as $\langle k \rangle \to \infty$, although, in contrast with the case of regular lattices, due to random distribution of edges the idea of critical dimension above which the MFA is strict is not applicable in models on random graphs. Unfortunately, the approach based on the TAP equations modified to take into account the structure of interactions of the DSG model on RRGs and ERGs fails in predicting quantitatively the critical temperature for the SG transition.

In this section, theoretical investigation of the nonequilibrium DSG model in contact with two thermal baths with different temperatures will be conducted only for the case of the structure of the exchange interactions in the form of a RRG, with the degree distribution $P(k) = \delta_{k,K}$. This is in order to avoid complications in the derivation of equations for the stationary values of the order parameter(s) corresponding to different thermodynamic phases of the model, which are related to small but existing heterogeneity of the degree distribution of ERGs. This simplification has no much importance since results of MC simulations (Sect. 4.2) are almost the same for the models on RRGs and ERGs with $\langle k \rangle = K$. Hence, concerning approximate methods, homogeneous MFA and homogeneous PA are extended and applied to the case of a model with coexisting FM and AFM exchange interactions (6); these approaches can be improved by taking into account the heterogeneity of the degree distribution $P(k)$, which, however, can lead to severe complications of the equations

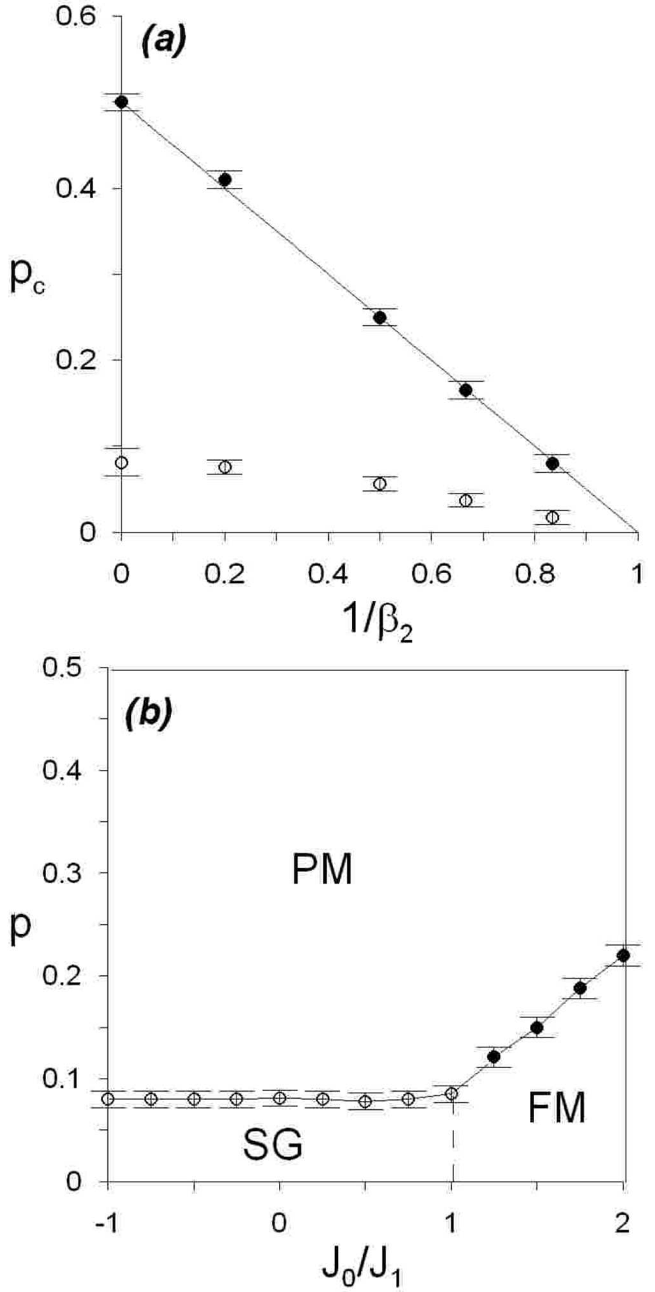

Fig. 2. (a) Critical values of the parameter p vs. $1/\beta_2$ from MC simulations of the Ising model on a complete graph in contact with two thermal baths with $\beta_1 = 0$ ($T_1 \to \infty$) and different $\beta_2 > 1$ ($T_2 < 1$), for the FM transition $p_{c,MC}^{(FM)}$ (•) in the model with $J_0 = 1$, $J_1 = 0$ and for the and SG-like transition $p_{c,MC}^{(SG)}$ (∘) in the model with $J_0 = 0$, $J_1 = 1$. Solid line shows theoretical predictions for the critical values of the parameter p vs. $1/\beta_2$ in the above-mentioned model, from the MFA for the FM transition $p_{c,MFA}^{(FM)}$ in the model with $J_0 = 1$, $J_1 = 0$, equation (14), and from the TAP equation for the SG-like transition $p_{c,TAP}^{(SG)}$ in the model with $J_0 = 0$, $J_1 = 1$, equation (22) (both lines overlap). (b) Phase diagram obtained from MC simulations of the MV model on a complete graph with $J_1 = 1$ and $-1 \le J_0 \le 2$, shown are critical values for the FM transition $p_{c,MC}^{(FM)}$ (•) or for the SG-like transition $p_{c,MC}^{(SG)}$ (∘), lines are guides to the eyes.

for the order parameter(s), in particular in the case of PA [45,46].

In the following, in Section 4.1.1 is shown that, by generalizing arguments for nonequilibrium models on regular lattices [28–30,32], in rare cases the nonequilibrium DSG model on RRGs is equivalent to an equilibrium model with certain effective temperature which can exhibit both FM and SG transition. This equivalence applies to a more general model in contact with two thermal baths with different temperatures. In the remaining part of this section attention is constrained to the MV model. In Section 4.1.2 homogeneous MFA is derived and the resulting critical value of the parameter p for the FM transition is evaluated. In Section 4.1.3 homogeneous PA is extended to the case of a model with a mixture of FM and AFM interactions and phase borders between the PM and FM phases for the MV model are obtained; derivation of the equations for the order parameters in the PA closely follows that in reference [47] where a simpler approach than that in reference [45,46] was used. PA is in general considered as more accurate than the MFA since it takes into account both the dynamics of the spins in the nodes and of the connecting edges (for details, see below). Here it is shown that predictions of both the MFA and PA concerning the location of the critical point for the FM transition are only qualitatively correct and deviate from the results of MC simulations for increasing fraction of the AFM exchange integrals r; moreover, unexpectedly, predictions of the PA are quantitatively worse than those from the MFA. Unfortunately, both MFA and PA can be used only to investigate the FM transition, and the evidence for the SG-like transition is based only on MC simulations of the MV model under study.

4.1.1 The case with an effective Hamiltonian

In exceptional cases apparently nonequilibrium models are equivalent to related equilibrium ones. This equivalence can be proven by demonstrating that the nonequilibrium model obeys the same Gibbs distribution as its equilibrium counterpart described by an appropriately constructed effective Hamiltonian [28–30,32]. In the case of models which possess Hamiltonian, as in equation (1), but are out of equilibrium due to contact with several thermal baths with different temperatures the above-mentioned equivalence can be revealed by finding effective temperature $T_{eff} = 1/\beta_{eff}$, and the respective Gibbs distribution is that with T_{eff} and energy given by the Hamiltonian of the model [28,29]. Most analytic and numerical results concerning such equivalence were obtained for spin models on regular d-dimensional lattices with competing spin flip mechanisms [28–37], but they can be easily extended to similar models on RRGs since in both cases the degrees of nodes obey a one-point distribution $P(k) = \delta_{k,K}$. For example, let us consider a model with the spin flip rate (2) being a combination of two Glauber rates with temperatures T_1, T_2. This model is equivalent to an equilibrium Ising model with the same structure of interactions defined by a given realization of a RRG and of the distribution of the exchange integrals J_{ij} if there is an effective temperature T_{eff} for which the usual (equilibrium) Glauber rate is equal to the combination of Glauber rates (2) for any node i, any value of the spin $s_i = \pm 1$ and for all possible values of the local field $I_i = \sum_{j\in \mathrm{nn}_i} J_{ij}s_j$, i.e.,

$$\frac{1}{2}\left[1 - s_i \tanh\left(\beta_{eff} I_i\right)\right] = \frac{1}{2}\left\{1 - s_i\left[2p\tanh\left(\beta_1 I_i\right) + (1-2p)\tanh\left(\beta_2 I_i\right)\right]\right\}. \quad (27)$$

For the model on a RRG, the possible values of the local field are $I_i = -K, -K+2, \ldots K$. This leads to approximately $K/2$ different equations (27) (equation with $I_i = 0$ is an identity, and equations for $\pm I_i$ are the same) which in general cannot be simultaneously fulfilled by a single value $\beta_{eff} = 1/T_{eff}$. However, for $K = 2$ only one (that for $I_i = 2$) non-trivial equation remains, which yields

$$\beta_{eff} = \frac{1}{2}\mathrm{atanh}\left[2p\tanh\left(2\beta_1\right) + (1-2p)\tanh\left(2\beta_2\right)\right], \quad (28)$$

and in particular for the MV model with $T_1 \to \infty$ ($\beta_1 \to 0$), $T_2 \to 0$ ($\beta_2 \to \infty$)

$$\beta_{eff} = \frac{1}{2}\mathrm{atanh}(1-2p). \quad (29)$$

Due to equivalence between the standard $q = 2$ Potts model and the Ising model it may be shown that the latter model on RRGs with purely AFM exchange integrals ($r = 1$ in Eq. (6)) exhibits SG transition at $T = -2J/\ln\left[1 - 2/(\sqrt{K-1}+1)\right]$ [44], which for $K = 2$ is $T = 0$. Substituting $\beta_{eff} \to \infty$ in equation (29) yields that the MV model on RRGs with $K = 2$ and with purely AFM interactions should exhibit SG transition at $p = 0$, i.e., when the probability that the spin is in contact with thermal bath with temperature $T_2 = 0$ is $1 - 2p = 1$. While not particularly interesting, this result shows that analytic methods based on the effective Hamiltonian introduced to study nonequilibrium models on regular lattices can be also applicable in models on complex networks, in particular on RRGs. It also raises some hope that SG-like transition in the MV model on RRGs with $K > 2$ reported below is not a numerical artifact.

4.1.2 Mean field approximation

In the case of the MV model on random graphs the averaging procedure leading to the MFA is much simpler than for the model on complete graphs, thus it is possible to consider directly the model with the spin flip rate (3). The equations for the local magnetizations $m_i = \langle s_i\rangle$ are

$$\frac{dm_i}{dt} = -2\langle s_i w_i(\mathbf{s})\rangle = -m_i + (1-2p)\langle \mathrm{sign}(I_i)\rangle, \quad (30)$$

where

$$\langle \mathrm{sign}(I_i)\rangle = (+1)\Pr(\mathrm{sign}(I_i) = +1) + (-1)\Pr(\mathrm{sign}(I_i) = -1). \quad (31)$$

In the case of a model on a RRG all nodes are equivalent, thus all local magnetizations are identical and equal to the magnetization of the model, $m_i = m$, $i = 1, 2, \dots N$, thus by definition for any node i there is $\Pr(s_i = +1) = 1 - \Pr(s_i = -1) = (1+m)/2$. Besides, the local field $I_i = \sum_{j\in \mathrm{nn}_i} J_{ij} s_j$, where the summation is over a set nn_i of K neighbors of the node i, is a sum of terms which are $J_{ij}s_j = +J$ if $J_{ij} = +J$ (which happens with probability $1-r$, see Eq. (6)) and $s_j = +1$ or if $J_{ij} = -1$ (which happens with probability r) and $s_j = -1$, or $J_{ij}s_j = -J$ if $J_{ij} = +J$ and $s_j = -1$ or if $J_{ij} = -J$ and $s_j = +1$. Thus, by introducing the probability

$$\Pi = \Pr(s_j = +1)(1-r) + \Pr(s_i = -1)r = \frac{1 + (1-2r)m}{2}, \quad (32)$$

it is obtained that

$$\Pr(\mathrm{sign}(I_i) = -1) = \Pr(I_i < 0) = \sum_{l=0}^{\lfloor K/2 \rfloor} \Pi^l (1-\Pi)^{K-l},$$
$$\Pr(\mathrm{sign}(I_i) = +1) = \Pr(I_i > 0) = \sum_{l=\lceil K/2 \rceil}^{K} \Pi^l (1-\Pi)^{K-l}, \quad (33)$$

where $\lfloor \cdot \rfloor$ ($\lceil \cdot \rceil$) denote the floor (ceil) function. For large K it is possible to approximate the binomial distribution by the normal one which for small magnetization m yields

$$\Pr(\mathrm{sign}(I_i) = \pm 1) = \frac{1}{2} \pm \frac{1}{2}\mathrm{erf}\left[\left(\Pi - \frac{1}{2}\right)\sqrt{2K}\right], \quad (34)$$

and finally from equations (30), (31), and (32) to obtain

$$\frac{dm}{dt} = -m + (1-2p)\mathrm{erf}\left[\sqrt{\frac{K}{2}}(1-2r)m\right]. \quad (35)$$

Equation (35) has a fixed point $m = 0$ corresponding to the PM phase. By elementary stability analysis it is easy to show that as p is decreased this point loses stability and a pair of symmetric stable solutions with $|m| > 0$ corresponding to the FM phase appears. This corresponds to a continuous FM transition which occurs at a critical value

$$p_{c,MFA}^{(FM)} = \frac{1}{2}\left[1 - \frac{\sqrt{2\pi}}{2(1-2r)}\frac{1}{\sqrt{K}}\right]. \quad (36)$$

For $r = 0$ (purely FM interactions) the result in equation (36) coincides with that in reference [10]. Thus, according to MFA the critical value $p_{c,MFA}^{(FM)}$ decreases with r and reaches zero at $r_{MFA}^{\star} = \frac{1}{2}\left(1 - \frac{\sqrt{2\pi}}{2}\frac{1}{\sqrt{K}}\right)$, and for $r > r_{MFA}^{\star}$ the PM solution is stable in the whole range of $0 < p < 1/2$ and the FM transition does not occur. However, MC simulations show that for increasing r the critical value of p for the FM transition deviates from the prediction of equation (36) and for larger r the PM phase is not stable and the SG-like transition occurs (see Sect. 4.2).

4.1.3 Pair approximation

The key idea behind the homogeneous PA is to describe a model consisting of interacting two-state spins using two dynamical, mutually dependent variables: the usual magnetization m (which is conveniently expressed by the concentration $c_\uparrow = c$ of spins with orientation up, so that $c = (1+m)/2$ and the concentration of spins with orientation down is $c_\downarrow = 1 - c = (1-m)/2$) and the concentration b of pairs of spins connected by the so-called active links [31,45–47]. In the case of models with only FM interactions ($J_{ij} > 0$) preferring parallel orientation of interacting spins, active links correspond to the edges of network connecting nodes at which spins with opposite orientations are located; in related equilibrium models described by the Hamiltonian (1) presence of such pairs of interacting spins increases energy of the model. Thus, in the model under study in which both FM and AFM ($J_{ij} = -J < 0$) interactions are present a natural generalization of the concept of active links is to assume that such links correspond to interactions between pairs of spins which increase the energy (1), i.e., to the edges with associated FM interactions connecting spins with opposite orientations or to the edges with associated AFM interactions connecting spins with the same orientations; the remaining links are termed inactive. Then, it is assumed that for any node orientations of spins in the neighboring nodes are not mutually correlated, thus the number of active links l ($l \le K$) attached to the node containing spin with orientation ν, ($\nu \in \{\uparrow, \downarrow\}$) obeys a binomial distribution $B_{K,l}(\theta_\nu) = \binom{K}{l}\theta_\nu^l (1-\theta_\nu)^{K-l}$. Here, θ_ν is conditional probability that a link is active provided that it is attached to a randomly chosen node containing spin with orientation ν; since for a model on RRGs all nodes are statistically equivalent this probability is the same for all nodes.

Expression for the conditional probabilities $\theta_\uparrow$, $\theta_\downarrow$ in terms of the dynamical variables, i.e., concentration of spins up c (normalized to the number of nodes N) and of active links b (normalized to the total number of edges in the graph) is a first step toward formulation of the dynamical equations in the homogeneous PA. Since the number of nodes in the graph is N, the total number of links is $NK/2$, the number of FM links with $J_{ij} = J > 0$ is $(1-r)NK/2$, the number of AFM links with $J_{ij} = -J < 0$ is $rNK/2$, the number of active links is $NKb/2$, each link has two tips (ends) attached to two different nodes, thus the total number of tips is NK, the number of tips of FM links attached to the nodes is $(1-r)NK$, the number of tips of AFM links attached to the nodes is rNK and the number of tips of active links attached to the nodes is NKb. Let us denote by $P(\nu, \nu')$, where $\nu, \nu' \in \{\uparrow, \downarrow\}$, concentration of tips of links attached to the nodes with spins with orientation ν such that the other tip of the link is attached to a node with spin with orientation ν', normalized to the total number of tips. Hence, the corresponding number of above-mentioned tips is $NKP(\nu, \nu')$. In order to proceed with calculation it should be assumed that signs of the exchange integrals associated with subsequent links are not correlated with orientations of spins in the nodes to which these links are attached. Then, using the definition of an active link it is obtained that the number of tips of active links can be expressed as

$$NKb = NK\left\{(1-r)\left[P(\downarrow,\uparrow)+P(\uparrow,\downarrow)\right] + r\left[P(\downarrow,\downarrow)+P(\uparrow,\uparrow)\right]\right\}. \quad (37)$$

Obviously, $P(\downarrow,\uparrow) = P(\uparrow,\downarrow)$ and $\sum_{\nu,\nu'\in\{\uparrow,\downarrow\}} P(\nu,\nu') = 1$, thus

$$P(\downarrow,\uparrow) = P(\uparrow,\downarrow) = \frac{b-r}{2(1-2r)}. \quad (38)$$

Besides, the number of tips of links attached to nodes with spin with orientation ν is $NKc_\nu = NK\sum_{\nu'\in\{\uparrow,\downarrow\}} P(\nu,\nu')$, hence

$$P(\nu,\nu) = c_\nu - \frac{b-r}{2(1-2r)} \quad (39)$$

for $\nu \in \{\uparrow,\downarrow\}$. The conditional probability θ_ν can be evaluated as the ratio of the number of (tips of) active links attached to nodes with spins with orientation ν to the total number of (tips of) links attached to such nodes. Using equations (38) and (39), it is eventually obtained that the conditional probabilities can be expressed as

$$\theta_\downarrow = \frac{NK\left[(1-r)P(\downarrow,\uparrow)+rP(\downarrow,\downarrow)\right]}{NK(1-c)} = \frac{b-r}{2(1-c)} + r$$
$$\theta_\uparrow = \frac{NK\left[(1-r)P(\uparrow,\downarrow)+rP(\uparrow,\uparrow)\right]}{NKc} = \frac{b-r}{2c} + r. \quad (40)$$

Another quantity of interest for the derivation of the dynamical equations in the PA is the average rate of spin flips in nodes with l active links attached $f(l,p)$: in the case of model on RRGs with $P(k) = \delta_{k,K}$ the average is over all nodes $i = 1,2,\ldots N$ and can be simply obtained from the spin flip rate, equation (3). It is easy to verify that in terms of the number of active links l the local field I_i acting on the spin in node i can be written as $I_i = \sum_{j\in \mathrm{nn}_i} J_{ij}s_j = Jl(-s_i) + J(K-l)s_i = J(K-2l)s_i$, thus

$$f(l,p) = w_i(\mathbf{s}) = \begin{cases} p & \text{for} \quad l \le \lfloor K/2 \rfloor \\ \frac{1}{2} & \text{for} \quad k = K/2 \\ 1-p & \text{for} \quad l \ge \lceil K/2 \rceil. \end{cases} \quad (41)$$

Then, expressions can be derived for the flip rates in the direction up and down, respectively, averaged over all nodes of the network, i.e., over the probability distribution to find a node with spin up or down and over the respective binomial distributions of the number of active links attached to the nodes with spins up and down,

$$\gamma^+ = (1-c)\sum_{l=0}^{K} B_{K,l}(\theta_\downarrow)\, f(l,p),$$
$$\gamma^- = c\sum_{l=0}^{K} B_{K,l}(\theta_\uparrow)\, f(l,p). \quad (42)$$

The above rates occur in the Master equation for the concentration c of spins with direction up. Moreover, a complementary equation for the concentration b of active links can be obtained by taking into account that each spin flip in the node with l active links changes active links attached to this node into inactive and vice versa, thus changing the concentration of active links b by $\frac{2}{NK}(K-2l)$, and again performing average over all nodes of the network as above. Finally, the rate equations for the macroscopic quantities c, b in the PA are [47]

$$\frac{\partial c}{\partial t} = \gamma^+ - \gamma^-, \quad (43)$$
$$\frac{\partial b}{\partial t} = \frac{2}{K}\sum_{\nu\in\{\uparrow,\downarrow\}} c_\nu \sum_{l=0}^{K} B_{K,l}(\theta_\nu)\, f(l,p)\,(K-2l). \quad (44)$$

Let us denote the right-hand side of equation (43) by $A(c,b)$ and of equation (44) by $B(c,b)$. The fixed point(s) of the above system of equations are solutions of a system of algebraic equations $A(c,b)=0$, $B(c,b)=0$. The (stable or unstable) fixed point with $c=1/2$ ($m=0$), which corresponds to the PM phase, exists in a whole range of p, $0\le p\le 1/2$. At this point $\theta_\downarrow = \theta_\uparrow \equiv \theta = b$ from equation (40), and, as a result, equation $A(c=1/2, b=\theta)=0$ is trivially fulfilled. The value of θ at the PM fixed point depends on p and is a solution of equation $B(c=1/2,\theta)=0$, i.e.,

$$\sum_{l=0}^{K} B_{K,l}(\theta)\, f(l,p)\,(K-2l) = 0. \quad (45)$$

Stability of the PM fixed point can be determined from the eigenvalues of the Jacobian matrix of the right-hand sides of equation (43), (44) evaluated at $c=1/2$, $b=\theta$. After some calculations it can be found that

$$\left.\frac{\partial A}{\partial b}\right|_{c=1/2,b=\theta} = \left.\frac{\partial B}{\partial c}\right|_{c=1/2,b=\theta} = 0, \quad (46)$$

thus the eigenvalues of the Jacobian matrix are

$$\lambda_1 = \left.\frac{\partial A}{\partial c}\right|_{c=1/2,b=\theta}$$
$$= \sum_{l=0}^{K}\binom{K}{l}\Big\{-2\theta^l(1-\theta)^{K-l} + 2(\theta-r)\Big[l\theta^{l-1}(1-\theta)^{K-l} - (K-l)\theta^l(1-\theta)^{K-l-1}\Big]\Big\} f(l,p), \quad (47)$$

$$\lambda_2 = \frac{\partial B}{\partial b}\Big|_{c=1/2,b=\theta} = \sum_{l=0}^{K} \binom{K}{l} \Big[l\theta^{l-1}(1-\theta)^{K-l} \\ -(K-l)\theta^{l}(1-\theta)^{K-l-1} \Big] (K-2l)f(l,p). \quad (48)$$

For the parameters used in the MC simulations below numerical analysis of equations (45), (47), and (48) reveals that for θ being a solution of equation (45) the eigenvalue $\lambda_2 < 0$ in the whole range $0 < p < 1/2$ while λ_1 can change sign provided that $r < r^{\star}_{PA}$, where $r^{\star}_{PA}$ is also determined numerically. Thus, the critical value of p at which the solution with $c = 1/2$ ($m = 0$) loses stability as well as the corresponding value of θ are determined from simultaneous solution of equations $B(c = 1/2, \theta) = 0$, equation (45), and $\lambda_1 = 0$, equation (47). In fact, PA predicts that for $r < r^{\star}_{PA}$ there are two critical values $p^{(FM)}_{c1,PA}$ and $p^{(FM)}_{c2,PA}$ ($p^{(FM)}_{c1,PA} < p^{(FM)}_{c2,PA}$) which approach each other with increasing r and converge at $r = r^{\star}_{PA}$, i.e., the solution with $c = 1/2$ is stable not only for $p > p^{(FM)}_{c2,PA}$, where it corresponds to a usual PM phase, but also for $0 < p < p^{(FM)}_{c1,PA}$. Apart from the fixed point with $m = 0$ for certain ranges of the parameters p, r a pair of stable fixed points of equations (43) and (44) with $|m| > 0$ exists corresponding to the FM phase. These FM fixed points can be found by solving numerically the system of equations $A(c, b) = 0$, $B(c, b) = 0$, or more directly as asymptotic solutions of the system of differential equations (43), (44), and their stability can be further verified by evaluating eigenvalues of the Jacobian matrix of the right-hand sides of equations (43) and (44) at these fixed points. Unexpectedly, PA predicts that apart from typical continuous transition from the PM to the FM phase also discontinuous transition may occur in the model, characterized by bistability of the fixed points with $|m| > 0$ and $m = 0$ over a narrow range of the parameter p. A complex phase diagram resulting from the PA is discussed in detail in Section 4.2. However, as in the case of the MFA, MC simulations of the model under study reveal that the predictions of the PA are invalid for larger fractions of the AFM interactions r due to deviation of the critical value of p for the FM transition from $p^{(FM)}_{c2,PA}$ and the occurrence of the SG-like phase.

4.2 Numerical results

Numerical investigation of the FM and SG-like transitions in the MV model on random graphs is performed in a similar way as in Section 3.2 in the case of a model on a complete graph, with the exception that averaging $[\cdot]_{av}$ is performed over different realizations of the graph with a given degree distribution $P(k)$ and of the distribution of the exchange integrals $P(J_{ij})$ (6). As an example, the MV model with $J = 1$ on RRGs with $K = 13$ and ERGs with $\langle k \rangle = 13$ is studied for different fraction of AFM interactions r and results are compared with theoretical predictions of Sections 4.1.2 and 4.1.3. For small r the model exhibits continuous FM transition at $p = p^{(FM)}_{c,MC}$ which decreases with increasing r, while for larger r it

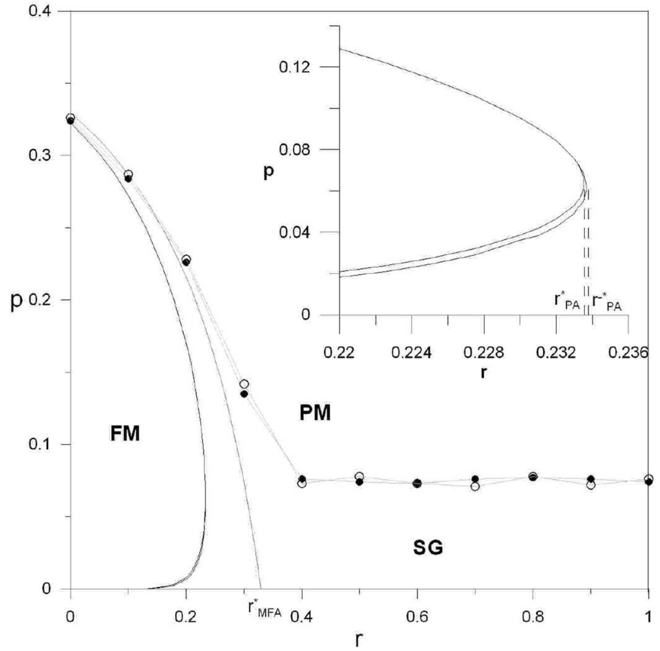

Fig. 3. Phase diagram for the MV model on a RRG with $K = 13$. Symbols: results of MC simulations, critical lines $p^{(FM)}_{c,MC}$ (•) for the FM transition, $p^{(SG)}_{c,MC}$ (○) for the SG-like transition, lines are guides to the eyes. Black solid lines: result of the PA, shown are $p^{(FM)}_{c1,PA}$, $p^{(FM)}_{c2,PA}$, the lower and upper borders of stability of the PM solution and $\bar{p}^{(FM)}_{c1,PA}$, $\bar{p}^{(FM)}_{c2,PA}$ the lower and upper borders of stability of the FM solution, obtained from the stability analysis of the fixed points of equations (43) and (44), as explained in Section 4.1.2. Gray solid line: result of the MFA, critical line for the FM transition $p^{(FM)}_{c,MFA}$ from equation (36). Inset: result of the PA, magnification of the "cusps" of the borders of stability of the PM and FM phases.

exhibits SG-like transition at $p = p^{(SG)}_{c,MC} \approx 0.075$ which is almost independent of r. The phase diagram obtained from MC simulations is shown in Figure 3; the PM-FM and PM-SG phase borders for the models on the RRG and on the ERG overlap with good accuracy. Moreover, the phase diagram in Figure 3 qualitatively resembles that for the equilibrium DSG model on ERGs [26], with the parameter p playing the role of temperature. It is interesting to compare this phase diagram with that for another version of the MV model (with anticonformists) on random graphs, which also exhibits SG-like transition [12]. In that model exchange interactions in general are not symmetric, thus the model does not have any Hamiltonian; simultaneously, the phase diagram is qualitatively different, with only small area occupied by the SG-like phase (see Appendix A). This, as well as the overall similarity of the phase diagram in Figure 3 to that for the equilibrium DSG model again enables one to speculate that in the MV model on random graphs studied in this paper the SG-like transition occurs since the spin configuration approaches that corresponding to a (possibly local) minimum of the energetic landscape given by the Hamiltonian (1).

Comparison between results of MC simulations and predictions obtained from the MFA and PA is summarized in Figure 3. For small r the MFA predicts occurrence of the FM transition at $p^{(FM)}_{c,MFA}$ given by equation (36), which is usually below $p^{(FM)}_{c,MC}$ obtained from MC simulations, decreases faster with r and reaches zero at $r = r^{\star}_{MFA} \approx 0.3289\ldots$, where in MC simulations the model still exhibits FM transition at $p^{(FM)}_{c,MC} > 0$. Unexpectedly, predictions of the PA are not only quantitatively, but even qualitatively worse. First, for $r < r^{\star}_{PA} = 0.2335\ldots$ the PA predicts two critical lines at which the PM phase becomes unstable: the instability occurs at $p = p^{(FM)}_{c2,PA}$ as p is lowered or at $p = p^{(FM)}_{c1,PA} < p^{(FM)}_{c2,PA}$ as p is increased from zero. The value of $p^{(FM)}_{c2,PA}$ is significantly smaller than $p^{(FM)}_{c,MC}$ and even smaller than $p^{(FM)}_{c,MFA}$ and decreases with r, and the value of $p^{(FM)}_{c1,PA}$ increases with r so that both curves meet at $r = r^{\star}_{PA}$ and annihilate after forming a "cusp" visible in the inset in Figure 3, and for $r > r^{\star}_{PA}$ the PM phase is stable for all $0 < p < 1/2$. For $r < 0.23325\ldots$ the PA predicts second-order FM transition with decreasing p at $p = p^{(FM)}_{c2,PA}$. However, in a narrow interval $0.23325\ldots < r < r^{\star}_{PA}$ first-order transition is predicted since for increasing p the FM phase loses stability at $p = \bar{p}^{(FM)}_{c2,PA} > p^{(FM)}_{c2,PA}$ and there is a narrow region of bistability of the PM and FM phases for $p^{(FM)}_{c2,PA} < p < \bar{p}^{(FM)}_{c2,PA}$. Moreover, for $r < r^{\star}_{PA}$ and for decreasing p the FM phase loses stability at $p = \bar{p}^{(FM)}_{c1,PA} < p^{(FM)}_{c1,PA}$, thus the predicted PM-FM transition at low p is also first-order. The borders of stability of the FM phase extend even beyond those for the PM phase and meet at $\bar{r}^{\star}_{PA} \approx 0.2337\ldots$, where they form another "cusp" visible in the inset in Figure 3. Hence, for $r^{\star}_{PA} < r < \bar{r}^{\star}_{PA}$ and $\bar{p}^{(FM)}_{c1,PA} < p < \bar{p}^{(FM)}_{c2,PA}$ there is still bistability of the PM and FM phases which, however, is not related to the PM-FM phase transition. In this region of the space of parameters stable fixed points of the system of equations (43,44) with $m = 0$ and $|m| > 0$ coexist, corresponding to the PM and FM phases, respectively, but the PM fixed point never loses stability as p is decreased.

The complex phase diagram following from the PA is not observed in MC simulations. This may be related to violation of assumptions of the PA, e.g., that about independence of the sign of the exchange integral and the relative orientation of the spins at the ends of the corresponding edge. Moreover, the PM-FM phase borders predicted by the PA are located significantly below the border obtained from simulations, and for larger r the SG-like transition occurs in MC simulations at relatively high critical value $p = p^{(SG)}_{c,MC}$ (Fig. 3), which cannot be analyzed in the framework of the PA. It is tempting to speculate that the predicted critical (spinodal) lines for the first-order PM-FM transition at $p = \bar{p}^{(FM)}_{c1,PA}$ and $p = p^{(FM)}_{c1,PA}$ are related to the de Almeida – Thouless line determining the border of stability of the replica-symmetric solution for the equilibrium SK model [21–23,25] which occurs also in the equilibrium DSG model. However, the de Almeida – Thouless instability of the FM phase leads to the occurrence of the re-entrant SG phase characterized by non-zero magnetization rather than the PM phase predicted by the PA in the case of the MV model on random graphs.

5 Summary and conclusions

FM and SG-like transitions in spin models in contact with two thermal baths with different temperatures were investigated, mainly in the MV model being a nonequilibrium counterpart of the SK and DSG models on complete and homogeneous or weakly heterogeneous random graphs, respectively. In particular, the MV model is often used as a toy model for the opinion formation. The models under study possessed the Hamiltonian (1) but in general were out of thermal equilibrium due to non-zero heat flux through the system. Nevertheless, MC simulations showed that apart from the well-known FM transition, corresponding to the transition to consensus in models for social behavior, the models under study exhibit SG-like transition for a wide range of parameters controlling the distribution of the exchange interactions, in particular the fraction of AFM interactions. Theoretical investigations were performed using the MFA and the PA which was extended to the case of models on random graphs with both FM and AFM interactions between two-state spins or agents. It was shown that the FM transition in all models under study could be qualitatively described in the framework of the MFA, though full quantitative agreement with results of MC simulations was not achieved, in particular in the MV model on RRGs or ERGs. Unexpectedly, in the latter case predictions of the apparently more elaborate PA showed more significant quantitative or even qualitative differences with results of MC simulations than those of the MFA. SG-like transition was investigated theoretically only in the model on a complete graph, using the modified TAP equation which, however, substantially overestimates the critical value of the parameter p for the occurrence of this transition. It is interesting to note that the phase diagrams for the MV model on complete and random graphs obtained from MC simulations show obvious resemblance to the diagrams for the corresponding equilibrium SK and DSG spin models. This suggests that the mechanism leading to the appearance of the SG-like phase in the nonequilibrium models under study may be similar to that in nonequilibrium models, e.g., the approach of the spin configuration to that corresponding to a local or global minimum of the energetic landscape determined by the Hamiltonian (1). More insight in the character of the FM and SG-like transitions in the models under study could probably be obtained by evaluating critical exponents, as in the case without AFM interactions [5], which is beyond the scope of this paper.

Investigation of the SG transition in nonequilibrium systems is not a completely new subject. In fact, much work using the concept of effective Hamiltonian was done for models named generally nonequilibrium SG models on regular lattices [30–32,34], although the main interest there was in the occurrence of the FM or AFM phase, or for neural networks in which the SG phase can occur [38]. However, in the context of models for the opinion formation and models on random graphs this problem has not received much attention. It should be emphasized that in the models studied in this paper, in particular the MV model on RRGs or ERGs with symmetric exchange interactions (6), the SG phase can occur in the range of the fraction r of the AFM (negative) exchange integrals which is realistic in the models for social interactions; e.g., in the case of the political stage divided between two parties each person can easily interact both with the followers of the same or another party and thus tend to follow or object their opinions. This is in contrast with the model of reference [12] in which the SG phase occurs only in an unrealistic society consisting almost exclusively of anticonformists (see Appendix A). This suggests that in real societies apart from the spectacular FM transition to consensus also a more subtle transition to the SG-like phase may occur, characterized by local rather than global ordering of agents' opinions.

Author contribution statement

The author (A.K.) is responsible for the whole content of the paper.

Appendix A: Comparison with the majority vote model with anticonformists on networks

In reference [12] a related MV model on random graphs was considered which also exhibits both FM and SG-like phase transitions. In that model a fraction r of nodes is occupied by agents called anticonformists who change their orientation with probability $1-p$ if the majority of their neighbors has the same opinion, with probability p if the majority has opposite opinion, and with probability $1/2$ if there is balance of opinions in their neighborhood; and the remaining fraction $1-r$ of nodes is occupied by conformists which follow the usual majority rule described in Section 1. Thus, for anticonformists the spin flip rate is

$$w_i^{(a)}(\mathbf{s}) = \frac{1}{2}\left[1 + (1-2p)s_i \text{sign}\left(I_i\right)\right], \qquad \text{(A.1)}$$

while for conformists the spin flip rate $w_i^{(c)}(\mathbf{s})$ is still given by equation (3), where in both cases the effective field is $I_i = J\sum_{j\in \text{nn}_i} s_j$, $J > 0$. Due to the difference in the spin flip rates conformists behave effectively as if all their interactions with the neighboring agents had FM character and anticonformists as if they had AFM character; moreover, interactions need not be symmetric, thus in contrast with the MV model on random graphs discussed in this paper the model from reference [12] has no underlying Hamiltonian at all and hence cannot be described as the Ising model in contact with two thermal baths with different temperatures.

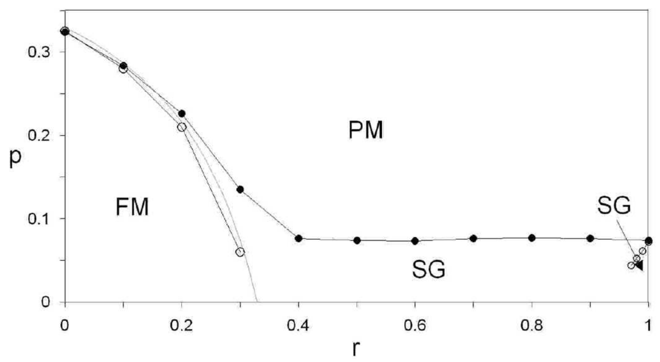

Fig. A.1. Comparison of phase diagrams obtained from MC simulations for the MV model considered in this paper (•) and for the MV model with anticonformists from reference [20] (○), both on RRG with $K = 13$, gray solid line shows prediction of the MFA, equations (36) and (A.6) (predictions overlap with each other).

Let us consider the model of reference [12] on a RRG with K edges attached to each node. Then the MFA for this model can be derived in a similar way as in Section 4.1.2. Let us denote the magnetization (average value of spins) of nodes containing conformist and anticonformist agents by $m^{(c)}$ and $m^{(a)}$, respectively; then the model magnetization is $m = rm^{(a)} + (1-r)m^{(c)}$, the probability that a conformist (anticonformist) has a given orientation is $\Pr(s_i = \pm 1|\, c) = (1 \pm m^{(c)})/2$ ($\Pr(s_i = \pm 1|\, a) = (1 \pm m^{(a)})/2$) and the probability that a randomly selected agent has a given orientation is $\Pr(s_i = \pm 1) = r\Pr(s_i = \pm 1|\, a) + (1-r)\Pr(s_i = \pm 1|\, a) = (1 \pm m)/2$. The equations for the magnetizations $m^{(c)}$, $m^{(a)}$ in the MFA have the same overall form as in equation (30)

$$\frac{dm_i^{(c,a)}}{dt} = -2\langle s_i w_i^{(c,a)}(\mathbf{s})\rangle = -m_i^{(c,a)} \pm (1-2p)\langle \text{sign}(I_i)\rangle, \qquad \text{(A.2)}$$

where $\langle \text{sign}(I_i)\rangle$ may be expressed in terms of the magnetizations $m^{(c)}$, $m^{(a)}$ using equation (31) and (33) with

$$\Pi = \Pr(s_i = \pm 1) = \frac{1+m}{2}. \qquad \text{(A.3)}$$

Under similar assumptions as in Section 4.1.2 for large K the equations for the magnetizations can be written as

$$\frac{dm^{(c,a)}}{dt} = -m^{(c,a)} \pm (1-2p)\text{erf}\left(\sqrt{\frac{K}{2}}m\right). \qquad \text{(A.4)}$$

Finally, multiplying the equation for $m^{(c)}$ ($m^{(a)}$) by $1-r$ (r) and summing the results yields the following equation for the magnetization in the MFA,

$$\frac{dm}{dt} = -m + (1-2r)(1-2p)\text{erf}\left(\sqrt{\frac{K}{2}}m\right). \tag{A.5}$$

Equation (A.5) has a solution $m = 0$, corresponding to the PM phase, which loses stability at

$$p_{c,MFA}^{(FM)} = \frac{1}{2}\left[1 - \frac{\sqrt{2\pi}}{2(1-2r)}\frac{1}{\sqrt{K}}\right], \tag{A.6}$$

and for $p < p_{c,MFA}^{(FM)}$ a pair of stable FM solutions with $|m| > 0$ occurs as a result of a second-order phase transition. The result of equation (A.6) is the same as that of equation (36) for the MV model on RRGs with a fraction r of AFM and $1-r$ of FM symmetric exchange interactions considered in this paper. Thus, according to the MFA transition from the PM to the FM phase for given r occurs in both models at the same critical value of the parameter p.

The latter prediction is confirmed qualitatively by MC simulations, which can be seen in Figure A.1, where the borders between the PM and FM phases for the MV model on RRGs considered in this paper and that with anticonformists deviate noticeably from each other for increasing r, but in general follow the trend predicted by the MFA. However, this is not the case for the border between the PM and SG-like phases. On the phase diagram for the MV model with symmetric exchange interactions the critical value $p_{c,MC}^{(SG)} \approx 0.075$ for the SG transition is independent of r, and there is a tricritical point where the PM-FM and PM-SG phase borders meet. In contrast, in the model of reference [12] the SG-like phase is observed only in a small interval of $r \to 1$ and $p_{c,MC}^{(SG)}$ quickly decreases to zero as r is slightly lowered. Only for $r = 1$, when the model of reference [12] consists only of anticonformists interacting via symmetric AFM exchange integrals the SG-like transition occurs at the same critical value $p_{c,MC}^{(SG)} \approx 0.075$ in both models. It should be recollected that in the latter case numerical investigation of the two-spin correlation function shows that the spins in neighboring nodes tend to have opposite orientations [12], thus the model exhibits local AFM ordering which provides additional evidence for the occurrence of the SG-like transition.

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