



${}^5\text{He}({}^3\text{He}, {}^4\text{He}){}^4\text{He}$ as a three-body reaction via a continuum resonance in the $n+{}^4\text{He}$ system

A tribute to Mahir S. Hussein

Stefan Typel^{1,2,a}

¹ Technische Universität Darmstadt, Fachbereich Physik, Institut für Kernphysik, Schlossgartenstraße 9, 64289 Darmstadt, Germany

² GSI Helmholtzzentrum für Schwerionenforschung GmbH, Theorie, Planckstraße 1, 64291 Darmstadt, Germany

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Abstract The ${}^5\text{He}({}^3\text{He}, {}^4\text{He}){}^4\text{He}$ reaction involving the unstable ${}^5\text{He}$ nucleus is a possible process in primordial nucleosynthesis to convert ${}^3\text{He}$ into ${}^4\text{He}$ in a neutron transfer reaction. Since experimental data for the reaction cross section are not available, a theoretical prediction is needed to estimate the relevance of this process in comparison to other reactions, e.g., ${}^3\text{He}({}^2\text{H}, p){}^4\text{He}$ or ${}^3\text{H}({}^2\text{H}, n){}^4\text{He}$. In this work the cross section and the Maxwellian-averaged transition rate of the ${}^5\text{He}({}^3\text{He}, {}^4\text{He}){}^4\text{He}$ reaction are calculated using a post-form distorted-wave Born approximation in a simple cluster model. For that purpose the reaction is treated as a genuine process with three particles, $n + {}^4\text{He} + {}^3\text{He}$, in the entrance channel proceeding through the $3/2^-$ resonance in the $n - {}^4\text{He}$ scattering continuum.

1 Introduction

The study of nuclear reactions was one of the most prominent topics in the work of Mahir S. Hussein where he made seminal contributions to the theoretical description of various processes. In recent years, he was interested in particular in three- and four-body systems, partly connected to applications in astrophysics or to halo nuclei, see, e.g., [1–8].

Nuclear reactions with three particles in the initial state have been investigated in the astrophysical context of nucleosynthesis for many years. The most prominent example is probably the triple- α reaction that proceeds through a sharp 0^+ state at 7.654 MeV excitation energy in ${}^{12}\text{C}$, the so-called Hoyle state, just above the 3α breakup threshold [9–12]. It is essential for the nucleosynthesis of carbon in red giant stars that burn ${}^4\text{He}$. For recent theoretical works see, e.g., [13, 14] and references therein.

Other three-body reactions might be relevant in certain astrophysical scenarios and environments, e.g., in primordial nucleosynthesis. In many cases, the reactants show a strong cluster structure and the reactions proceed at rather low energies. In general, such type of reactions are strongly suppressed as compared to reactions with two nuclei in the initial state because of the tiny likelihood of finding three particles close together, in particular if they are all charged and repel each other due to the Coulomb interaction. Only very high densities or a resonant enhancement of the cross section can help to increase the reaction rates substantially.

A special situation occurs if at least one of the particles is uncharged, i.e., a neutron. However, free neutrons are unstable and decay with a mean lifetime of 879 ± 0.6 s [15]. Thus they are usually not available as participants in a three-body reaction. Only during Big Bang nucleosynthesis within the first few minutes of the Universe, a considerable amount of neutrons was available, see, e.g., [16, 17]. Under these conditions they can form unstable ${}^5\text{He}$ nuclei with the already synthesized ${}^4\text{He}$ nuclei. The ${}^5\text{He}$ ground state is a $3/2^-$ resonance of 0.648 MeV width at 0.735 MeV excitation energy above the $n - \alpha$ threshold [18]. Then the ${}^5\text{He}({}^3\text{He}, {}^4\text{He}){}^4\text{He}$ reaction could be a possible path to synthesize ${}^4\text{He}$ in Big Bang nucleosynthesis via a neutron transfer reaction. Hence, an estimate of the reaction rate is of interest. It requires a theoretical calculation since a direct measurement of the cross section is not possible in the laboratory. Only an indirect experimental approach like the Trojan-Horse method using the ${}^9\text{Be}({}^3\text{He}, {}^2\text{He}){}^4\text{He}$ reaction [19, 20] could help to constrain the cross section of this reaction.

Considering the above issues, a simple attempt is made in this work to calculate the rate of the ${}^5\text{He}({}^3\text{He}, {}^4\text{He}){}^4\text{He}$ reaction treating the ${}^5\text{He}$ system as a resonance in the $n+{}^4\text{He}$ continuum. A post-form distorted-wave Born approximation

^a e-mail: stypel@ikp.tu-darmstadt.de (corresponding author)

(DWBA) is used to find the relevant T-matrix element that enters in the calculation of the transition rate of this neutron-transfer reaction. This approach shares some similarities with the reaction theory applied in the Trojan-Horse method where transfer reactions to the continuum are employed to extract cross sections of a certain sub-process in an indirect approach [21–24]. The results can serve as an input to reaction network calculations for primordial nucleosynthesis.

This work is structured as follows. In Sect. 2 the basic definitions of astrophysical reaction rates and their connection to cross sections is presented with particular emphasis on the differences between reactions with two or three particles in the entrance channel. The connection of transition rates and cross sections with the T-matrix elements describing the process is established for general cases in Sect. 3. The T-matrix elements for the reaction $n + {}^4\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + {}^4\text{He}$ are calculated in post-from DWBA using appropriate cluster and scattering wave functions in Sect. 4. Also the notation for spatial coordinates and momenta is introduced there, numerical details for the representation of the scattering wave functions and the calculation of integrals are given, and selection rules due to angular momentum coupling are discussed. The nuclear potentials to calculate the scattering wave functions and corresponding phase shifts are given in Sect. 5. Results for the cross section and the astrophysical S factor of the ${}^5\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + {}^4\text{He}$ reaction are presented in Sect. 6 as well as the Maxwellian-averaged transition rate that is needed to calculate the astrophysical reaction rate of the $n + {}^4\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + {}^4\text{He}$ reaction. Finally, conclusions are given in Sect. 7.

2 Reaction rates in astrophysics

In an astrophysical environment, participants of nuclear reactions move with various velocities which follow in the non-degenerate case a Maxwellian distribution that is determined by the temperature T [25,26]. For a reaction



with two particles in the initial state, the reaction rate is given by

$$R_{a+b \rightarrow c+d+\dots}(T) = \frac{n_a n_b}{N_{id}!} \langle \sigma v \rangle_{a+b \rightarrow c+d+\dots} \tag{2}$$

with densities n_a and n_b of the particles a and b with masses m_a and m_b . N_{id} is the number of identical particles in the entrance channel, i.e., $N_{id}! = 1 + \delta_{ab}$. The last factor is the so-called Maxwellian-averaged cross section (MACS)

$$\begin{aligned} \langle \sigma v \rangle_{a+b \rightarrow c+d+\dots} &= \int \frac{d^3 P_{ab}}{N_{ab}} \exp\left(-\frac{E_{ab}}{k_B T}\right) w_{a+b \rightarrow c+d+\dots}(P_{ab}) \end{aligned} \tag{3}$$

with the Boltzmann constant k_B . It is given by an integral over all momenta

$$\mathbf{P}_{ab} = \mu_{ab} \left(\frac{\mathbf{p}_a}{m_a} - \frac{\mathbf{p}_b}{m_b} \right) \tag{4}$$

of relative motion with energies $E_{ab} = P_{ab}^2 / (2\mu_{ab})$ where $\mu_{ab} = m_a m_b / (m_a + m_b)$ denotes the reduced mass and

$$N_{ab} = \int d^3 P_{ab} \exp\left(-\frac{E_{ab}}{k_B T}\right) = (2\pi \mu_{ab} k_B T)^{3/2} \tag{5}$$

is a normalization factor. The total transition rate

$$w_{a+b \rightarrow c+d+\dots}(P_{ab}) = \sigma_{a+b \rightarrow c+d+\dots}(E_{ab}) v_{ab} \tag{6}$$

is a product of the total reaction cross section $\sigma_{a+b \rightarrow c+d+\dots}$ and the relative velocity $v_{ab} = P_{ab} / \mu_{ab}$ of the particles. The MACS can be transformed easily to the often used form

$$\begin{aligned} \langle \sigma v \rangle_{a+b \rightarrow c+d+\dots} &= \sqrt{\frac{8}{\pi \mu_{ab}}} (k_B T)^{-3/2} \\ &\int dE_{ab} E_{ab} \exp\left(-\frac{E_{ab}}{k_B T}\right) \sigma_{a+b \rightarrow c+d+\dots}(E_{ab}) \end{aligned} \tag{7}$$

with a simple energy integral. This is possible because the total reaction cross section $\sigma_{a+b \rightarrow c+d+\dots}$ depends only on the energy E_{ab} or modulus P_{ab} of the momentum \mathbf{P}_{ab} .

The situation is different for a reaction



with three particles in the entrance channel because a single energy is not sufficient to characterize the kinematics in the initial state. The relative motion of the three particles can be specified by two relative momenta, e.g., \mathbf{P}_{ab} , see Eq. (4), and

$$\mathbf{P}_{(ab)c} = \mu_{(ab)c} \left(\frac{\mathbf{p}_a + \mathbf{p}_b}{m_a + m_b} - \frac{\mathbf{p}_c}{m_c} \right) \tag{9}$$

with $\mu_{(ab)c} = (m_a + m_b)m_c / (m_a + m_b + m_c)$. Then the equivalent of the MACS, Eq. (3), is given by the Maxwellian-averaged transition rate

$$\begin{aligned} \langle w \rangle_{a+b+c \rightarrow d+e+\dots} &= \int \frac{d^3 P_{ab}}{(2\pi \mu_{ab} k_B T)^{3/2}} \frac{d^3 P_{(ab)c}}{(2\pi \mu_{(ab)c} k_B T)^{3/2}} \\ &\exp\left(-\frac{E_{ab} + E_{(ab)c}}{k_B T}\right) w_{a+b+c \rightarrow d+e+\dots}(\mathbf{P}_{ab}, \mathbf{P}_{(ab)c}) \end{aligned} \tag{10}$$

where $E_{(ab)c} = P_{(ab)c}^2 / (2\mu_{(ab)c})$ and $w_{a+b+c \rightarrow d+e+\dots}$ is the corresponding total transition rate which will be defined in Eq. (26) below. It does not really depend on six kinematics variables but only on three due to rotational symmetries: the moduli of the momenta, P_{ab} and $P_{(ab)c}$, and a single angle, e.g., the angle between \mathbf{P}_{ab} and $\mathbf{P}_{(ab)c}$. Thus the six-dimensional integral in (10) can be reduced effectively to an integration over only three variables. For a given total energy

in the entrance channel, only two independent variables are left and the state of the entrance channel can be visualized with the help of a Dalitz plot [27, 28]. Finally, the astrophysical reaction rate assumes the form

$$R_{a+b+c \rightarrow d+e+\dots}(T) = \frac{n_a n_b n_c}{N_{id}!} \langle w \rangle_{a+b+c \rightarrow d+e+\dots} \quad (11)$$

with a product of three particle densities and $N_{id}! = 1 + \delta_{ab} + \delta_{ac} + \delta_{bc} + 2\delta_{ab}\delta_{ac}$ in analogy to Eq. (2).

It is worthwhile to check the units of the various quantities X introduced above. They will be denoted $[X]$ in the following. The symbols E and L are used for the energy and length unit and c is the speed of light. Then $[P_{ab}] = [P_{(ab)c}] = E/c$, $[\mu_{ab}] = [\mu_{(ab)c}] = E/c^2$ and for the reaction (1) one has $[\sigma_{a+b \rightarrow c+d+\dots}] = L^2$ and hence

$$[(\sigma v)_{a+b \rightarrow c+d+\dots}] = [w_{a+b \rightarrow c+d+\dots}] = L^2 c. \quad (12)$$

With the densities $[n_a] = [n_b] = L^{-3}$ expressed as particles per volume, the reaction rate (2) carries the unit

$$[R_{a+b \rightarrow c+d+\dots}] = L^{-4} c \quad (13)$$

i.e., reactions per volume (L^3) and time (L/c). In case of reaction (8) the units are

$$[w_{a+b+c \rightarrow d+e+\dots}] = L^5 c \quad (14)$$

with an additional factor of L^3 and again

$$[R_{a+b+c \rightarrow d+e+\dots}] = L^{-4} c \quad (15)$$

as for reaction (1). In astrophysical applications, particle densities are sometimes expressed as amount of substance per volume instead of particles per volume. Then it is convenient to multiply the MACS with the Avogadro constant $N_A \approx 6.022 \cdot 10^{23} \text{ mol}^{-1}$ [15] and give the quantity $N_A (\sigma v)_{a+b \rightarrow c+d+\dots}$ in tables with numerical data. Similarly, the quantity $N_A^2 \langle w_{a+b+c \rightarrow d+e+\dots} \rangle$ is suitable in the tabulation of data for a process with three particles in the entrance channel.

3 Transition rates and cross sections

In the previous section the connection between the temperature dependent reaction rates for astrophysical applications and the total transition rates of reactions with two or three particles in the initial state was presented. For that purpose it was not necessary to specify the final states in detail, e.g., the number of ejectiles and their momenta. In the actual calculation of transition rates, however, this has to be taken into account. In the present work, only reactions of the type

$$a + b \rightarrow c + d + e \quad (16)$$

and its inverse

$$c + d + e \rightarrow a + b \quad (17)$$

will be covered i.e., two-body \rightarrow three-body and three-body \rightarrow two-body reactions. Besides the kinematic quantities, also spin degrees of freedom will be considered. The total angular momentum of a particle i will be denoted J_i and its projection M_i .

3.1 Two-body \rightarrow three-body reaction

The differential transition rate of reaction (16) has the general form

$$\begin{aligned} dw_{a+b \rightarrow c+d+e}(\mathbf{P}_{ab}) &= \frac{2\pi}{\hbar} \frac{1}{(2J_a + 1)(2J_b + 1)} \sum_{M_a M_b} \sum_{M_c M_d M_e} \\ &\int \frac{d^3 P_{de}}{(2\pi\hbar)^3} \frac{d^3 P_{c(de)}}{(2\pi\hbar)^3} |T_{a+b \rightarrow c+d+e}|^2 \\ &\times \delta(E_{ab} - E_{de} - E_{c(de)} + Q_{a+b \rightarrow c+d+e}) \end{aligned} \quad (18)$$

with obvious notation of the momenta and energies [29]. It contains the usual averaging over initial spin projections and summation over final spin projections. The Q value

$$Q_{a+b \rightarrow c+d+e} = m_a + m_b - m_c - m_d - m_e \quad (19)$$

contains the rest masses of the particles and appears in the δ function for energy conservation. The formulation with relative momenta guarantees already the conservation of the total momentum. The expression (18) is written in a form where the scattering wave functions are normalized to plane-wave states as $\exp(i\mathbf{P}_{ij} \cdot \mathbf{R}_{ij}/\hbar) + \dots$ without any additional prefactor. The information on the reaction process is contained in the T-matrix element $T_{a+b \rightarrow c+d+e}$ which is the central quantity to be calculated, see section 4.

Since there are three particles in the final state, there are integrations over two relative momenta \mathbf{P}_{de} and $\mathbf{P}_{c(de)}$. Energy conservation reduces the number of independent variables in the final state from six to five. Using

$$d^3 P_{de} = d\Omega_{de} P_{de}^2 dP_{de} = d\Omega_{de} \mu_{de} P_{de} dE_{de} \quad (20)$$

(and similar for $d^3 P_{c(de)}$) the integration over $E_{c(de)}$ can be performed immediately. Then the fully differential transition rate assumes the form

$$\begin{aligned} \frac{dw_{a+b \rightarrow c+d+e}}{dE_{de} d\Omega_{de} \Omega_{c(de)}}(\mathbf{P}_{ab}) &= \frac{2\pi}{\hbar} \frac{\mu_{de} P_{de}}{(2\pi\hbar)^3} \frac{\mu_{c(de)} P_{c(de)}}{(2\pi\hbar)^3} \frac{1}{(2J_a + 1)(2J_b + 1)} \\ &\times \sum_{M_a M_b} \sum_{M_c M_d M_e} |T_{a+b \rightarrow c+d+e}|^2. \end{aligned} \quad (21)$$

An integration over the energy E_{de} and the angles Ω_{de} and $\Omega_{c(de)}$ gives the total transition rate

$$w_{a+b \rightarrow c+d+e}(\mathbf{P}_{ab}) = \int dE_{de} d\Omega_{de} \Omega_{c(de)} \frac{dw_{a+b \rightarrow c+d+e}}{dE_{de}d\Omega_{de}\Omega_{c(de)}}(\mathbf{P}_{ab}). \quad (22)$$

After a division by the the flux factor (= relative velocity for the plane-wave normalization used here) $v_{ab} = P_{ab}/\mu_{ab}$ the total reaction cross section

$$\sigma_{a+b \rightarrow c+d+e}(\mathbf{P}_{ab}) = \frac{\mu_{ab}}{P_{ab}} w_{a+b \rightarrow c+d+e}(\mathbf{P}_{ab}) \quad (23)$$

is obtained. Since the direction of the momentum \mathbf{P}_{ab} is usually chosen to be parallel to the z -axis, only a dependence on the modulus P_{ab} or the energy E_{ab} remains.

3.2 Three-body \rightarrow two-body reaction

The transition rate of the reaction (17) with three particles in the entrance channel is given by

$$dw_{c+d+e \rightarrow a+b}(\mathbf{P}_{de}, \mathbf{P}_{c(de)}) = \frac{2\pi}{\hbar} \frac{1}{(2J_c + 1)(2J_d + 1)(2J_e + 1)} \sum_{M_c M_d M_e} \sum_{M_a M_b} \int \frac{d^3 P_{ab}}{(2\pi\hbar)^3} |T_{c+d+e \rightarrow a+b}|^2 \times \delta(E_{de} + E_{c(de)} - E_{ab} + Q_{c+d+e \rightarrow a+b}) \quad (24)$$

with the Q value $Q_{c+d+e \rightarrow a+b} = -Q_{a+b \rightarrow c+d+e}$ in close analogy to Eq. (18) keeping the full dependence on the momenta \mathbf{P}_{de} and $\mathbf{P}_{c(de)}$ in the initial state. An integration over E_{ab} gives the differential transition rate

$$\frac{dw_{c+d+e \rightarrow a+b}}{d\Omega_{ab}}(\mathbf{P}_{de}, \mathbf{P}_{c(de)}) = \frac{2\pi}{\hbar} \frac{\mu_{ab} P_{ab}}{(2\pi\hbar)^3} \frac{1}{(2J_c + 1)(2J_d + 1)(2J_e + 1)} \sum_{M_c M_d M_e} \sum_{M_a M_b} |T_{c+d+e \rightarrow a+b}|^2 \quad (25)$$

and a further integration over Ω_{ab} the total transition rate

$$w_{c+d+e \rightarrow a+b}(\mathbf{P}_{de}, \mathbf{P}_{c(de)}) = \int d\Omega_{ab} \frac{dw_{c+d+e \rightarrow a+b}}{d\Omega_{ab}}(\mathbf{P}_{de}, \mathbf{P}_{c(de)}) \quad (26)$$

that enters in the integral (10) to obtain the Maxwellian-averaged transition rate (note the change of the particle indices).

It is possible to establish a theorem of detailed balance for the differential transition rates

$$(2J_a + 1)(2J_b + 1)\mu_{ab} P_{ab} \frac{dw_{a+b \rightarrow c+d+e}}{dE_{cd}d\Omega_{cd}\Omega_{c(de)}}(\mathbf{P}_{ab}) = (2J_c + 1)(2J_d + 1)(2J_e + 1) \frac{\mu_{de} P_{de} \mu_{c(de)} P_{c(de)}}{(2\pi\hbar)^3} \frac{dw_{c+d+e \rightarrow a+b}}{d\Omega_{ab}}(\mathbf{P}_{de}, \mathbf{P}_{c(de)}) \quad (27)$$

because

$$|T_{a+b \rightarrow c+d+e}|^2 = |T_{c+d+e \rightarrow a+b}|^2. \quad (28)$$

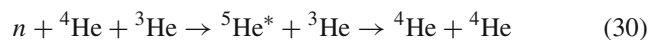
There is no need to define a total reaction cross section $\sigma_{c+d+e \rightarrow a+b}(\mathbf{P}_{de}, \mathbf{P}_{c(de)})$ for the calculation of the Maxwellian-averaged transition rate (10) and, in contrast to reaction (16), it is not immediately obvious how to define the appropriate flux factor for the conversion. A division of $w_{c+d+e \rightarrow a+b}$ by a velocity would lead to a quantity with units L^5 , cf. Eq. (14), and not to an area. However, a cross section can be introduced as

$$\sigma_{c+(d+e) \rightarrow a+b}(\mathbf{P}_{c(de)}) = \frac{\mu_{c(de)}}{P_{c(de)}} \int \frac{d^3 P_{de}}{(2\pi\hbar)^3} w_{c+d+e \rightarrow a+b}(\mathbf{P}_{de}, \mathbf{P}_{c(de)}) \quad (29)$$

by integrating over all possible momenta in the $d + e$ subsystem and by dividing by the flux factor for the relative motion $c - (de)$. It has the unit of an area as required.

4 T-matrix elements and wave functions

The T-matrix element contains the full information of the reaction. It can be calculated using the appropriate potentials with the wave functions in the initial and final states that are found by solving the many-body problem. In general, exact solutions are not known and approximations have to be used. The reaction ${}^5\text{He}({}^3\text{He}, {}^4\text{He}){}^4\text{He}$ is actually treated as a process



where the transition proceeds via the $3/2^-$ resonance in ${}^5\text{He}$ with a resonance energy $E_{n\alpha'}^{\text{res}} = 0.735$ MeV above the $n + {}^4\text{He}$ threshold. The symbols α, t and n will be used to identify ${}^4\text{He}, {}^3\text{He}$ and the neutron. Then the T-matrix element can be written in a symbolic way without explicit angular momentum coupling in the usual form of a transfer reaction

$$T_{n+\alpha+t \rightarrow \alpha+\alpha} = \langle \Phi_\alpha \Phi_{\alpha'} \chi_{\alpha\alpha'}^{(-)}(\mathbf{P}_{\alpha\alpha'}) | W | \Phi_t \Phi_{n\alpha'}^{(+)}(\mathbf{P}_{n\alpha'}) \chi_{t(n\alpha')}^{(+)}(\mathbf{P}_{t(n\alpha')}) \rangle \quad (31)$$

in distorted-wave Born approximation with a transition potential W , and distorted waves $\chi_{\alpha\alpha'}^{(-)}$ and $\chi_{t(n\alpha')}^{(+)}$ for the relative motion in the ${}^4\text{He}-{}^4\text{He}$ and ${}^3\text{He}-{}^5\text{He}$ systems, respectively. $\Phi_\alpha, \Phi_{\alpha'}$, and Φ_t are the many-body wave functions of the clusters. The main difference as compared to the description of a neutron transfer reaction to a bound state is that $\Phi_{n\alpha'}^{(+)}$ is the many-body wave function of the ${}^5\text{He}$ system in a continuum state. The T-matrix element is formulated in relative coordinates leaving out the common cm motion because it does not affect the reaction process.

The transition potential can be given as

$$W = V_{t(n\alpha')} - U_{t(n\alpha')} \tag{32}$$

in prior form or

$$W = V_{\alpha\alpha'} - U_{\alpha\alpha'} \tag{33}$$

in post form with the full potentials $V_{t(n\alpha')}$ and $V_{\alpha\alpha'}$ depending on all the individual nucleon coordinates and the optical potentials $U_{t(n\alpha')}$ and $U_{\alpha\alpha'}$ depending only on the relative coordinates between the cm of the clusters. In the present application it is useful to use the post form of the matrix element as in the Trojan-Horse method [21]. Then we can use the approximation

$$W = V_{\alpha\alpha'} - U_{\alpha\alpha'} = V_{n\alpha'} + V_{t\alpha'} - U_{\alpha\alpha'} \approx V_{n\alpha'} \approx U_{n\alpha'} \tag{34}$$

which is frequently used in the theory of transfer reactions. This choice simplifies the calculation since the potential W depends now only on the relative coordinate $\mathbf{R}_{n\alpha'}$ between n and α' . The calculation of the T-matrix element proceeds in several steps. First, the notation for the coordinates, momenta and spins has to be introduced.

4.1 Coordinates, momenta and spins

Eight nucleons participate in the reaction under consideration. Thus for the calculation of matrix elements single-particle coordinates

$$\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, \mathbf{r}'_1, \mathbf{r}'_2, \mathbf{r}'_3, \mathbf{r}'_4 \tag{35}$$

and the corresponding conjugated momenta

$$\mathbf{p}_i = \hbar\mathbf{k}_i = M\dot{\mathbf{r}}_i \quad \mathbf{p}'_i = \hbar\mathbf{k}'_i = M\dot{\mathbf{r}}'_i \tag{36}$$

are introduced for all nucleons using an average nucleon mass M . Furthermore the cm coordinates

$$\mathbf{R}_t = \frac{1}{3}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) \tag{37}$$

of the ^3He nucleus,

$$\mathbf{R}_\alpha = \frac{1}{4}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4) \tag{38}$$

$$\mathbf{R}_{\alpha'} = \frac{1}{4}(\mathbf{r}'_1 + \mathbf{r}'_2 + \mathbf{r}'_3 + \mathbf{r}'_4) \tag{39}$$

of the two ^4He nuclei and

$$\mathbf{R}_{(n\alpha')} = \frac{1}{5}(\mathbf{r}_4 + \mathbf{r}'_1 + \mathbf{r}'_2 + \mathbf{r}'_3 + \mathbf{r}'_4) \tag{40}$$

of the ^5He system are defined. Thus the position of the neutron that is transferred from ^5He to ^3He is denoted with $\mathbf{r}_4 = \mathbf{R}_n$. It is also convenient to use the relative coordinate vectors

$$\mathbf{R}_{\alpha\alpha'} = \mathbf{R}_\alpha - \mathbf{R}_{\alpha'} \tag{41}$$

$$\mathbf{R}_{t(n\alpha')} = \mathbf{R}_t - \mathbf{R}_{(n\alpha')} \tag{42}$$

$$\mathbf{R}_{n\alpha'} = \mathbf{R}_n - \mathbf{R}_{\alpha'} \tag{43}$$

and the Jacobi coordinates

$$\mathbf{x} = \mathbf{r}_2 - \mathbf{r}_1 \tag{44}$$

$$\mathbf{y} = \mathbf{r}_3 - \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2) \tag{45}$$

$$\mathbf{z} = \mathbf{r}_4 - \frac{1}{3}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) = \mathbf{R}_n - \mathbf{R}_t \tag{46}$$

and similarly for the primed vectors. The total center-of-mass coordinate is given by

$$\mathbf{R} = \frac{1}{2}\mathbf{R}_\alpha + \frac{1}{2}\mathbf{R}_{\alpha'} = \frac{5}{8}\mathbf{R}_{(n\alpha')} + \frac{3}{8}\mathbf{R}_t \tag{47}$$

The condition $\mathbf{R} = 0$ reduces the number of independent variables from eight to seven.

The momenta conjugated to the position vectors defined above are given by

$$\mathbf{P}_\alpha = \hbar\mathbf{K}_\alpha = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 \tag{48}$$

$$\mathbf{P}_{\alpha'} = \hbar\mathbf{K}_{\alpha'} = \mathbf{p}'_1 + \mathbf{p}'_2 + \mathbf{p}'_3 + \mathbf{p}'_4 \tag{49}$$

$$\mathbf{P}_t = \hbar\mathbf{K}_t = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \tag{50}$$

$$\mathbf{P}_{(n\alpha')} = \hbar\mathbf{K}_{(n\alpha')} = \mathbf{p}_4 + \mathbf{p}'_1 + \mathbf{p}'_2 + \mathbf{p}'_3 + \mathbf{p}'_4 \tag{51}$$

$$\mathbf{P}_{\alpha\alpha'} = \hbar\mathbf{K}_{\alpha\alpha'} = \mu_{\alpha\alpha'} \left(\frac{\mathbf{P}_\alpha}{m_\alpha} - \frac{\mathbf{P}_{\alpha'}}{m_{\alpha'}} \right) \tag{52}$$

$$\mathbf{P}_{t(n\alpha')} = \hbar\mathbf{K}_{t(n\alpha')} = \mu_{t(n\alpha')} \left(\frac{\mathbf{P}_t}{m_t} - \frac{\mathbf{P}_{(n\alpha')}}{m_n + m_{\alpha'}} \right) \tag{53}$$

$$\mathbf{P}_{n\alpha'} = \hbar\mathbf{K}_{n\alpha'} = \mu_{n\alpha'} \left(\frac{\mathbf{P}_n}{m_n} - \frac{\mathbf{P}_{\alpha'}}{m_{\alpha'}} \right) \tag{54}$$

with the reduced masses $\mu_{ij} = m_i m_j / (m_i + m_j)$.

In the calculation of the transition rates, the spins of the particles have to be considered. The total angular momenta and parities are $J_n^{\pi_n} = 1/2^+$, $J_t^{\pi_t} = 1/2^+$, $J_5^{\pi_5} = 3/2^-$, and $J_{\alpha\alpha'}^{\pi_{\alpha\alpha'}} = J_{\alpha\alpha'}^{\pi_{\alpha\alpha'}} = 0^+$, of the neutron, ^3He , ^5He , and the ^4He nuclei, respectively. The ^5He resonance is described as a neutron in the $p_{3/2}$ scattering continuum coupled to the spinless ^4He core.

4.2 Cluster wave functions

The ^5He system in the ket state is described by the wave function

$$\Phi_{(n\alpha')}^{(+)}(\mathbf{P}_{n\alpha'}) = \psi_{n\alpha'}^{(+)}(\mathbf{P}_{n\alpha'}) \chi_{J_n M_n} \Phi_{\alpha'} \tag{55}$$

with the scattering function $\psi_{cd}^{(+)}$ of the neutron with respect to the ^4He core nucleus in ^5He , the neutron spin function $\chi_{J_n M_n}$, and the cluster wave function $\Phi_{\alpha'}$ of the α particle. The intrinsic wave functions of the clusters are given by

$$\Phi_j(J_j, M_j) = C_j \exp \left[-\frac{B_j}{2} \sum_{i=1}^{A_j} (\mathbf{r}_i - \mathbf{R}_j)^2 \right] \chi_{J_j M_j} \tag{56}$$

with mass numbers A_j for $j = \alpha, \alpha', t$. Using the Jacobi coordinates \mathbf{x}, \mathbf{y} , and \mathbf{z} introduced in Sect. 4.1 they assume the explicit form

$$\Phi_t = C_3 \exp \left[-\frac{B_3}{2} \left(\frac{1}{2} \mathbf{x}^2 + \frac{2}{3} \mathbf{y}^2 \right) \right] \chi_{J_t M_t} \tag{57}$$

for the ^3He nucleus and

$$\Phi_\alpha = C_4 \exp \left[-\frac{B_4}{2} \left(\frac{1}{2} \mathbf{x}^2 + \frac{2}{3} \mathbf{y}^2 + \frac{3}{4} \mathbf{z}^2 \right) \right] \chi_{J_\alpha M_\alpha} \tag{58}$$

$$\Phi_{\alpha'} = C_4 \exp \left[-\frac{B_4}{2} \left(\frac{1}{2} (\mathbf{x}')^2 + \frac{2}{3} (\mathbf{y}')^2 + \frac{3}{4} (\mathbf{z}')^2 \right) \right] \chi_{J_{\alpha'} M_{\alpha'}} \tag{59}$$

for the ^4He nuclei with spin functions $\chi_{J_t M_t}$, $\chi_{J_\alpha M_\alpha}$, and $\chi_{J_{\alpha'} M_{\alpha'}}$. The normalization constants

$$C_3 = \left[\left(\frac{\pi}{B_3} \right)^3 3^{3/2} \right]^{-1/2} \tag{60}$$

$$C_4 = \left[\left(\frac{\pi}{B_4} \right)^{9/2} 4^{3/2} \right]^{-1/2} \tag{61}$$

can be determined from the point rms radii

$$r_t = \sqrt{\frac{1}{B_3}} = \sqrt{[r_t^{(c)}]^2 - [r_p^{(c)}]^2} \tag{62}$$

$$r_\alpha = \sqrt{\frac{9}{8B_4}} = \sqrt{[r_\alpha^{(c)}]^2 - [r_p^{(c)}]^2} \tag{63}$$

that are equal to the point rms charge radii. They can be extracted from the experimental rms charge radii [30]

$$r_t^{(c)} = 1.9661 \text{ fm} \quad r_\alpha^{(c)} = 1.6755 \text{ fm} \tag{64}$$

by correcting with the proton rms charge radius $r_p^{(c)} = 0.8783 \text{ fm}$.

4.3 Reduction of T-matrix element

The T-matrix element can be written explicitly as

$$\begin{aligned} T_{n+\alpha+t \rightarrow \alpha+\alpha} &= \int d^3 r_1 d^3 r_2 d^3 r_3 d^3 r_4 d^3 r'_1 d^3 r'_2 d^3 r'_3 d^3 r'_4 \delta(\mathbf{R}) \\ &\quad \Phi_\alpha^\dagger(\mathbf{x}, \mathbf{y}, \mathbf{z}) \Phi_{\alpha'}^\dagger(\mathbf{x}', \mathbf{y}', \mathbf{z}') \chi_{\alpha\alpha'}^{(-)\dagger}(\mathbf{P}_{\alpha\alpha'}, \mathbf{R}_{\alpha\alpha'}) W(\mathbf{R}_{n\alpha'}) \\ &\quad \Phi_t(\mathbf{x}, \mathbf{y}) \Phi_{\alpha'}(\mathbf{x}', \mathbf{y}', \mathbf{z}') \psi_{n\alpha'}^{(+)}(\mathbf{P}_{n\alpha'}, \mathbf{R}_{n\alpha'}) \chi_{J_n M_n} \\ &\quad \chi_{t(n\alpha')}^{(+)}(\mathbf{P}_{t(n\alpha')}, \mathbf{R}_{t(n\alpha')}) \end{aligned} \tag{65}$$

where the δ function fixes the cm coordinate reducing the integral to seven intrinsic coordinates. Since

$$\frac{\partial(\mathbf{R}_t, \mathbf{x}, \mathbf{y})}{\partial(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)} = 1 \tag{66}$$

$$\frac{\partial(\mathbf{R}_\alpha, \mathbf{x}, \mathbf{y}, \mathbf{z})}{\partial(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)} = 1 \tag{67}$$

and similar for the primed vectors, a change of variables in the integral is easily achieved and it is straightforward to integrate over the intrinsic spatial and spin coordinates of one of the ^4He nuclei with the result

$$\begin{aligned} T_{n+\alpha+t \rightarrow \alpha+\alpha} &= \int d^3 z d^3 R_\alpha d^3 R_{\alpha'} \delta(\mathbf{R}) \\ &\quad \varphi_{\alpha t}^*(\mathbf{z}) \chi_{J_\alpha M_\alpha}^\dagger \chi_{\alpha\alpha'}^{(-)\dagger}(\mathbf{P}_{\alpha\alpha'}, \mathbf{R}_{\alpha\alpha'}) W(\mathbf{R}_{n\alpha'}) \\ &\quad \psi_{n\alpha'}^{(+)}(J_n, M_n, \mathbf{P}_{n\alpha'}, \mathbf{R}_{n\alpha'}) \chi_{J_n M_n} \\ &\quad \chi_{t(n\alpha')}^{(+)}(\mathbf{P}_{t(n\alpha')}, \mathbf{R}_{t(n\alpha')}) \chi_{J_t M_t} . \end{aligned} \tag{68}$$

Here the overlap function

$$\begin{aligned} \varphi_{\alpha t}^*(\mathbf{z}) &= C_4^* C_3 \int d^3 x d^3 y \exp \left[-B_4 \left(\frac{1}{2} \mathbf{x}^2 + \frac{2}{3} \mathbf{y}^2 + \frac{3}{4} \mathbf{z}^2 \right) \right] \\ &\quad \exp \left[-B_3 \left(\frac{1}{2} \mathbf{x}^2 + \frac{2}{3} \mathbf{y}^2 \right) \right] \\ &= D_{43} \exp \left(-\frac{3B_4}{4} \mathbf{z}^2 \right) \end{aligned} \tag{69}$$

with the constant

$$D_{43} = C_4^* C_3 3^{3/2} \left(\frac{\pi}{B_3 + B_4} \right)^3 \tag{70}$$

appears. It only depends on \mathbf{z} and represents the wave function of the transferred neutron. With the definition of the cm coordinate \mathbf{R} we have

$$\mathbf{R}_{n\alpha'} = \frac{3}{4} \mathbf{z} + \mathbf{R}_{\alpha\alpha'} \tag{71}$$

$$\mathbf{R}_{t(n\alpha')} = \frac{4}{5} \mathbf{R}_{\alpha\alpha'} - \frac{2}{5} \mathbf{z} \tag{72}$$

and

$$\frac{\partial(\mathbf{R}_{n\alpha'}, \mathbf{R}_{t(n\alpha')}, \mathbf{R})}{\partial(\mathbf{z}, \mathbf{R}_\alpha, \mathbf{R}_{\alpha'})} = 1 . \tag{73}$$

The T-matrix element reduces to an integral over two relative coordinates

$$\begin{aligned} T_{n+\alpha+t \rightarrow \alpha+\alpha} &= \int d^3 R_{n\alpha'} d^3 R_{t(n\alpha')} \varphi_{\alpha t}^*(\mathbf{z}) \chi_{\alpha\alpha'}^{(-)\dagger}(\mathbf{P}_{\alpha\alpha'}, \mathbf{R}_{\alpha\alpha'}) \\ &\quad W(\mathbf{R}_{n\alpha'}) \psi_{n\alpha'}^{(+)}(\mathbf{P}_{n\alpha'}, \mathbf{R}_{n\alpha'}) \chi_{t(n\alpha')}^{(+)}(\mathbf{P}_{t(n\alpha')}, \mathbf{R}_{t(n\alpha')}) \\ &\quad \chi_{J_\alpha M_\alpha}^\dagger \chi_{J_n M_n} \chi_{J_t M_t} . \end{aligned} \tag{74}$$

Since the potential W depends only on $\mathbf{R}_{n\alpha'}$ in the present approximation and the overlap function $\varphi_{\alpha t}$ is analytic, it is useful to choose $\mathbf{R}_{n\alpha'}$ and $\mathbf{R}_{t(n\alpha')}$ as integration variables. Although the integrand contains three scattering wave functions, the integral is finite because the overlap function $\varphi_{\alpha t}$

and the potential W are of short range and limit the range of integration.

The cluster wave functions carry the units

$$[\Phi_\alpha] = [\Phi_{\alpha'}] = L^{-9/2} \quad [\Phi_T] = L^{-3} \tag{75}$$

thus $[\varphi_{\alpha T}] = L^{-3/2}$ for the overlap function. The scattering wave functions are normalized as

$$[\chi_{\alpha\alpha'}^{(-)}] = [\chi_{T(n\alpha')}^{(+)}] = [\psi_{n\alpha'}^{(+)}] = 1 \tag{76}$$

and the unit of the T matrix becomes

$$[T] = E L^{9/2} \tag{77}$$

as anticipated in Sect. 3.

4.4 Partial-wave expansions

In the next step we consider the scattering wave functions in more detail introducing partial-wave expansions and include the angular momentum coupling. In the ket state the orbital angular momentum $l_{n\alpha'}$ of relative motion of the neutron with respect to ${}^4\text{He}$ in the ${}^5\text{He}$ system is coupled with the spin $J_n = 1/2$ of the neutron to the angular momentum J_5 of ${}^5\text{He}$. For the resonance of interest we have $l_{n\alpha'} = 1$ and $J_5 = 3/2$. In a next step the spin $J_t = 1/2$ of ${}^3\text{He}$ and $J_{(n\alpha')}$ are first coupled to the channel spin S and then the orbital angular momentum $l_{T(n\alpha')}$ is coupled with S to the total angular momentum J . In the bra state, there is no explicit angular momentum coupling required because $J_\alpha = J_{\alpha'} = 0$ and only the orbital angular momentum $l_{\alpha\alpha'}$ appears.

The bra state is defined by the simple product

$$\begin{aligned} \Psi_{\alpha\alpha'}^{(-)}(J_\alpha, M_\alpha, \mathbf{z}, \mathbf{P}_{\alpha\alpha'}, \mathbf{R}_{\alpha\alpha'}) \\ = \varphi_{\alpha T}(\mathbf{z}) \chi_{J_\alpha M_\alpha} \chi_{\alpha\alpha'}^{(-)}(\mathbf{P}_{\alpha\alpha'}, \mathbf{R}_{\alpha\alpha'}) \end{aligned} \tag{78}$$

but does not depend explicitly on the integration variables $\mathbf{R}_{n\alpha'}$ and $\mathbf{R}_{T(n\alpha')}$. The overlap function $\varphi_{\alpha T}$ depends on

$$\mathbf{z} = \frac{4}{5} \mathbf{R}_{n\alpha'} - \mathbf{R}_{T(n\alpha')} \tag{79}$$

and can be easily expanded in the variables $\mathbf{R}_{n\alpha'}$ and $\mathbf{R}_{T(n\alpha')}$ as

$$\begin{aligned} \varphi_{\alpha T}(\mathbf{z}) = \sum_{L_{n\alpha'} M_{n\alpha'}} u_{L_{n\alpha'}}(R_{n\alpha'}, R_{T(n\alpha')}) \\ Y_{L_{n\alpha'} M_{n\alpha'}}^*(\hat{R}_{n\alpha'}) Y_{L_{n\alpha'} M_{n\alpha'}}(\hat{R}_{T(n\alpha')}) \end{aligned} \tag{80}$$

with the spherical harmonics Y_{LM} and the radial function

$$\begin{aligned} u_{L_{n\alpha'}}(R_{n\alpha'}, R_{T(n\alpha')}) \\ = 4\pi D_{43} \exp\left[-\frac{3B_4}{4} \left(\frac{16}{25} R_{n\alpha'}^2 + R_{T(n\alpha')}^2\right)\right] \\ i_{L_{n\alpha'}}^{(1)} \left(\frac{6B_4}{5} R_{n\alpha'} R_{T(n\alpha')}\right) \end{aligned} \tag{81}$$

that contains modified spherical Bessel functions of the first kind [31]

$$i_l^{(1)}(z) = \sqrt{\frac{\pi}{2z}} I_{l+1/2}(z). \tag{82}$$

For the scattering wave function

$$\begin{aligned} \chi_{\alpha\alpha'}^{(-)}(\mathbf{P}_{\alpha\alpha'}, \mathbf{R}_{\alpha\alpha'}) = \frac{1}{K_{\alpha\alpha'} R_{\alpha\alpha'}} \sum_{l_{\alpha\alpha'}} (2l_{\alpha\alpha'} + 1) i_{l_{\alpha\alpha'}}^{l_{\alpha\alpha'}} \\ \psi_{l_{\alpha\alpha'}}^{(-)}(K_{\alpha\alpha'} R_{\alpha\alpha'}) P_{l_{\alpha\alpha'}}(\cos \theta_{\alpha\alpha'}) \end{aligned} \tag{83}$$

in (78) with argument

$$\cos \theta_{\alpha\alpha'} = \frac{\mathbf{P}_{\alpha\alpha'} \cdot \mathbf{R}_{\alpha\alpha'}}{P_{\alpha\alpha'} R_{\alpha\alpha'}} \tag{84}$$

of the Legendre polynomial $P_{l_{\alpha\alpha'}}$ and radial wave functions $\psi_{l_{\alpha\alpha'}}^{(-)}$, a new expansion

$$\begin{aligned} \chi_{\alpha\alpha'}^{(-)}(\mathbf{P}_{\alpha\alpha'}, \mathbf{R}_{n\alpha'}, \mathbf{R}_{T(n\alpha')}) = \sum_{L_{\alpha\alpha'}} (2L_{\alpha\alpha'} + 1) i_{L_{\alpha\alpha'}}^{L_{\alpha\alpha'}} \\ v_{L_{\alpha\alpha'}}(\mathbf{P}_{\alpha\alpha'}, R_{n\alpha'}, R_{T(n\alpha')}) P_{L_{\alpha\alpha'}}(x) \end{aligned} \tag{85}$$

can be introduced with

$$x = \frac{\mathbf{R}_{n\alpha'} \cdot \mathbf{R}_{T(n\alpha')}}{R_{n\alpha'} R_{T(n\alpha')}} \tag{86}$$

using

$$\mathbf{R}_{\alpha\alpha'} = \frac{2}{5} \mathbf{R}_{n\alpha'} + \frac{3}{4} \mathbf{R}_{T(n\alpha')}. \tag{87}$$

The expansion functions $v_{L_{\alpha\alpha'}}$ are found numerically from

$$\begin{aligned} v_{L_{\alpha\alpha'}}(\mathbf{P}_{\alpha\alpha'}, R_{n\alpha'}, R_{T(n\alpha')}) \\ = \frac{1}{2i^{L_{\alpha\alpha'}} K_{\alpha\alpha'}} \sum_{l_{\alpha\alpha'}} (2l_{\alpha\alpha'} + 1) i_{l_{\alpha\alpha'}}^{l_{\alpha\alpha'}} \\ \int_{-1}^1 \frac{dx}{R_{\alpha\alpha'}} \psi_{l_{\alpha\alpha'}}^{(-)}(K_{n\alpha'} R_{n\alpha'}) P_{l_{\alpha\alpha'}}(\cos \theta_{\alpha\alpha'}) P_{L_{\alpha\alpha'}}(x) \end{aligned} \tag{88}$$

using the orthogonality relation of the Legendre polynomials. In the actual calculation, a coordinate system is chosen so that the momentum $\mathbf{P}_{\alpha\alpha'}$ is in the direction of the z -axis and only the dependence on the modulus $P_{\alpha\alpha'}$ remains.

The full wave function in the ket state including angular momentum couplings can be expressed in the form

$$\begin{aligned} \Psi_{T(n\alpha')}^{(+)}(J_n, M_n, J_t, M_t, \mathbf{P}_{n\alpha'}, \mathbf{R}_{n\alpha'}, \mathbf{P}_{T(n\alpha')}, \mathbf{R}_{T(n\alpha')}) \\ = \frac{4\pi}{K_{n\alpha'} R_{n\alpha'}} \frac{4\pi}{K_{T(n\alpha')} R_{T(n\alpha')}} \sum_{l_{n\alpha'}} \sum_{J_5 M_5} \sum_{l_{T(n\alpha')}} \sum_{S \Sigma} \sum_{J M} \\ i_{l_{n\alpha'}}^{l_{n\alpha'}} \psi_{l_{n\alpha'} J_5}^{(+)}(K_{n\alpha'} R_{n\alpha'}) \mathcal{Z}_{J_5 M_5 M_n}^{J_n^*}(\hat{P}_{n\alpha'}) \\ i_{l_{T(n\alpha')}}^{l_{T(n\alpha')}} \psi_{l_{T(n\alpha')} S J}^{(+)}(K_{T(n\alpha')} R_{T(n\alpha')}) \mathcal{Z}_{J M \Sigma}^{l_{T(n\alpha')} S^*}(\hat{P}_{T(n\alpha')}) \\ (J_t M_t J_5 M_5 | S \Sigma) \mathcal{X}_{J M \Sigma}^{l_{n\alpha'} J_n J_t J_5 l_{T(n\alpha')} S}(\hat{R}_{n\alpha'}, \hat{R}_{T(n\alpha')}) \end{aligned} \tag{89}$$

with radial wave functions $\psi_{l_{n\alpha'} J_5}^{(+)}$ and $\psi_{l_{t(n\alpha')} S J}^{(+)}$ for the relative motion of neutron and ^4He in the ^5He system and the relative motion of ^3He and ^5He , respectively. The angular dependence is contained in the functions

$$\mathcal{Z}_{JMm}^{lj}(\hat{P}) = \sum_{m_l} (l m_l j m | J M) Y_{lm_l}(\hat{P}) \tag{90}$$

and

$$\begin{aligned} &\chi_{JM\Sigma}^{lJ_n J_t J_5 L S}(\hat{R}_1, \hat{R}_2) \\ &= \sum_{\bar{M} \bar{\Sigma}} \sum_{\bar{M}_1 \bar{M}_5} (L \bar{M} S \bar{\Sigma} | J M) (J_t \bar{M}_t J_5 \bar{M}_5 | S \Sigma) \\ &\quad Y_{L\bar{M}}(\hat{R}_2) \mathcal{Y}_{J_5 \bar{M}_5}^{lJ_n}(\hat{R}_1) \chi_{J_t \bar{M}_t} \end{aligned} \tag{91}$$

with the tensor spherical harmonics

$$\mathcal{Y}_{jm}^{ls}(\hat{R}) = \sum_{m_l m_s} (l m_l s m_s | j m) Y_{lm}(\hat{R}) \chi_{sm_s} \tag{92}$$

and Clebsch-Gordan coefficients.

Combining these results, the T matrix becomes

$$\begin{aligned} &T_{n+\alpha+t \rightarrow \alpha+\alpha}(P_{\alpha\alpha'}, \mathbf{P}_{n\alpha'}, \mathbf{P}_{t(n\alpha')}) \\ &= \int d^3 R_{n\alpha'} d^3 R_{t(n\alpha')} \\ &\quad \Psi_{\alpha\alpha'}^{(-)\dagger}(J_\alpha, M_\alpha, \mathbf{z}, \mathbf{P}_{\alpha\alpha'}, \mathbf{R}_{\alpha\alpha'}) W(\mathbf{R}_{n\alpha'}) \\ &\quad \Psi_{t(n\alpha')}^{(+)}(J_n, M_n, J_t, M_t, \mathbf{P}_{n\alpha'}, \mathbf{R}_{n\alpha'}, \mathbf{P}_{t(n\alpha')}, \mathbf{R}_{t(n\alpha')}) \\ &= \sum_{L_{n\alpha'} L_{\alpha\alpha'} l_{n\alpha'}} \sum_{J_5} \sum_{l_{t(n\alpha')}} \sum_S \sum_J \delta_{l_{n\alpha'} l_{t(n\alpha')}} \delta_{J J_\alpha} \\ &\quad C(l_{n\alpha'}, L_{n\alpha'}, L_{\alpha\alpha'}, J_n, J_t, J_\alpha, J_5, S) \\ &\quad I(L_{n\alpha'}, L_{\alpha\alpha'}, l_{n\alpha'}, J_5, l_{t(n\alpha')}, S, J, P_{\alpha\alpha'}, P_{n\alpha'}, P_{t(n\alpha')}) \\ &\quad W_{M_n, M_t, M_\alpha}(l_{n\alpha'}, l_{t(n\alpha')}, J_n, J_t, J_\alpha, J_5, S, \hat{P}_{n\alpha'}, \hat{P}_{t(n\alpha')}) \end{aligned} \tag{93}$$

after integration over the directions of $\mathbf{R}_{n\alpha'}$ and $\mathbf{R}_{t(n\alpha')}$ with three factors. The first

$$\begin{aligned} &C(l_{n\alpha'}, L_{n\alpha'}, L_{\alpha\alpha'}, J_n, J_t, J_\alpha, J_5, S) \\ &= (-1)^{J_n - J_5 - S - J_\alpha - l_{n\alpha'}} \sqrt{(2S + 1)(2J_5 + 1)} \\ &\quad \frac{(2L_{n\alpha'} + 1)(2L_{\alpha\alpha'} + 1)}{4\pi(2l_{n\alpha'} + 1)} \\ &\quad (L_{n\alpha'} 0 L_{\alpha\alpha'} 0 | l_{n\alpha'} 0)^2 \left\{ \begin{matrix} J_n & J_t & J_\alpha \\ S & l_{n\alpha'} & J_5 \end{matrix} \right\} \end{aligned} \tag{94}$$

is related to the angular momentum coupling, the second is a two-dimensional integral

$$\begin{aligned} &I(L_{n\alpha'}, L_{\alpha\alpha'}, l_{n\alpha'}, J_5, l_{t(n\alpha')}, J, S) \\ &= i^{l_{n\alpha'} + l_{t(n\alpha')} - L_{\alpha\alpha'}} \frac{(4\pi)^3}{K_{n\alpha'} K_{t(n\alpha')}} \\ &\quad \int dR_{n\alpha'} dR_{t(n\alpha')} R_{n\alpha'} R_{t(n\alpha')} W(R_{n\alpha'}) \end{aligned}$$

$$\begin{aligned} &u_{L_{n\alpha'}}(R_{n\alpha'}, R_{t(n\alpha')}) v_{L_{\alpha\alpha'}}^*(P_{\alpha\alpha'}, R_{n\alpha'}, R_{t(n\alpha')}) \\ &\psi_{l_{n\alpha'} J_5}^{(+)}(K_{n\alpha'} R_{n\alpha'}) \psi_{l_{t(n\alpha')} S J}^{(+)}(K_{t(n\alpha')} R_{t(n\alpha')}) \end{aligned} \tag{95}$$

with the radial wave functions, and the third

$$\begin{aligned} &W_{M_n, M_t, M_\alpha}(l_{n\alpha'}, l_{t(n\alpha')}, J_n, J_t, J_\alpha, J_5, S) \\ &= \sum_{M_5 \Sigma} \sum_{m_{n\alpha'}} \sum_{m_{t(n\alpha')}} (J_t M_t J_5 M_5 | S \Sigma) \\ &\quad (l_{n\alpha'} m_{n\alpha'} J_n M_n | J_5 M_5) Y_{l_{n\alpha'} m_{n\alpha'}}^*(\hat{P}_{n\alpha'}) \\ &\quad (l_{t(n\alpha')} m_{t(n\alpha')} S \Sigma | J_\alpha M_\alpha) Y_{l_{t(n\alpha')} m_{t(n\alpha')}}^*(\hat{P}_{t(n\alpha')}) \end{aligned} \tag{96}$$

describes the angular dependence. The dependence on the momenta was suppressed in the arguments of (95) and (96).

4.5 Numerical details

The radial wave functions $\psi_{l_{n\alpha'} J_5}^{(+)}$, $\psi_{l_{t(n\alpha')} S J}^{(+)}$, and $\psi_{l_{\alpha\alpha'}}^{(-)}$ are discretized on a grid with spacing $h = 0.05$ fm for radii up to 25 fm. They are calculated by an outward numerical integration with the Numerov method [32] and normalized to the asymptotic form at 10 fm which is beyond the range of the nuclear potentials. The partial wave expansion of $\psi_{l_{\alpha\alpha'}}^{(-)}$ is obtained with an angular integration using a 20-point Gauss-Legendre method [31] in the integrals (88). Partial waves with $L_{\alpha\alpha'}$ and $L_{n\alpha'}$ from 0 to 15 are considered in the sum (93). The double radial integral (95) is calculated including radii R_{14} and R_{35} up to 10 fm. This proved to be sufficient to achieve convergence.

Since only the $3/2^-$ resonance in the ^5He system is considered here, certain selection rules for the angular momenta apply. We have $J_n = J_t = 1/2$, $J_\alpha = 0$ and $J_5 = 3/2$ with $l_{n\alpha'} = 1$. Then $l_{t(n\alpha')} = l_{n\alpha'} = 1$ and the channel spin can be $S = J_5 \pm J_t = 1$ or 2. The channel spin S couples with $l_{t(n\alpha')}$ to $J = J_\alpha$. Hence $J = 0$ and $l_{t(n\alpha')} = S = 1$ For the final channel we have the condition that $|L_{n\alpha'} - l_{n\alpha'}| \leq L_{\alpha\alpha'} \leq L_{n\alpha'} + l_{n\alpha'}$ with $(-1)^{L_{n\alpha'} + L_{\alpha\alpha'} - l_{n\alpha'}} = 1$. Hence we have the possible pairs $(L_{n\alpha'}, L_{\alpha\alpha'}) = (0, 1), (1, 0), (1, 2), (2, 1), (2, 3), (3, 2), \dots$

5 Potentials and phase shifts

The potentials for the calculation of the scattering wave functions are a sum $U_{ij}(r_{ij}) = V_{ij}^N(r_{ij}) + V_{ij}^C(r_{ij})$ of a nuclear potential $V_{ij}^N(r_{ij})$ and a Coulomb potential $V_{ij}^C(r_{ij})$. In order to simplify the description and to have only a small number of parameters, the nuclear potentials in the $n+^4\text{He}$, $^4\text{He}+^4\text{He}$, and $^3\text{He}+^5\text{He}$ systems are assumed to be of Gaussian shape

$$V_{ij}^N(r_{ij}) = -V_{ij}^{(0)} \exp\left(-\frac{r_{ij}^2}{R_{ij}^2}\right) \tag{97}$$

with the two parameters depth $V_{ij}^{(0)}$ and radius R_{ij} . The Coulomb potential

$$V_{ij}^C(r_{ij}) = Z_i Z_j e^2 \begin{cases} \frac{1}{2R_{ij}^C} \left(3 - \frac{r_{ij}}{R_{ij}^C} \right) & \text{if } r_{ij} \leq R_{ij}^C \\ \frac{1}{r_{ij}} & \text{if } r_{ij} > R_{ij}^C \end{cases} \quad (98)$$

is that of a homogeneously charge sphere of radius $R_{ij}^C = 1.25 (A_i + A_j)^{1/3}$ fm with the mass numbers A_i and A_j of the nuclei. It only contributes in the ${}^4\text{He}+{}^4\text{He}$ and ${}^3\text{He}+{}^5\text{He}$ scattering.

5.1 $n+{}^4\text{He}$ system

Here the $\frac{3}{2}^-$ resonance at energy $E = 0.735$ MeV above the $n - \alpha$ threshold with width $\Gamma = 0.648$ MeV [18] has to be described. These experimental values are obtained with $V_{n\alpha'}^{(0)} = 48.1313888$ MeV and $R_{n\alpha'} = 2.4494$ fm. The phase shifts for elastic p-wave scattering in the $n + {}^4\text{He}$ system are shown as a black dashed line in Fig. 1. The resonance is clearly visible with a rapid increase of the phase shifts.

5.2 ${}^4\text{He}+{}^4\text{He}$ system

In this case all partial waves with orbital angular momentum $l_{\alpha\alpha'} = 0, 2, 4, 6, 8$ are considered using the same potential. The potential depth and width are fitted to energy of the ${}^8\text{Be}$ ground state with $J^\pi = 0^+$ at 91.84 MeV above the ${}^4\text{He}+{}^4\text{He}$ threshold and of the first excited state with $J^\pi = 2^+$ at 3.12184 MeV [33]. This leads to $V_{\alpha\alpha'}^{(0)} = 50.77696$ MeV and $R_{\alpha\alpha'} = 2.226655$ fm. With these values the resonance widths of $\Gamma(0^+) = 4.743$ eV and $\Gamma(2^+) = 1.048$ MeV are found that can be compared to the experimental values of 5.57 eV and 1.513 MeV. The phase shifts in the partial waves with $l_{\alpha\alpha'} = 0, 2, 4$ are depicted in Fig. 1 as full lines. The 0^+ and 2^+ resonances appear prominently with small widths. The phase shifts of the partial waves with larger $l_{\alpha\alpha'}$ are too small to be visible on the scale of the figure.

5.3 ${}^3\text{He}+{}^5\text{He}$ system

Here the potential in the channel with $J_{l(n\alpha')} = 0, S = l_{l(n\alpha')} = 1$ is needed. However, it is not clear how to fix the potential parameters without any experimental information. Since it concerns again the ${}^8\text{Be}$ system, a transformation of the ${}^4\text{He}+{}^4\text{He}$ potential is used with an increased radius due to the more diffuse surface structure of the ${}^3\text{He}+{}^5\text{He}$ system. With a standard value of $R_{l(n\alpha')} = 1.25 \text{ fm} \times 8^{1/3} = 2.5$ fm one sets $V_{l(n\alpha')}^{(0)} = V_{\alpha\alpha'}^{(0)} R_{\alpha\alpha'}^3 / R_{l(n\alpha')}^3$ assuming identical volume integrals of the potentials. The phase shifts decrease smoothly with energy and do not show any resonant behavior as can be seen in Fig. 1.

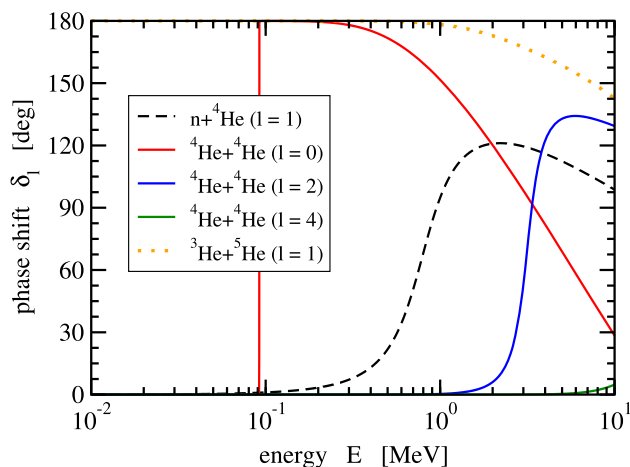


Fig. 1 Phase shifts for the elastic scattering of $n + {}^4\text{He}$ (dashed line), ${}^4\text{He}+{}^4\text{He}$ (full lines), and ${}^3\text{He}+{}^5\text{He}$ (dotted line) as a function of the cm energy E for partial waves $l_{n\alpha'} = 1, l_{\alpha\alpha'} = 0, 2, 4$, and $l_{l(n\alpha')} = 1$, respectively

6 Results

Following Eq. (25) and respecting $J_\alpha = 0$ in the final state, the differential transition rate of the reaction ${}^5\text{He}({}^3\text{He}, {}^4\text{He}){}^4\text{He}$ can be calculated from

$$\frac{dw_{n+\alpha+t \rightarrow \alpha+\alpha}}{d\Omega_{\alpha\alpha'}}(\mathbf{P}_{n\alpha'}, \mathbf{P}_{t(n\alpha')}) = \frac{2\pi}{\hbar} \frac{\mu_{\alpha\alpha'} P_{\alpha\alpha'}}{(2\pi\hbar)^3} \frac{1}{(2J_n + 1)(2J_t + 1)(2J_\alpha + 1)} \sum_{M_n, M_t, M_\alpha} |T_{n+\alpha+t \rightarrow \alpha+\alpha}(\mathbf{P}_{\alpha\alpha'}, \mathbf{P}_{n\alpha'}, \mathbf{P}_{t(n\alpha')})|^2 \quad (99)$$

once the T-matrix element is known.

In a scattering reaction with two particles in the entrance channel, the direction of the initial momentum is usually fixed and an integration over the scattering angles in the exit channel has to be performed to arrive at the total cross section. In the present case, the direction of the final momentum $\mathbf{P}_{\alpha\alpha'}$ was fixed to a given direction in the calculation of the T-matrix element in Sect. 4.4. Thus the total reaction rate is now found from an integration over the directions of the initial momenta

$$w_{n+\alpha+t \rightarrow \alpha+\alpha}(P_{n\alpha'}, P_{t(n\alpha')}) = \frac{2\pi}{\hbar} \frac{\mu_{\alpha\alpha'} P_{\alpha\alpha'}}{(2\pi\hbar)^3} \frac{1}{(2J_n + 1)(2J_t + 1)(2J_\alpha + 1)} \sum_{M_n, M_t, M_\alpha} \int d\Omega_{n\alpha'} d\Omega_{t(n\alpha')} |T_{n+\alpha+t \rightarrow \alpha+\alpha}(P_{\alpha\alpha'}, \mathbf{P}_{n\alpha'}, \mathbf{P}_{t(n\alpha')})|^2 \quad (100)$$

and a dependence on $P_{n\alpha'}$ and $P_{t(n\alpha')}$ is left. The summation over the angular momentum projections can be performed explicitly. The result is

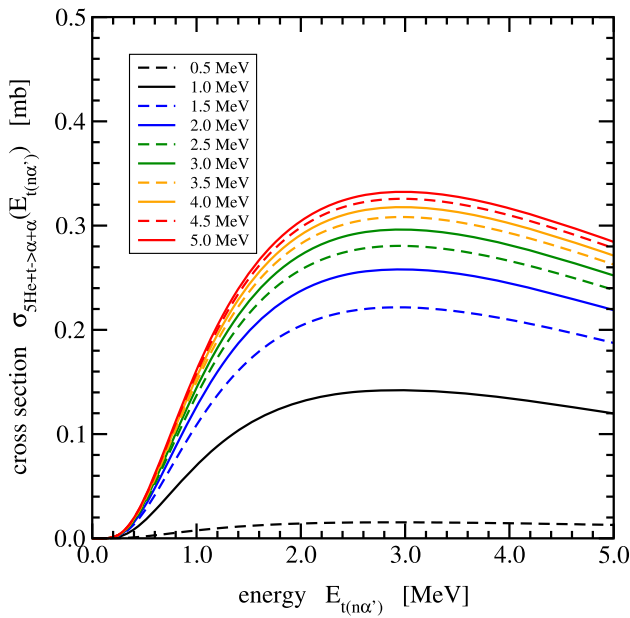


Fig. 2 Dependence of the cross section $\sigma_{5He+t \rightarrow \alpha + \alpha}(E_{t(n\alpha')})$ of the reaction $n+{}^4He+{}^3He \rightarrow {}^4He+{}^4He$ via the $p_{3/2}$ resonance in 5He on the energy $E_{t(n\alpha')}$ after an integration over all energies $E_{n\alpha'}$ up to a maximum between 0.5 MeV and 5.0 MeV in steps of 0.5 MeV

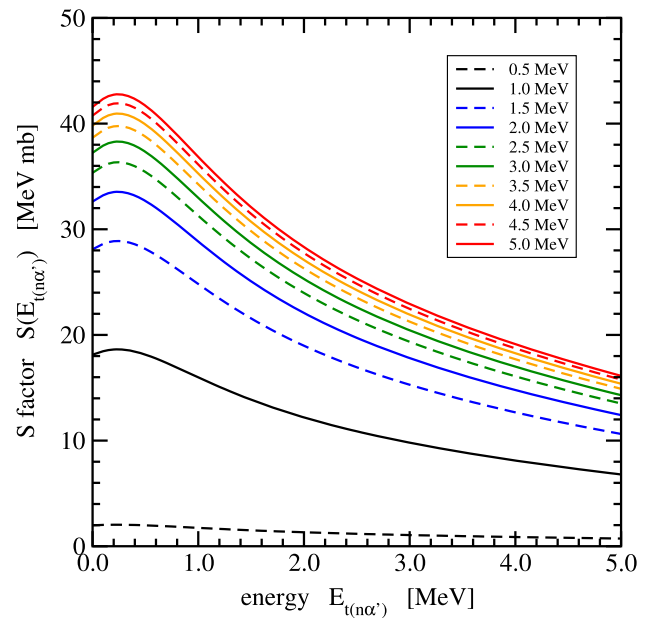


Fig. 3 Dependence of the S factor $S(E_{t(n\alpha')})$ of the reaction $n+{}^4He+{}^3He \rightarrow {}^4He+{}^4He$ via the $p_{3/2}$ resonance in 5He on the energy $E_{t(n\alpha')}$ after an integration over all energies $E_{n\alpha'}$ up to a maximum between 0.5 MeV and 5.0 MeV in steps of 0.5 MeV

$$\begin{aligned}
 & w_{n+\alpha+t \rightarrow \alpha+\alpha}(P_{n\alpha'}, P_{t(n\alpha')}) \\
 &= \frac{2\pi \mu_{\alpha\alpha'} P_{\alpha\alpha'}}{\hbar (2\pi\hbar)^3} \frac{1}{(2J_n + 1)(2J_t + 1)} \\
 &\times \sum_{l_{n\alpha'}} \sum_{l_{t(n\alpha')}} \sum_{J_5} \sum_S \sum_J \delta_{l_{n\alpha'} l_{t(n\alpha')}} \delta_{J J_\alpha} \\
 &\left| \sum_{L_{n\alpha'} L_{\alpha\alpha'}} I(L_{n\alpha'}, L_{\alpha\alpha'}, l_{n\alpha'}, J_5, l_{t(n\alpha')}, J, S) \right. \\
 &\left. C(l_{n\alpha'}, L_{n\alpha'}, L_{\alpha\alpha'}, J_n, J_t, J_\alpha, J_5, S) \right|^2 \quad (101)
 \end{aligned}$$

with several summations over angular momenta quantum numbers.

Integrating over the continuum states in the 5He system up to a maximum energy $E_{n\alpha'}^{max}$, the cross section

$$\begin{aligned}
 \sigma_{5He+t \rightarrow \alpha + \alpha}(P_{t(n\alpha')}, E_{n\alpha'}^{max}) &= \frac{\mu_{t(n\alpha')}}{P_{t(n\alpha')}} \\
 \int_0^{E_{n\alpha'}^{max}} dE_{n\alpha'} \frac{\mu_{n\alpha'} P_{n\alpha'}}{(2\pi\hbar)^3} & w_{n+\alpha+t \rightarrow \alpha+\alpha}(P_{n\alpha'}, P_{t(n\alpha')}) \quad (102)
 \end{aligned}$$

is obtained. It carries the proper unit L^2 of a cross section for a reaction with two particles in the entrance channel. The energy dependence of $\sigma_{5He+t \rightarrow \alpha + \alpha}$ is presented in Fig. 2 for different maximum energies $E_{n\alpha'}^{max}$. A considerable variation of the absolute values with this cutoff is found.

Since the cross section has a very strong energy dependence, it is reasonable to convert it to the astrophysical S factor that is defined as

$$\begin{aligned}
 S(E_{t(n\alpha')}) &= \sigma_{5He+t \rightarrow \alpha + \alpha}(E_{t(n\alpha')}) E_{t(n\alpha')} \exp(2\pi\eta_{t(n\alpha')}) \quad (103)
 \end{aligned}$$

with the Sommerfeld parameter

$$\eta_{t(n\alpha')} = \frac{Z_t Z_5 He e^2}{\hbar v_{t(n\alpha')}} = \frac{Z_t Z_5 He e^2}{\hbar} \sqrt{\frac{\mu_{t(n\alpha')}}{2E_{t(n\alpha')}}} \quad (104)$$

The energy dependence of the S factor is depicted in Fig. 3, again for different values of $E_{n\alpha'}^{max}$. A maximum close 0.25 MeV is found with a smooth decrease to higher energies. The shape is almost independent of the cutoff energy. There is a general rise of the cross section and S factor with increasing $E_{n\alpha'}^{max}$. It is less strong above the energy of the $3/2^-$ resonance but does not show a convergence up to 5 MeV.

The cross section (102) could be used to calculate the MACS

$$\begin{aligned}
 \langle \sigma v \rangle_{5He+t \rightarrow \alpha + \alpha} &= \sqrt{\frac{8}{\pi \mu_{t(n\alpha')}}} (k_B T)^{-3/2} \int dE_{t(n\alpha')} E_{t(n\alpha')} \\
 \exp\left(-\frac{E_{n\alpha'}^{res} + E_{t(n\alpha')}}{k_B T}\right) & \sigma_{5He+t \rightarrow \alpha + \alpha} \quad (105)
 \end{aligned}$$

as if it was a reaction with two particles in the entrance channel. However, an uncertainty originating from $E_{n\alpha'}^{max}$ remains and the density of the unstable 5He nucleus is needed to obtain the reaction rate (2) in astrophysical applications.

The proper Maxwellian-averaged transition rate is actually given by

$$\begin{aligned} \langle w \rangle_{n+\alpha+t \rightarrow \alpha+\alpha} &= \frac{1}{(2\pi k_B T)^3} \int dE_{n\alpha'} dE_{t(n\alpha')} \frac{P_{n\alpha'} P_{t(n\alpha')}}{(\mu_{n\alpha'} \mu_{t(n\alpha')})^{1/2}} \\ &\exp\left(-\frac{E_{n\alpha'} + E_{t(n\alpha')}}{k_B T}\right) w_{n+\alpha+t \rightarrow \alpha+\alpha}(P_{n\alpha'}, P_{t(n\alpha')}) \end{aligned} \tag{106}$$

with a double integral where the distribution of strength across the resonance is correctly taken into account and there is no ambiguity concerning a cutoff energy. The quantity $N_A^2 \langle w \rangle_{n+\alpha+t \rightarrow \alpha+\alpha}$ is depicted in Fig. 4 as a function of T_9 , the temperature in units of 10^9 K. A strong rise over many orders of magnitude is found. This dependence can be parametrised as

$$N_A^2 \langle w \rangle_{n+\alpha+t \rightarrow \alpha+\alpha} = \frac{A_0}{T_9^3} f(x) \exp\left(A_1 - \frac{A_2}{x + A_3}\right) \tag{107}$$

with the function

$$f(x) = 1 + \frac{B_1}{1 + B_2(x + B_3)^2} - \frac{C_1(x - C_3)}{1 + C_2(x - C_3)^4} \tag{108}$$

that depends on $x = \ln T_9$. The coefficients are

$$A_0 = 0.842612 \text{ cm}^6 \text{ mol}^{-2} \text{ s}^{-1} \tag{109}$$

$$A_1 = 59.0088 \quad A_2 = 898.718 \quad A_3 = 10.6587 \tag{110}$$

$$B_1 = 0.392179 \quad B_2 = 1.75921 \quad B_3 = 5.36199 \tag{111}$$

$$C_1 = 1.24285 \quad C_2 = 0.45668 \quad C_3 = 1.80872 \tag{112}$$

and the rms deviation of the approximation (107) is about 2.7 % for $10^{-3} \leq T_9 \leq 10^1$. With the densities n_n , n_α , and n_t , the astrophysical reaction rate (11) can be calculated for applications, e.g., in reaction network calculations of primordial nucleosynthesis. Then the significance of the reaction $n + {}^4\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + {}^4\text{He}$ can be assessed. This is, however, outside the scope of the present work.

7 Conclusions

The ${}^5\text{He}$ nucleus in the entrance channel of the ${}^5\text{He}({}^3\text{He}, {}^4\text{He}){}^4\text{He}$ reaction is not a stable nucleus but a $3/2^-$ p-wave resonance in the $n - {}^4\text{He}$ scattering continuum. Since there are no experimental data available for the cross section of this reaction, a theoretical estimate is necessary to judge its importance in comparison to other nuclear reactions that synthesize ${}^4\text{He}$ in astrophysical environments. What is actually needed is the Maxwellian-averaged transition rate with three particles, n , ${}^4\text{He}$, and ${}^3\text{He}$, in the initial state and a proper treatment of this state to calculate the temperature dependence of the process. The pertinent differences in the theoret-

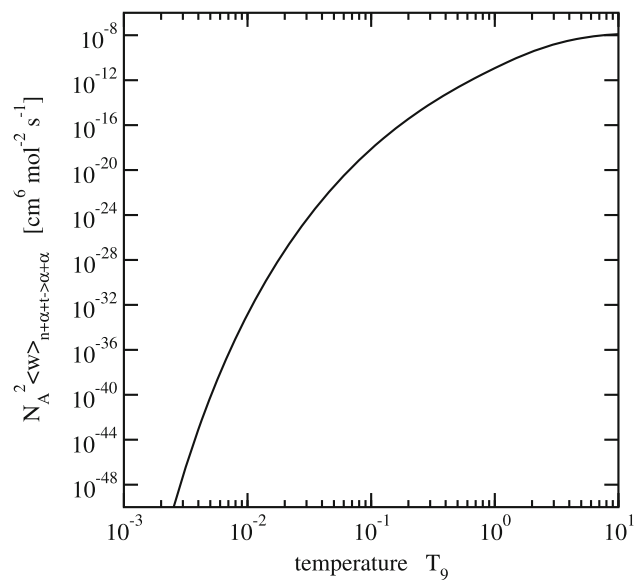


Fig. 4 Dependence of the Maxwellian-averaged transition rate $N_A^2 \langle w \rangle_{n+\alpha+t \rightarrow \alpha+\alpha}$ on the temperature T_9

ical formulation as compared to reactions with two particles in the entrance channel were discussed in detail.

In this work, the transition rate is calculated using a post-form DWBA for the T-matrix element, simple cluster wave functions for ${}^4\text{He}$ and ${}^3\text{He}$ and a particular three-body scattering wave function in the initial state. This allows to reduce the full integral of the many-body matrix element considerably to two dimensions. Only transitions via the $3/2^-$ p-wave resonance in ${}^5\text{He}$ were considered. A 'pseudo' cross section of the ${}^5\text{He}({}^3\text{He}, {}^4\text{He}){}^4\text{He}$ reaction as a function of the ${}^3\text{He}-{}^5\text{He}$ energy of relative motion was defined and presented together with the corresponding S factor. This can be used to normalize experimental data that can be obtained in an indirect approach like the Trojan-horse method. However, there is a non-negligible dependence on the cutoff energy in the $n - {}^4\text{He}$ subsystem. The full dependence on both relative energies is included in the calculation of the Maxwellian-averaged transition rate without ambiguities. It was parametrized with a suitable function of the temperature. This first theoretical estimate can be used in future astrophysical applications.

The theoretical calculation present here can be improved in several aspects. For instance, the cluster wave functions of Gaussian type can be replaced by more microscopic many-body wave functions. Also a better treatment of the three-particle scattering state can be envisaged. More appropriate and better constrained potentials can be used to describe the scattering in the $n - {}^4\text{He}$, ${}^3\text{He} - {}^5\text{He}$, and ${}^4\text{He} - {}^4\text{He}$ systems. But these changes would require larger numerical efforts in the calculation.

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Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: The Maxwellian-averaged transition rate for use in future applications is given in parametrised form in equation (107).]

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