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Review

Consequences of temperature fluctuations in observables measured in high-energy collisions

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Abstract. We review the consequences of intrinsic, nonstatistical temperature fluctuations as seen in observables measured in high-energy collisions. We do this from the point of view of nonextensive statistics and Tsallis distributions. Particular attention is paid to multiplicity fluctuations as a first consequence of temperature fluctuations, to the equivalence of temperature and volume fluctuations, to the generalized thermodynamic fluctuations relations allowing us to compare fluctuations observed in different parts of the phase space, and to the problem of the relation between Tsallis entropy and Tsallis distributions. We also discuss the possible influence of conservation laws on these distributions and provide some examples of how one can get them *without* considering temperature fluctuations.

1 Introduction

Nowadays the statistical approach is a standard procedure used to model high-energy multiparticle production processes [1]. However, it has been realized that data on many single-particle distributions deviate in a visible way from what one expects from the usual statistical models, based on Boltzman-Gibbs (BG) statistics. These frequently show power-like rather than exponential behavior, and, in addition, multiparticle distributions are broader than naively expected. These observations prompted the idea of a suitable modification of a simple statistical approach used by including in it the possibility of accounting for possible intrinsic, nonstatistical fluctuations. These were identified as the source of the deviations. Such fluctuations are important as possible signals of phase transition(s) taking place in an hadronizing system [2–4]. Therefore it is important to be able to include them. In this way the Tsallis statistical approach [5–7], already known in other branches of physics, was successfully introduced to the field of multiparticle production processes[1]. In this approach a new parameter, the nonextensivity parameter q appears, which is identified with fluctuations of the parameter T identified with the "temperature" of the hadronizing fireball [9,10].

It was shown there that such a situation can only occur when the heat bath is not homogeneous and must be described by a local temperature, T, fluctuating from point to point around some equilibrium value, T_0. Assuming some simple diffusion picture as being responsible for equalization of this temperature [8–10] one obtains the evolution of T in the form of a Langevin stochastic equation with the distribution of $1/T$, $g(1/T)$, emerging as a solution of the corresponding Fokker-Planck equation. It turns out that, in this case, $g(1/T)$ takes the form of a gamma distribution,

$$g(1/T) = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)}\frac{T_0}{q-1}\left(\frac{1}{q-1}\frac{T_0}{T}\right)^{\frac{2-q}{q-1}} \cdot \exp\left(-\frac{1}{q-1}\frac{T_0}{T}\right). \tag{1}$$

Convoluting the usual Boltzman-Gibbs exponential factor $\exp(-E/T)$ with this $g(1/T)$, one immediately gets a Tsallis distribution, $h_q(E)$, with a new parameter q, which, for $q \to 1$, becomes the usual BG distribution[2],

$$h_q(E) = \frac{2-q}{T}\exp_q\left(-\frac{E}{T}\right) = \frac{2-q}{T}\left[1-(1-q)\frac{E}{T}\right]^{\frac{1}{1-q}} \tag{2}$$

$$\stackrel{q\to 1}{\Longrightarrow} \frac{1}{T}\exp\left(-\frac{E}{T}\right), \tag{3}$$

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[1] For details see our previous review [8]. Here we present recent developments in this field not covered there.

[2] Notice that all distributions used here are defined as probability density functions with standard normalization, $\int dE h_q(E) = 1$. This results in the presence of the prefactor $(2-q)/T$.

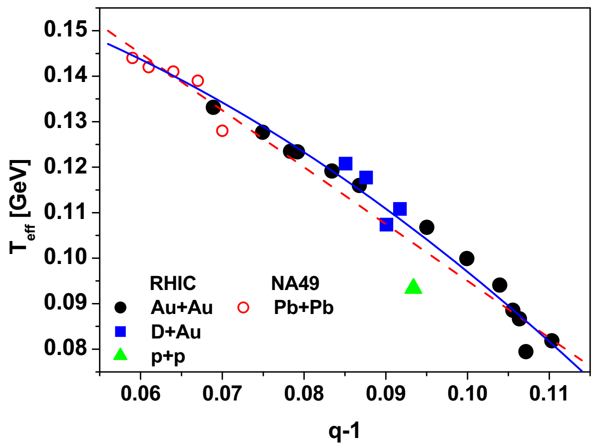

Fig. 1. (Color online) Dependence of T_{eff} on q for different energies. RHIC data points are from [11] whereas NA49 points are from [12–14] (for, respectively, $\sqrt{s}$ = 6.3, 7.6, 8.8, 12.3, 17.3 GeV (negative pions)). Fits are: $T_{eff} = 0.17–7.3(q-1)^2$ (full line, and $T_{eff} = 0.22–1.25(q-1)$ (dashed line)). In both cases T_{eff} is in GeV.

with

$$q = 1 + \omega_T^2 \qquad \text{where} \qquad \omega_T^2 = \frac{\mathrm{Var}(T)}{\langle T\rangle^2}\,, \tag{4}$$

directly connected to the variance of T. This idea was further developed in [15–17] (where problems connected with the notion of temperature in such cases were addressed). This forms a basis for so-called *superstatistics* [18,19]. In what follows, we shall use this approach when discussing Tsallis distributions (except for sect. 4 in which we compare it with the distribution obtained from Tsallis entropy).

It must be mentioned that temperature fluctuations (visualized by $q > 1$ values of the nonextensivity parameter) also allow for a description of the possible energy transfer from or to the heat bath [8]. Namely, if T_v is a new parameter characterizing such an energy transfer, then

$$T \to T_{eff} = T_0 + (q-1)T_v. \tag{5}$$

Figure 1 shows that such an effect is indeed observed [20]. It is caused mainly by the possible energy transfer between the central fireball (participants) and nuclear fragments passing by without interaction (spectators)[3]. Notice that this energy transfer is only possible in the presence of fluctuations, *i.e.*, for $q > 1$, when there are no fluctuations and $q = 1$ one has $T_{eff} = T_0$.

It is worth mentioning, at this point, that fluctuation phenomena as discussed here can be incorporated into a traditional presentation of thermodynamics [22]. In such a general approach, the Tsallis distribution (2) belongs to the class of general admissible distributions which satisfy thermodynamic consistency conditions and present a natural extension of the usual BG canonical distribution (3).

[3] A similar effect is also expected in the propagation of cosmic rays through the outer space, cf. [21]; we shall not discuss this issue here.

This, together with a recent generalization of classical thermodynamics to a nonextensive case presented in [23–25], form a constructive answer to the critical remarks we encountered concerning the consistency of Tsallis statistics with the usual thermodynamics in [26–30].

Applications of Tsallis distributions to multiparticle production processes are now numerous. To those quoted previously in [8] one should add some new results from refs. [20,31] and presented in [17]. The most recent applications of this approach come from the STAR and PHENIX Collaborations at RHIC [32,33] and from the CMS [34,35], ALICE [36,37] and ATLAS [38] Collaborations at LHC (see also a recent compilation [39,40])[4]. In sect. 2 we report on new results concerning the consequences of temperature fluctuations in observables measured in high-energy collisions obtained since our previous review [8]. In sect. 3 the influence of conservation laws, forcing the use of conditional probabilities and resulting in $q < 1$, is discussed. In sect. 4, the differences between Tsallis distributions as obtained from Tsallis entropy and the concept of superstatistics is discussed. A possible experimental check is proposed. Section 5 is devoted to yet another, not based on statistical models, derivation of Tsallis distribution. Section 6 is our summary.

2 Imprints of superstatistic in multiparticle processes

2.1 Multiplicity distributions

In [47] (cf. also [8]) we saw that T fluctuations in the form of eq. (1) not only result in power-like behavior of single particle distributions, but also in a specific broadening of the corresponding multiplicity distributions, $P(N)$, which evolve from the Poissonian form characteristic of BG distributions to the negative binomial (NB) form for Tsallis distributions. In short: whenever we have N independently produced secondaries with energies $\{E_{i=1,\dots,N}\}$ taken from the exponential distribution $f(E)$, cf. eq. (3), in which case the corresponding joint distribution is given by

$$f\left(\{E_{i=1,\dots,N}\}\right) = \frac{1}{\lambda^N}\exp\left(-\frac{1}{\lambda}\sum_{i=1}^{N}E_i\right), \tag{6}$$

and whenever

$$\sum_{i=0}^{N}E_i \le E \le \sum_{i=0}^{N+1}E_i, \tag{7}$$

then the corresponding multiplicity distribution is Poissonian,

$$P(N) = \frac{(\bar{N})^N}{N!}\exp(-\bar{N}) \qquad \text{where} \qquad \bar{N} = \frac{E}{\lambda}. \tag{8}$$

[4] In addition to the applications presented in this review, the nonextensive approach has also been applied to hydrodynamical models [41–43] and to investigations of dense nuclear matter [44–46].

But whenever in a given process N particles with energies $\{E_{i=1,\dots,N}\}$ are distributed according to the joint N-particle Tsallis distribution,

$$h\left(\{E_{i=1,\dots,N}\}\right) = C_N\left[1-(1-q)\frac{\sum_{i=1}^{N} E_i}{\lambda}\right]^{\frac{1}{1-q}+1-N} \tag{9}$$

(for which the corresponding one particle Tsallis distribution function in eq. (2) is the marginal distribution), then, under the same condition (7), the corresponding multiplicity distribution is the NB distribution [48,49]

$$P(N) = \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)}\frac{\left(\frac{\langle N\rangle}{k}\right)^{N}}{\left(1+\frac{\langle N\rangle}{k}\right)^{(N+k)}}\,, \tag{10}$$

where

$$k = \frac{1}{q-1}\,.$$

For $q \to 1$ one has $k \to \infty$ and (10) becomes a Poissonian distribution (8), whereas for $q \to 2$ one has $k \to 1$ and (10) becomes a geometrical distribution. For large values of N and $\langle N\rangle$ eq. (10) can be written in the following scaling form:

$$\langle N\rangle P(N) \cong \psi\left(z = \frac{N}{\langle N\rangle}\right) = \frac{k^k}{\Gamma(k)} z^{k-1}\exp(-kz), \tag{11}$$

known as the Koba-Nielsen-Olesen (KNO) scaling [50,51][5].

Note that, if in the Poisson distribution (8) one fluctuates the mean value, $\bar{N} = E/T$ (valid for the one-dimensional, $D = 1$, case), using its distribution in the form

$$g(\bar{N}) = g\left(\frac{1}{T} = \frac{\bar{N}}{E}\right)\left|\frac{\mathrm{d}\bar{N}}{\mathrm{d}(1/T)}\right| \tag{12}$$

(where $g(1/T)$ is given by eq. (1)) then the resulting multiplicity distribution,

$$P(N) = \int \mathrm{d}\bar{N} g(\bar{N})\frac{\bar{N}}{N!}\exp(-\bar{N}), \tag{13}$$

is the NB distribution given by eq. (10)[6].

[5] The connection between q and k was first discovered when fitting $p\bar{p}$ data for different energies by means of the Tsallis formula (2) [52,53]. The resulting energy dependence of the parameter q turned out to coincide with that of $1/k$ from the respective NB distribution fits to the corresponding $P(N)$. It was then realized that fluctuations of $\bar{N}$ in the Poissonian distribution (8) taken in the form of $\psi(\bar{N}/\langle N\rangle)$, eq. (11), lead to the NB distribution (10).

[6] Actually this has been also noted in [54,17] and recently discussed in [55] where the credit in what concerns the origin of the discussion of such connection between the Poisson and NB distributions has been given to [56].

2.2 Equivalence of temperature fluctuations and volume fluctuations

The KNO scaling form (11), with assumed identification

$$z = \left(\frac{V}{\langle V\rangle}\right)^{1/4} \tag{14}$$

(where V is the volume of the interaction region) has been used in [57] as a starting point for a description of particle spectra by means of *fluctuations of volume*. In this way it was hoped to avoid the notion of *fluctuating temperature* discussed here. The results were encouraging. However, for the constant total energy as assumed in [57], $E = \mathrm{const}$, both the volume V and the temperature T are related,

$$E \sim VT^4, \tag{15}$$

this means that

$$T = \langle T\rangle\left(\frac{\langle V\rangle}{V}\right)^{\frac{1}{4}} \tag{16}$$

and the mean multiplicity in the microcanonical ensemble (MCE), $\bar{N}$, can be written as

$$\bar{N} = \langle N\rangle \cdot \frac{V}{\langle V\rangle}\left(\frac{T}{\langle T\rangle}\right)^{3} = \langle N\rangle\frac{\langle T\rangle}{T}\,. \tag{17}$$

This implies that both approaches are equivalent and that fluctuations of V *assumed* in [57] in the form given by eq. (16) arise as an effect of fluctuations of T considered here with $g(1/T)$ given by eq. (1). This is not assumed but *derived* from the properties of the underlying physical process in the nonhomogeneous heat bath. One should also remember that UA5 data [58] show that the KNO scaling is broken due to the energy dependence of the parameter k[7]. In fact, as shown in [48,49], $k^{-1} = -0.104 + 0.058\ln\sqrt{s}$. Therefore, in the scenario with fluctuations of the volume V, the scaling KNO form of the $P(N)$ used to model these fluctuations is a somewhat rough simplification. On the contrary, in the scenario of temperature T fluctuations, $P(N)$ is given by a NB distribution, which adequately describes the data.

2.3 Relation between fluctuations observed in different parts of the phase space

2.3.1 q sum rules

So far, fluctuations of T as introduced in [9,10] and measured by the corresponding parameter q were discussed using examples of distributions of the longitudinal phase space (in the rapidity variable y and integrated over transverse momenta), $\mathrm{d}N/\mathrm{d}y$, and in transverse phase space,

[7] A possible solution to solve the breakdown of the KNO scaling in multiplicity distributions measured in e^+e^- and pp collisions has been proposed in [59].

$\mathrm{d}N/\mathrm{d}\boldsymbol{p}_T$. It was found that the corresponding parameters q, $q = q_T$ and $q = q_L$, respectively, are different. Whereas $q_L - 1 \sim 0.1$–0.3 and grows with the energy of collision (measured mainly in pp and $\bar{p}p$ collisions), transverse fluctuations are much weaker, $q_T - 1 \sim 0.01$–0.1 and vary slowly with energy (depending only slightly on whether one observes elementary collisions or collisions between nuclei) [60–63]. As shown in [47,8] the same fluctuations of T result in the broadening of multiplicity distributions resulting in its NB form as given by eq. (10). This time the corresponding q describes fluctuations in the whole of the phase space, with $p = \sqrt{|\boldsymbol{p}^2|} = \sqrt{p_L^2 + p_T^2}$.

In [60,61] it was proposed that, because $q-1 = \sigma^2(T)/\langle T\rangle^2$ (*i.e.*, is given by fluctuations of the total temperature T), and assuming that $\sigma^2(T) = \sigma^2(T_L) + \sigma^2(T_T)$, the resulting values of q should not be too different from

$$q = \frac{q_L\langle T_L\rangle^2 + q_T\langle T_T\rangle^2}{\langle T\rangle^2} - \frac{\langle T_L\rangle^2 + \langle T_T\rangle^2}{\langle T\rangle^2} + 1. \qquad (18)$$

Therefore, because of the dominance of longitudinal (partition) temperature over transverse, $T_L \gg T_T$, one should expect that $q \sim q_L$. This is indeed observed [60,61]. This is the first sum rule for parameters q obtained from different measurements.

Fluctuations of temperature are usually deduced either from data averaged over all other possible fluctuations or from data also accounting for fluctuations of other measured variables. In this case one can refine the experimentally evaluated q and, for example, when extracting q from distributions of $\mathrm{d}N/\mathrm{d}y$, one finds that (cf. [64], for details)

$$q - 1 \stackrel{def}{=} \frac{\mathrm{Var}(T)}{\langle T\rangle^2} = \frac{\mathrm{Var}(z)}{\langle z\rangle^2} - \frac{\mathrm{Var}\,(m_T)}{\langle m_T\rangle^2}\,, \qquad (19)$$

where $z = m_T/T$ (with $m_T = \sqrt{m^2 + p_T^2}$). This is the second sum rule for the nonextensivity parameters q obtained from different measurements. It connects total q, which can be obtained from an analysis of the NB form of the measured multiplicity distributions, $P(N)$, with $q_L - 1 = \mathrm{Var}(z)/\langle z\rangle^2$, obtained from fitting rapidity distributions and $\mathrm{Var}(m_T)/\langle m_T\rangle^2$ obtained from data on transverse mass distributions. When extracting q from distributions of $\mathrm{d}N/\mathrm{d}m_T$, we proceed analogously but now with $z = \cosh y/T$.

2.3.2 Generalized thermodynamic fluctuation relations

So far, we concentrated only on fluctuations of T. We shall continue the discussion by allowing the energy (U), temperature (T) and multiplicity (N) of the system to fluctuate and propose to express these fluctuations by the corresponding parameter q [65]. Our discussion is based on the notion of thermodynamic uncertainty relations discussed in [66]. It was suggested there that the temperature T and energy U could be regarded as complementary, similarly as are energy and time in quantum mechanics. One expects from simple dimensional analysis that (k is Boltzman's constant)

$$\Delta U\,\Delta\beta \geq k, \qquad \text{where} \qquad \beta = 1/T. \qquad (20)$$

Definite U (isolation) and definite T (contact with a heat bath) to represent the two extreme cases of this complementarity. This leads to the so-called Lindhard's uncertainty relation between the fluctuations of U and T [67–70][8],

$$\omega_U^2 + \omega_T^2 = \frac{1}{\langle N\rangle} \qquad \text{where} \qquad \omega_x^2 = \mathrm{Var}(x)/\langle x\rangle^2, \quad (21)$$

and this, as was shown in [65], can be generalized to include all variables: U, T and N by using the nonextensive approach. One can then study an ensemble in which the energy (U), temperature (T) and multiplicity (N), can all fluctuate. These fluctuations are then connected by the following relation:

$$\left|\omega_N^2 - \frac{1}{\langle N\rangle}\right| = \omega_U^2 + \omega_T^2 - 2\rho\omega_U\omega_T$$
$$= (\omega_U - \omega_T)^2 + 2\omega_U\omega_T(1-\rho) = |q-1|, \qquad (22)$$

where $\rho = \rho(U,T) \in [-1,1]$ is the correlation coefficient between U and T. This generalizes Linhard's thermodynamic uncertainty relation, eq. (21). The correlation coefficient enters since when all variables, U, N and T fluctuate, the pairs of variables, (U,N) and (U,T), cannot all be independent because

$$\mathrm{Var}(U) = \langle T\rangle\,\mathrm{Cov}(U,N) + \langle N\rangle\,\mathrm{Cov}(U,T) \qquad (23)$$

(cf. [47]). This means that, in general,

$$\omega_U = \rho(U,N)\omega_N + \rho(U,T)\omega_T, \qquad (24)$$

where $\rho(X,Y)$ denotes the corresponding correlation coefficients between variables X and Y. It should be noticed at this point that in the literature [71] there is a similar relation connecting the volume, V, pressure, P and temperature, T,

$$\omega_P^2 = \omega_V^2 + \omega_T^2, \qquad (25)$$

but we shall not discuss it here.

The observed systematics in the energy dependence of the parameter q, deduced from presently available data, is shown in fig. 2. From the measurements of different observables one observes that, for high enough energies, $q > 1$ and that values of q found from different observables are different. The latter is caused either by technical (methodical) problems or else by a physical cause. The former arises when, for example, fluctuations of the temperature are deduced either from data averaged over other fluctuations, or from more refined data also accounting for fluctuations of other variables (as in [64], see eq. (19)). The latter case is connected with the fact that the observed q's were obtained in different parts of the phase space. In this case one gets an uncertainty relation (22) with the help of which one can connect fluctuations observed in different parts of the phase space. For example, one can recalculate

[8] This idea is still disputable, see [67–70], nevertheless we shall treat these increments as a measure of fluctuations of the corresponding physical quantities.

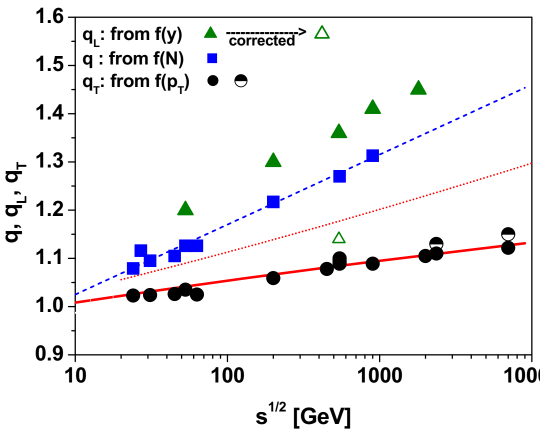

Fig. 2. (Color online) Energy dependencies of the parameters q, q_L and q_T as obtained from different observables. Triangles: q_L obtained from an analysis of rapidity distributions [52, 53]; solid triangles show the uncorrected values, whereas open triangle indicates the corrected value [64]. Squares: q obtained from multiplicity distributions $P(N)$ (fitted by $q = 1+1/k$ with $1/k = -0.104 + 0.029\ln(s)$) [72]. Circles: q_T obtained from a different analysis of transverse momenta distributions, $f(p_T)$. Data points in this case come, respectively, from the [48,49] compilation of data (full symbols) and from CMS data (half-filled circles at high energies) [34,35]. The full and dotted lines come from eq. (27) and show, respectively, the energy dependence of q_T and energy dependence of q_L (for $\rho = 0$, $\alpha = 2/3$ and $\kappa = 1$).

q obtained from $P(N)$ (*i.e.*, obtained from the whole phase space, see dashed line in fig. 2) and compare it with q evaluated from $f(p_T)$ (*i.e.*, obtained from only the transverse part of the phase space, see full line in fig. 2)[9].

The correlation parameter ρ appearing here bears important information on the details of the production process. For example, $\rho < 0$ means that a large energy U (*i.e.*, large inelasticity of reaction, K) results in a large number of secondaries of lower energies, whereas $\rho > 0$ means the opposite, one gets a smaller number of larger energies. From eq. (22) one finds that the coefficient ρ is a function of all the nonextensivity parameters involved. Denoting by α the part of fluctuations of T in the transverse direction, one finds

$$q_T - 1 = \alpha\omega_T^2, \qquad q_L - 1 = \omega_U^2 + (1-\alpha)\omega_T^2, \qquad (26)$$

and further

$$q - 1 = (q_L - 1) + (q_T - 1) - 2\rho\omega_U\omega_T. \qquad (27)$$

[9] See [65] for details. A comment is in order concerning the results of fig. 2 obtained from $f(y)$. Namely, it turns out that, in the fitting procedure, parameters T and q are strongly correlated [39,40,64]. As a result q values evaluated in a different analysis of rapidity distributions [60,61] differ slightly from those presented here (they give q values comparable or somewhat higher that one obtained from multiplicity distribution).

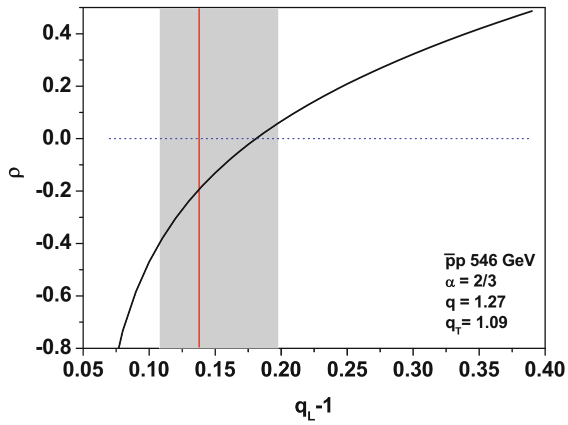

Fig. 3. (Color online) Example of ρ obtained from eq. (29). The shaded area shows the extent of possible error, due to the uncertainty in fixing q_L.

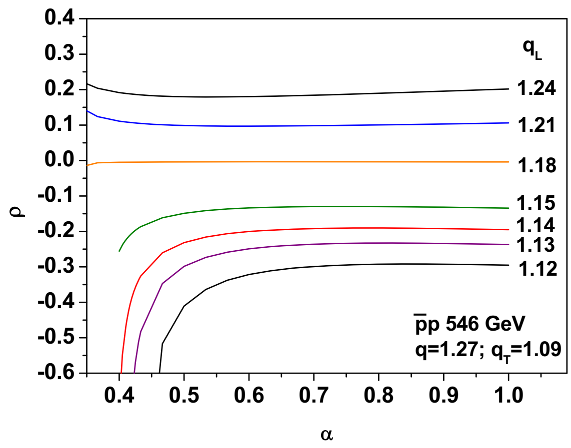

Fig. 4. (Color online) Dependence of the correlation coefficient ρ on the parameter α for different values of q_L.

It can be shown that

$$\kappa = \frac{\omega_U}{\omega_T} = \sqrt{\alpha\left(\frac{q_L - 1}{q_T - 1} + 1\right) - 1}. \qquad (28)$$

Finally, one obtains the correlation coefficient ρ expressed in terms of different fluctuations (in principle *measured*) (cf. [73])[10],

$$\rho = \frac{1 - \frac{(q-1)-(q_L-1)}{q_T-1}}{\frac{2}{\alpha}\sqrt{\alpha\left(\frac{q_L-1}{q_T-1} + 1\right) - 1}}; \qquad \alpha = \frac{q_T - 1}{\omega_T^2}. \qquad (29)$$

An example of the feasibility of deducing ρ from data is presented in fig. 3 for data on $\bar{p}+p$ at 546 GeV [58]. In this case one takes from $P(N)$ $q = 1.27$, from the distribution of p_T one has $q_T = 1.09$, whereas from the original $q_L = 1.36$ one obtains, after correction, $q_L = 1.14$ (cf. fig. 2).

[10] In [47] we used $\alpha = 2/3$ and $\kappa = 1$; for $\rho = 0$. However, the actual values of α and κ parameters are irrelevant in this case.

To summarize this part, note that, to get the correlation coefficient ρ, one has to know *all the fluctuations*, *i.e.*, both in the entire phase space, q, as separately in its transverse, q_T, and longitudinal, q_L, parts (cf. fig. 4). The best known is q (no corrections needed), for q_T the corrections are small and can be neglected, finally, for q_L the corrections are large and must be accounted for (cf. fig. 3).

2.4 Energy fluctuations —heat capacity

We now present energy fluctuations resulting from Tsallis statistics and emerging from our analysis [74,47]. This subject already has its history (cf. [75,76]) and was also recently under investigation (cf. [77,78]).

In Boltzman statistics [74] (with $kT=1/\beta=$const and $N=$const) the energy $U=\sum_{i=1}^{N} E_i$ of N particles is distributed according to

$$g_{T,N}=\frac{\beta}{\Gamma(N)}(\beta U)^{N-1}\exp(-\beta U), \tag{30}$$

for which

$$\frac{\mathrm{Var}(U)}{\langle U\rangle^2}=\frac{k}{C_V^{(B)}}=\frac{1}{N} \qquad \text{where} \qquad C_V=\frac{\partial\langle U\rangle}{\partial T}. \tag{31}$$

In Tsallis statistics [47] one has, respectively,

$$h_N(U)=\frac{\Gamma\left(N+\frac{2-q}{q-1}\right)}{\Gamma(N)\Gamma\left(\frac{2-q}{q-1}\right)}(q-1)^N\beta(\beta U)^{N-1} \cdot\left[1-(1-q)\beta U\right]^{\frac{1}{1-q}+1-N}, \tag{32}$$

for which

$$\begin{aligned}\frac{\mathrm{Var}(U)}{\langle U\rangle^2}&=\frac{1}{4-3q}\left(\frac{k}{C_V^{(T)}}+q-1\right)\\&=\frac{1}{N}+\frac{q-1}{4-3q}\left(1+\frac{1}{N}\right),\end{aligned} \tag{33}$$

where

$$C_V^{(T)}=\frac{\partial\langle U\rangle}{\partial T}=Nk\frac{1}{3-2q}=C_V^{(B)}\frac{1}{3-2q}. \tag{34}$$

Notice that fluctuations of the energy U are, in general, given by the sum of two components: one obtained in the case of no fluctuations and given by the heat capacity $C_V^{(B)}$ (which we call the *kinetic component*) and one originating in fluctuations, and given by the heat capacity $C^{(f)}$ (vanishing when fluctuations vanish, we call it the *potential component*),

$$\frac{\mathrm{Var}(U)}{\langle U\rangle^2}=\frac{k}{C_V^{(B)}}+\frac{k}{C^{(f)}}, \tag{35}$$

where (cf. [75,76])

$$C_V^{(B)}=kN \qquad \text{and} \qquad C^{(f)}=k\frac{N}{N+1}\frac{4-3q}{q-1}. \tag{36}$$

From the analysis of nuclear collisions we know [20] that q depends on N,

$$q-1=\frac{\alpha}{N}, \tag{37}$$

where α is some constant of order unity depending on the reaction considered. We can therefore write

$$\begin{aligned}\frac{\mathrm{Var}(U)}{\langle U\rangle^2}&=\frac{1}{N}+\frac{q-1}{4-3q}\left(1+\frac{1}{N}\right)\\&=\frac{1}{N}\left[\frac{N(\alpha+1)-2\alpha}{N-3\alpha}\right]\overset{N\to\infty}{\Longrightarrow}0.\end{aligned} \tag{38}$$

For small values of $q-1$ (in practice already for $q-1\ll 0.5$) one has

$$\frac{\mathrm{Var}(U)}{\langle U\rangle^2}\geq(q-1)\frac{1+\alpha}{\alpha}=\frac{1+\alpha}{N}. \tag{39}$$

3 Conditional probability —influence of conservation laws

Let $\{E_{1,\dots,N}\}$ be a set of N independent identically distributed random variables described by some parameter λ and let $g_N(E,\lambda)$ denote the gamma density function with parameters N and λ. For independent energies, $\{E_{i=1,\dots,N}\}$, each distributed according to the simple Boltzman distribution,

$$g_1(E_i)=\frac{1}{\lambda}\exp\left(-\frac{E_i}{\lambda}\right), \tag{40}$$

the sum

$$E=\sum_{i=1}^{N}E_i \tag{41}$$

is then distributed according to the following gamma distribution:

$$g_N(E)=\frac{1}{\lambda(N-1)!}\left(\frac{E}{\lambda}\right)^{N-1}\exp\left(-\frac{E}{\lambda}\right). \tag{42}$$

If the available energy is limited, for example, if $E=\sum_{i=1}^{N}E_i=N\alpha=$ const, then we have the following conditional probability for the single particle distribution, $f(E_i)$,

$$\begin{aligned}f(E_i|E=N\alpha)&=\frac{g_1(E_i)\,g_{N-1}(N\alpha-E_i)}{g_N(N\alpha)}\\&=\frac{(N-1)}{N}\frac{1}{\alpha}\left(1-\frac{1}{N}\frac{E_i}{\alpha}\right)^{N-2}.\end{aligned} \tag{43}$$

This is nothing else than the well-known Tsallis distribution

$$f(E_i|E=\text{const})=\frac{2-q'}{\lambda}\left[1-(1-q')\frac{E_i}{\lambda}\right]^{\frac{1}{1-q'}}, \tag{44}$$

with

$$q'=\frac{N-3}{N-2}<1 \qquad \text{and} \qquad \lambda=(3-2q')\,\alpha, \tag{45}$$

which is always less than unity. Here λ = const and do not fluctuate.

Now consider a situation in which the parameter λ in the joint probability distribution

$$g\left(\{E_{1,\dots,N}\}\right) = \prod_{i=1}^{N} g_i\left(E_i\right)$$

fluctuates according to a Gamma distribution, eq. (1). In this case we have the single-particle Tsallis distribution

$$h_i\left(E_i\right) = \frac{2-q}{\lambda}\left[1-(1-q)\frac{E_i}{\lambda}\right]^{\frac{1}{1-q}}, \qquad (46)$$

and the distribution of $E=\sum_{i=1}^{N} E_i$ is given by (cf. [47])

$$h_N(E) = \frac{(q-1)^N \Gamma\left(N+\frac{2-q}{q-1}\right)}{\lambda\Gamma(N)\Gamma\left(\frac{2-q}{q-1}\right)} \cdot \left(\frac{E}{\lambda}\right)^{N-1}\left[1-(1-q)\frac{E}{\lambda}\right]^{1-N+\frac{1}{1-q}}. \qquad (47)$$

If the energy is limited, *i.e.*, if $E=\sum_{i=1}^{N} E_i = N\alpha = \text{const}$, we have the following conditional probability:

$$f\left(E_i|E\right) = \frac{h_i(E_i)h_{N-1}(E-E_i)}{h_N(E)} = \frac{(N-1)(2-q)}{E[(3-2q)-N(1-q)]}\frac{\lambda'}{\lambda}\left(\frac{E-E_i}{E}\right)^{N-1} \cdot\left[1-(1-q)\frac{E_i}{\lambda}\right]^{\frac{1}{1-q}}\left[1+(1-q)\frac{E_i}{\lambda'}\right]^{2-N+\frac{1}{1-q}}, \qquad (48)$$

where

$$\lambda' = \lambda-(1-q)E. \qquad (49)$$

For $q \to 1$, eq. (48) reduces to eq. (43). On the other hand, for large energy ($E\to\infty$) and large multiplicity ($N\to\infty$), the conditional probability distribution (48) reduces to the single-particle distribution given by eq. (46). Introducing the parameter q' defined in eq. (45) the conditional probability (48) can be rewritten as

$$f\left(E_i|E\right) = \frac{\left(2-q'\right)\left(2-q\right)}{E[(3-2q)\left(1-q'\right)-\left(3-2q'\right)(1-q)]}\cdot\frac{\lambda'}{\lambda} \cdot\left(\frac{E-E_i}{E}\right)^{\frac{1}{1-q'}}\cdot\left[1-(1-q)\frac{E_i}{\lambda}\right]^{\frac{1}{1-q}} \cdot\left[1+(1-q)\frac{E_i}{\lambda'}\right]^{\frac{1}{1-q}-\frac{1}{1-q'}}. \qquad (50)$$

For $E_i \ll E$ it becomes

$$f\left(E_i|E\right) \simeq \frac{\left(2-q'\right)(2q-1)(q-1)}{\lambda[(3-2q)\left(1-q'\right)-\left(3-2q'\right)(1-q)]} \cdot\left[1-(1-q)\frac{E_i}{\lambda}\right]^{\frac{1}{1-q}}, \qquad (51)$$

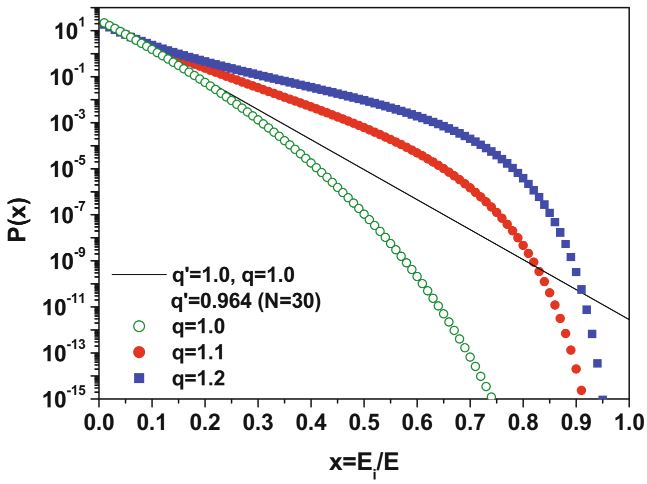

Fig. 5. (Color online) Conditional probability distribution, $P(x=E_i/E)$, for $q=1$ (eq. (44)) and $q>1$ eq. (51), in both cases $N=30$ ($q'=0.964$), compared to exponential distribution ($q=1$, $q'=1$).

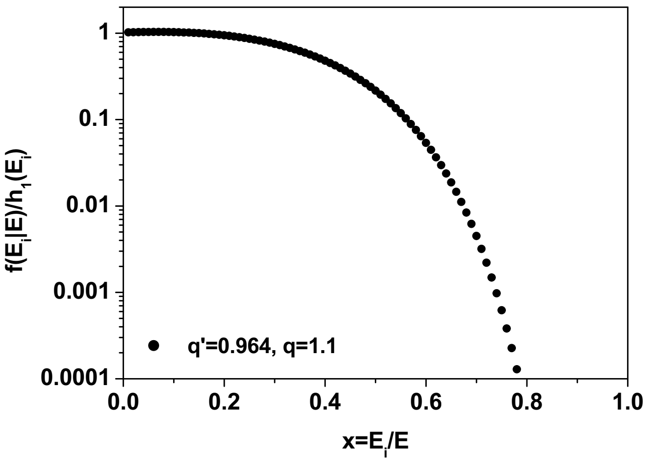

Fig. 6. (Color online) Ratio of conditional distribution function $f(E_i|E)$ and single-particle distribution $h_1(E_i)$ as a function of $x=E_i/E$ for Tsallis statistics ($q=1.1$ and $N=30$).

which, when additionally $N\gg 1$ (or $q'\to 1$) reduces to eq. (46).

The results presented here are summarized in figs. 5 and 6 which show how large differences are (in $x=E_i/E$) between the *conditional* Tsallis distribution $f(E_i|E)$ and the *usual* $h_1(E_i)$[11].

[11] We would like to stress that eq. (43) has the form of a microcanonical distribution in the one dimensional case, $D=1$. In [79,80] it was shown that smearing this distribution over a Gamma-type multiplicity distribution results in a microcanonical generalization of the Tsallis distribution which fits the fragmentation functions measured in e^+e^- experiments with similar $q(s)$ evolution to that presented in fig. 2. It was demonstrated that this type of energy dependence seems to be consistent with the DGLAP evolution equations [81].

4 Tsallis entropy and the Tsallis distribution function —nonadditivity in nuclear collisions

In all the examples discussed so far we treated the Tsallis distribution, eq. (2), as a kind of superstatistics [18, 19] without really resorting to Tsallis entropy [5–7]. However, closer inspection of both approaches reveals that the corresponding nonextensivity parameters (say q and q', respectively) are not identical. In fact one encounters a sort of duality, like $q = 2 - q'$ discussed, for example, in [82, 15–17]. We shall now address this problem in more detail (cf. [31] for details).

When starting from Tsallis entropy [5–7],

$$S_q = \frac{1}{1-q}\left[\int \mathrm{d}x f^q(x) - 1\right], \tag{52}$$

one can obtain the probability density function $f(x)$ either by optimizing it with constraints

$$\int \mathrm{d}x f(x) = 1; \qquad \int \mathrm{d}x x f^q(x) = \langle x\rangle_q, \tag{53}$$

in which case [83]

$$f(x) = (2-q)\left[1-(1-q)x\right]^{\frac{1}{1-q}}, \quad 0 \le x < \infty; \quad 1 \le q \le 3/2, \tag{54}$$

or else by using as constraints

$$\int \mathrm{d}x f(x) = 1; \qquad \int \mathrm{d}x x f(x) = \langle x\rangle, \tag{55}$$

in which case [83]

$$f(x) = \frac{q}{\left[1+(1-q)x\right]^{\frac{1}{1-q}}}; \tag{56}$$

$$0 \le x < \infty; \quad 1/2 < q \le 1. \tag{57}$$

Out of these two possibilities, only (54) is the same as the distribution obtained in superstatistics and used above, cf. eq. (2). On the other hand, the second distribution, eq. (56), which seems to be more natural from the point of view of a physical interpretation of the constraint used, becomes the first one if expressed in terms of q' given by

$$q' = 2 - q. \tag{58}$$

Namely, in this case, one has

$$f(x) = (2-q')\left[1-(1-q')x\right]^{\frac{1}{1-q'}}, \tag{59}$$

which, as show in fig. 2, when compared to single-particle distributions, results in $q' > 1$.

It turns out that there are data allowing the above duality (at least in principle, considering the present status of their quality). They are provided by nuclear collisions in which one observes the apparent nonadditivities which, as will be shown, allow us to compare and discuss both q and q'[12].

We start with the phenomenological approach used to describe nuclear collisions which is based on the superposition model with main ingredients being nucleons that have interacted at least once [84]. In this case, when sources are identical and independent of each other, the total (N) and the mean ($\langle N\rangle$) multiplicities are supposed to be given by

$$N = \sum_{i=1}^{\nu} n_i, \qquad \text{and} \qquad \langle N\rangle = \langle \nu\rangle\langle n_i\rangle, \tag{60}$$

where ν denotes the number of sources and n_i the multiplicity of secondaries from the i-th source. Albeit at present nuclear collisions are mostly described by different kinds of statistical models [1], which automatically account for possible collective effects, nevertheless a surprisingly large amount of data can still be described by assuming the above superposition of independent nucleon-nucleon collisions (possibly slightly modified) as the main mechanism for the production of secondaries. The question of the range of its validity is a legitimate one [85, 86].

Using the notion of entropy, and considering ν-independent systems for which the corresponding individual probabilities are combined as

$$p_q^{(\nu)}\left(x_1,\dots,x_\nu\right) = \prod_{k=1}^{\nu} p_q^{(1)}\left(x_k\right), \tag{61}$$

and assuming that all $p_q^{(1)}(x_k)$ are the same for all k (*i.e.*, their corresponding entropies $S_q^{(1)}$ are equal), one finds

$$S_q^{(\nu)} = \sum_{k=1}^{\nu}\frac{\nu!}{(\nu-k)!k!}(1-q)^{k-1}\left[S_q^{(1)}\right]^k = \frac{\left[1+(1-q)S_q^{(1)}\right]^{\nu}-1}{1-q}. \tag{62}$$

Notice that

$$\ln\left[1+(1-q)S_q^{(\nu)}\right] = \nu\ln\left[1+(1-q)S_q^{(1)}\right] \tag{63}$$

and that

$$S_q^{(\nu)} \xrightarrow{q\to 1} \nu\cdot S_1^{(1)}. \tag{64}$$

For $q < 1$ one has

$$\frac{S_q^{(\nu)}}{\nu} \xrightarrow{\nu\to\infty} \infty, \tag{65}$$

i.e., entropy $S_q^{(\nu)}$ is nonextensive. For $q > 1$ one has

$$S_q^{(\nu)} \ge 0 \quad \text{only for} \quad q < 1 + \frac{1}{S_q^{(1)}}, \tag{66}$$

[12] Apparently similar duality occurs in nonextensive treatment of fermions for which the particle-hole correspondence, $n_q(E,T,\mu) = 1 - n_{2-q}(-E,T,-\mu)$ (where μ is the chemical potential), must be preserved by the q-Fermi distributions [44–46]. However, here we are facing a different problem, namely that parameter q in entropy S_q differs from parameter q' in probability distribution $f_{q'}$ with $q = 2 - q'$.

and

$$\frac{S_q^{(\nu)}}{\nu} \xrightarrow{\nu\to\infty} 0, \tag{67}$$

i.e., entropy is extensive,

$$0 \le \frac{S_q^{(\nu)}}{\nu} \le S_q^{(1)}. \tag{68}$$

In the following we put $\nu = N_W/2 = N_P$ (N_W is the number of wounded nucleons and N_P is the number of participants from a projectile). Assuming naively that the total entropy is proportional to the mean number of produced particles,

$$S = \alpha\langle N\rangle, \tag{69}$$

one obtains the following relation between mean multiplicities in AA and NN collisions:

$$\alpha\langle N\rangle_{AA} = \frac{[1+(1-q)\alpha\langle N\rangle_{pp}]^{N_P} - 1}{1-q}. \tag{70}$$

At this point we stress the following observation, so far not discussed in detail. Namely, because (as shown in [20]), $\langle N\rangle_{AA}$ increases nonlinearly with N_P and $\langle N\rangle_{AA} > N_P \cdot \langle N\rangle_{pp}$, the nonextensivity parameter obtained here from considering the corresponding entropies must be smaller than unity, $q < 1$. On the other hand, all estimations of the nonextensivity parameter (let us denote it by q') discussed before lead to $q' > 1$. This is the q *duality in nonextensive statistics* mentioned above, on which we shall concentrate in more detail.

To start with, the relation (70) is not exactly correct for S_q. In what follows we denote entropy on the level of particle production by s (and the corresponding nonextensivity parameter by $\tilde{q}$), whereas the corresponding entropies and nonextensivity parameter on the level of NN collisions by S and q. From eq. (62) we have that, for N particles,

$$s_{\tilde{q}}^{(N)} = \frac{\left[1+(1-\tilde{q})\,s_{\tilde{q}}^{(1)}\right]^N - 1}{1-\tilde{q}} \xrightarrow{\tilde{q}\to 1} N\cdot s_{\tilde{q}}^{(1)} = \alpha N, \tag{71}$$

where $s_{\tilde{q}}^{(1)} = \alpha$ is the entropy of a single particle. In a $A+A$ collision with ν nucleons participating, eq. (62) results in

$$S_q^{(\nu)} = \frac{\left[1+(1-q)\,S_q^{(1)}\right]^\nu - 1}{1-q}, \tag{72}$$

where $S_q^{(1)}$ is the entropy of a single nucleon.

Denoting multiplicity in a single $N+N$ collision by n, the respective entropy is

$$S_q^{(1)} = S_{\tilde{q}}^{(1)} = \frac{\left[1+(1-\tilde{q})\,s_{\tilde{q}}^{(1)}\right]^n - 1}{1-\tilde{q}}, \tag{73}$$

whereas the entropy in a $A+A$ collision for N produced particles is

$$S_{\tilde{q}}^{(N)} = \frac{\left[1+(1-\tilde{q})\,s_{\tilde{q}}^{(1)}\right]^N - 1}{1-\tilde{q}}. \tag{74}$$

This means that

$$S_{\tilde{q}}^{(N)} = S_q^{(\nu)}. \tag{75}$$

Notice that parameters q and $\tilde{q}$ are usually not identical. Moreover, from the relation

$$q-1 = \frac{1}{aN_P}\left(1-\frac{N_P}{A}\right), \qquad a = \frac{C_V}{N_P}, \tag{76}$$

one finds that for NN collisions (where $N_P = A$) $\tilde{q} = 1$. On the other hand, for $\tilde{q} = q$, eq. (75) corresponds to the situation encountered in superpositions, as in this case one has

$$\left[1+(1-q)s_q^{(1)}\right]^N = \left[1+(1-q)s_q^{(1)}\right]^{n\nu}, \tag{77}$$

and so

$$N = n\nu. \tag{78}$$

Consider now the general case and denote

$$c_1 = 1+(1-\tilde{q})\,s_{\tilde{q}}^{(1)}; \qquad c_2 = \frac{1-q}{1-\tilde{q}}. \tag{79}$$

These quantities are not independent because

$$c_2c_1^N + 1 - c_2 = \left(c_2c_1^n + 1 - c_2\right)^\nu. \tag{80}$$

From relation (80)

$$\frac{N}{\nu\cdot n} = \frac{1}{\nu n\cdot\ln c_1}\ln\left[\frac{\left(c_2c_1^n+1-c_2\right)^\nu - (1-c_2)}{c_2}\right], \tag{81}$$

which for $N = \langle N_{AA}\rangle$, $n = \langle N_{pp}\rangle$ and $\nu = N_P$ is presented in fig. 7 for different reactions. As can be seen there, one can describe experimental data by using $c_2 = 1.7$ and with c_1 depending on energy $\sqrt{s}$ according to $c_1(s) = 1.0006\text{–}0.036s^{-1.035}$. Notice that for energies $\sqrt{s} > 7\,\text{GeV}$ one has $c_1 > 1$. This means that $\tilde{q} < 1$ and (because $c_2 > 0$) also $q < 1$.

To summarize this section, we have shown that non additivity in the superposition model described using the notion of entropy clearly requires $q < 1$, cf. fig. 7. This means that q' is not the same as q. The conclusion one can derive from these considerations is that the second way of deriving $f(x)$, which uses a linear condition, cf. eq. (56), is the correct one, and that q' in the distribution is not the same as q in the entropy. The problem is that, whereas from distributions one can easily deduce a numerical value of q', this is not the case when one uses entropy (at least not when deduced from presently available data). There are too many variables to play with (cf. considerations using the superposition model as above). For example, in the definition of c_1 in eq. (79), one has $s_{\tilde{q}}^{(1)}$, which is not known *a priori*. The only thing one can deduce in this case is that $q < 1$. We cannot therefore check numerically that relation (58) really holds. But, if one agrees that the Tsallis distribution comes from Tsallis entropy, we have only two options: either $q' = q$ or $q' - 1 = 1 - q$. Our conclusion presented here, that $q' > 1$ and $q < 1$, therefore supports the second option, *i.e.*, eq. (58).

However, this final observation calls for comment. Namely, the probability density function (PDF) is usually

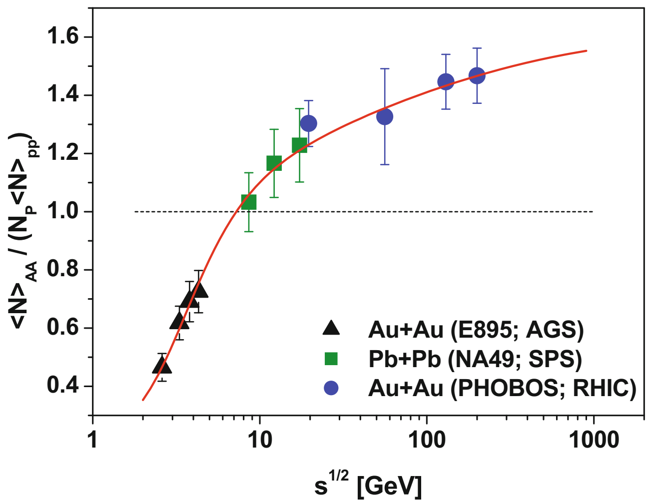

Fig. 7. (Color online) Energy dependence of the charged multiplicity for nucleus-nucleus collisions divided by the superposition of multiplicities from proton-proton collisions (cf. eq. (81)). Experimental data on multiplicity are taken from the compilation [87].

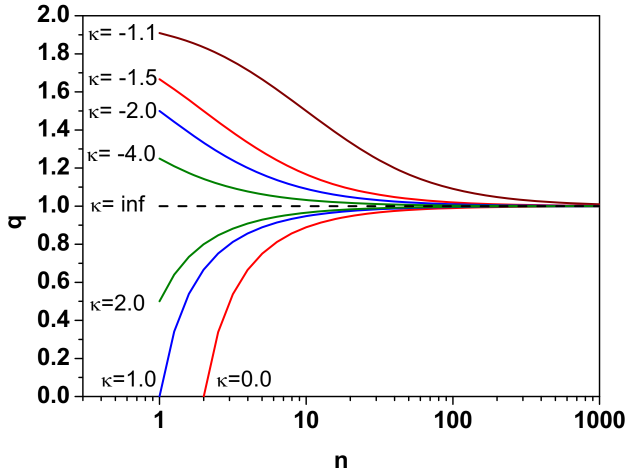

Fig. 8. (Color online) q as function of n given by eq. (84) for different values of κ.

evaluated by the Maximum Entropy Method (MEM) for Tsallis entropy with some constraints [88][13]. Therefore the situation is not unique since there are four possible well-documented MEMs [89,90] using two kinds of definition for an expectation value of the physical quantities: the normal average (55) and the q-average (53) (with normal, as here, or the so-called escort PDFs [91–93]). Although various arguments justifying it have been given [94–96] it was also been pointed out that, for a small change of the PDF, thermodynamic averages obtained by the q-averages are unstable, whereas those obtained by the normal average are stable [97,98]. On the other hand, it is claimed that for the escort PDF, the Tsallis entropy and thermodynamical averages are robust [99]. All this means that the stability (robustness) of thermodynamical averages as well as the Tsallis entropy is still a controversial issue [100].

5 Examples of nonfluctuating (nonthermal) mechanisms leading to the Tsallis distribution

It should be realized that the so far discussed origins of the Tsallis distribution, based either on superstatistics or on Tsallis entropy, are by no means the only possibilities. Therefore we end with short discussions of two examples of obtaining eq. (2) in a completely nonthermal way, these are the application of *order statistics* and the use of *stochastic networks*.

5.1 Order statistics

Order statistics is based on the observation [101] that the selection of the minimal value of the ordered variables leads in a natural way to its distribution being given eq. (2) (with q both greater and smaller than unity, depending on the circumstances), *i.e.*, in fact by the Tsallis distribution, the same as that resulting from Tsallis nonextensive statistics. The distribution of the minimal values of some specific choices of the variable E is known in the literature as order statistics [102][14].

We now present a generalized version of what was proposed in [101]. We start with a set of n virtual particles (so-called *ghost-particles*) with energies ε_i taken from some distribution $f(\varepsilon)$. Ordering the values of ε_i (*i.e.*, introducing in this set *rank statistics*), $\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n$, we choose a *real particle* with minimal energy $E = \varepsilon_1 = \min(\{\varepsilon_i\})$. It is straightforward to find a function $g(E)$ describing the energy distribution of real particles. The probability density to find a particle with energy E among n elements is $nf(E)$. The probability to find particles with energy greater than E is $1-F(E)$, where $F(E) = \int_0^E d\varepsilon f(\varepsilon)$ is the distribuant of f If a particle of energy E is already that of the minimal energy it means that the remaining $n-1$ particles have to poses higher energies. The probability of such an event is equal to $[1-F(E)]^{n-1}$. This means that the distribution of the minimal value in sample of n elements is[15]

$$g(E) = nf(E)[1-F(E)]^{n-1}. \tag{82}$$

[13] Notice that Tsallis entropy is a monotonic function of the Renyi entropy, $S_q = \ln_q[\exp(R_q)]$, and both lead to the same equilibrium statistics of particles (with coinciding maxima in equilibrium for similar constraints on the expectation value).

[14] Actually, one can easily invent a nonthermal scenario leading to a thermal-like form of the observed spectra, see, for example, recent work [103]. In such an approach the resultant distribution emerges not because of the equilibration of energies due to some collisions (*i.e.*, because of the kinematic thermalization), but rather because of the process of erasing of memory of the initial state and is the result of the approaching to a state of maximal entropy (called in [103] stochastic thermalization).

[15] More formally, the cumulative distribution function is $G(E) = 1 - [1-F(E)]^n$ and the density distribution is $g(E) = dG(E)/dE = n[1-F(E)]^{n-1} dF(E)/dE = nf(E)[1-F(E)]^{n-1}$.

Because $f(E) = \mathrm{d}F(E)$, the distribution $g(E)$ is properly normalized if $f(E)$ is normalized. For

$$f(\varepsilon) = -\alpha(\kappa+1)(1+\alpha\varepsilon)^{\kappa}, \tag{83}$$

where $\kappa \neq -1$ (because of the normalization requirement) and $\alpha = -\operatorname{sign}(\kappa+1)\beta$ ($\beta = 1/T > 0$) one gets $g(E)$ in the form of the Tsallis distribution, eq. (2), with

$$q = \frac{n(\kappa+1)-2}{n(\kappa+1)-1}, \tag{84}$$

$q > 1$ for $\kappa < -1$ and $q < 1$ for $\kappa > -1$. Figure 8 shows $q(n)$ dependence for different values of the parameter κ (special cases of $\kappa = -2$ and $\kappa = 0$ were discussed in [101]).

5.2 Stochastic networks

Stochastic network structures occur in almost all branches of modern science (including sociology and economy). They have therefore been the subject of intensive research, also by means of Tsallis statistics (cf. [104–107] for details and full list of references; in [108] this approach has been applied to multiparticle production processes[16]). There are two basic types of stochastic networks:

– Networks with a constant number of nods, M, for which probability that given node has k connections with other nodes (k links) is Poissonian [116],

$$P(k) = \frac{\kappa_0^k}{k!}\cdot e^{-\kappa_0}; \qquad \kappa_0 = \langle k\rangle. \tag{85}$$

– Networks in which the number of nodes is not stationary and the distribution of links $P(k)$ is given by dynamics of the growth of network [117–119]. It varies between being *exponential*,

$$P(k) = \frac{1}{\kappa}\cdot\exp\left(-\frac{k}{\kappa}\right), \tag{86}$$

and *power-like*,

$$P(k) = \frac{2\kappa^2 t}{\kappa_0 + t}\cdot k^{-3}, \tag{87}$$

behavior. In the former case each new node connects with the already existing ones with equal probability, $\Pi(k_i) = 1/(\kappa_0 + t - 1)$, independent of k_i. In the latter case one has preferential attachment (the so-called "rich-get-richer" mechanism, here $\kappa < \kappa_0$ is the number of new nodes added in each time step) with, in this case, $\Pi(k_i) = k_i/(2\kappa t)$ choice.

[16] In [108] the "power laws", assumed *ad hoc* in [109] (as a kind of opposition to Tsallis statistics), was explained using a stochastic networks approach presented here. Actually, this "power laws" idea is continued recently in [110,111] as an apparent new observation. It must be mentioned therefore that this idea is actually quite old; such a type of parametrization of p_T distributions has been proposed (and was shown to be phenomenological successful) already in [112–115].

Let us remind ourselves that, whereas

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = -\frac{1}{\lambda}f(x) \quad\Longrightarrow\quad f(x) = \frac{1}{\lambda}\exp\left(-\frac{x}{\lambda}\right), \tag{88}$$

for the x-dependent scale parameter,

$$\lambda \to \lambda(x) = \lambda_0 - (q-1)x, \tag{89}$$

the exponential solution takes a power-like form,

$$f(x) = \frac{2-q}{\lambda_0}\left[1-(1-q)\frac{x}{\lambda_0}\right]^{\frac{1}{1-q}}. \tag{90}$$

For preferential attachment used in [104,105], dividing the "master equation"

$$\frac{\partial P(k)}{\partial t} = -cP(k), \tag{91}$$

by the assumed "growth of the network"

$$\frac{\partial k}{\partial t} = a + bk, \tag{92}$$

one obtains the following evolution equation for the network considered:

$$\frac{\partial P(k)}{\partial k} = -cP(k)\frac{\partial t}{\partial k} = -\frac{c}{a+bk}P(k). \tag{93}$$

For $c = 1$, $a = \kappa_0$ and $b = q-1$ one has $\kappa(k) = \kappa_0 + (q-1)k$ and a solution of eq. (93) in the form of the Tsallis distribution, eq. (90),

$$P(k) = \frac{2-q}{\kappa_0}\left[1-(1-q)\frac{k}{\kappa_0}\right]^{\frac{1}{1-q}}. \tag{94}$$

For $q \to 1$ eq. (94) recovers eq. (86) whereas for $k \gg \kappa_0/(q-1)$ it leads to "scale-free" power distribution

$$P_q(k) \propto k^{-\gamma}, \qquad \text{with} \qquad \gamma = \frac{1}{q-1}. \tag{95}$$

The frequently observed value $\gamma = 3$ therefore corresponds to $q = 4/3$. At this value of q the variance of distribution $P(k)$ diverges,

$$\mathrm{Var}(k) = \frac{\kappa_0^2(2-q)}{(3-2q)^2(4-3q)} \overset{q\to 4/3}{\Longrightarrow} \infty. \tag{96}$$

We close this section by noticing that formally we can interpret eq. (93) as the stationary solution, of the following Fokker-Planck equation:

$$\frac{\mathrm{d}(K_2P(k))}{\mathrm{d}k} = K_1P(k), \tag{97}$$

where $K_1 = q-2$ and $K_2 = \kappa_0 + (q-1)k$. This corresponds (cf. the network growth given by eq. (92)) to the Langevin equation with multiplicative noise (η) in the form [15,16],

$$\frac{\partial k}{\partial t} + \eta k = \xi, \tag{98}$$

where ξ is the traditional noise term. In this case both noises have nonzero mean values: $\langle\xi(t)\rangle = \kappa_o$ and $\langle\eta(t)\rangle = 1-q$, and correlations: $\mathrm{Cov}(\xi(t),\xi(t')) = 2\kappa_0\delta(t-t')$, $\mathrm{Cov}(\eta(t),\eta(t')) = 0$ and $\mathrm{Cov}(\eta(t),\xi(t')) = (1-q)\delta(t-t')$.

6 Summary

The possibility of the occurrence of intrinsic, nonstatistical temperature fluctuations has far-reaching consequences which we have attempted to present in this review (covering results obtained since [8] or not covered there but worth mentioning). Our work in this field started with a realization that in a nonhomogeneous heat bath one can expect some heat diffusion process to operate. This then results in specific fluctuations of the temperature T, eventually resulting in a Tsallis distribution, eq. (2) [9,10]. Notwithstanding vivid discussions concerning the legitimacy of such a possibility [120–124], this idea has been further elaborated and generalized in [8,15–17].

The results presented here can be summarized as follows:

– Fluctuations of T (of any kind) result in Tsallis distributions (2) with $q > 1$.
– Observables from different parts of the phase space are characterized by different values of $q - 1$. We understand why this is so and are able to connect q as obtained from different observables.
– Constraints imposed by the conservation laws result in a distortion of the Tsallis distribution. In the limiting case (when unconditional distributions are of BG type) conditional distributions become of Tsallis type with $q < 1$.
– Tsallis distributions with $q > 1$ correspond to Tsallis entropy with $q' < 1$.
– The so-called "power law", propositions which occur in the literature [112–115,109–111], are nothing else but Tsallis distributions *in disguise*.

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References

1. M. Gaździcki, M. Gorenstein, P. Seyboth, Acta Phys. Polon. B **42**, 307 (2011) and references therein, for example.
2. L. Stodolsky, Phys. Rev. Lett. **75**, 1044 (1995).
3. H. Heiselberg, Phys. Rep. **351**, 161 (2001).
4. S. Mrówczyński, Acta Phys. Polon. B **40**, 1053 (2009).
5. C. Tsallis, Stat. Phys. **52**, 479 (1988).
6. C. Tsallis, Eur. Phys. J. A **40**, 257 (2009).
7. C. Tsallis, *Introduction to Nonextensive Statistical Mechanics* (Springer, 2009), cf. also; C. Tsallis, *From nolinear statistical mechanics to nonlinear quantum mechanics - Concepts and applications*, arXiv:1202.3178.
8. G. Wilk, Z. Włodarczyk, Eur. Phys. J. A **40**, 299 (2009).
9. G. Wilk, Z. Włodarczyk, Phys. Rev. Lett. **84**, 2770 (2000).
10. G. Wilk, Z. Włodarczyk, Chaos Solitons Fractals **13**, 581 (2001).
11. B. De, S. Bhattacharyya, G. Sau, S.K. Biswas, Int. J. Mod. Phys. E **16**, 1687 (2007).
12. C. Alt *et al.*, Phys. Rev. C **77**, 034906 (2008).
13. C. Alt *et al.*, Phys. Rev. C **77**, 024903 (2008).
14. S.V. Afanasiev *et al.*, Phys. Rev. C **66**, 054902 (2002).
15. T.S. Biró, A. Jakovác, Phys. Rev. Lett. **94**, 132302 (2005).
16. T.S. Biró, G. Purcsel, K. Ürmösy, Eur. Phys. J. A **40**, 325 (2009).
17. T.S. Biró, *Is there a temperature? Conceptual Challenges at High Energy, Acceleration and Complexity* (Springer, 2011).
18. C. Beck, E.G.D. Cohen, Physica A **322**, 267 (2003).
19. F. Sattin, Eur. Phys. J. B **49**, 219 (2006).
20. G. Wilk, Z. Włodarczyk, Phys. Rev. C **79**, 054903 (2009).
21. G. Wilk, Z. Włodarczyk, Cent. Eur. J. Phys. **8**, 726 (2010).
22. O.J.E. Maroney, Phys. Rev. E **80**, 061141 (2009).
23. T.S. Biró, K. Ürmössy, Z. Schram, J. Phys. G **37**, 094027 (2010).
24. T.S. Biró, P. Ván, Phys. Rev. E **83**, 061147 (2011).
25. T.S. Biró, Z. Schram, Eur. Phys. J. Web Conf. **13**, 05004 (2011).
26. M. Nauenberg, Phys. Rev. E **67**, 036114 (2003).
27. M. Nauenberg, Phys. Rev. E **69**, 038102 (2004).
28. C. Tsallis, Phys. Rev. E **69**, 038101 (2004).
29. R. Balian, M. Nauenberg, Europhys. News **37**, 9 (2006).
30. R. Luzzi *et al.*, Europhys. News **37**, 11 (2006).
31. G. Wilk, Z. Włodarczyk, Cent. Eur. J. Phys. **10**, 568 (2012) cf. also arXiv: 1110.4220v2 [hep-ph].
32. STAR Collaboration (B.I. Abelev *et al.*), Phys. Rev. C **75**, 064901 (2007).
33. PHENIX Collaboration (A. Adare *et al.*), Phys. Rev. D **83**, 052004 (2011).
34. CMS Collaboration (V. Khachatryan *et al.*), JHEP **02**, 041 (2010).
35. CMS Collaboration (V. Khachatryan *et al.*), Phys. Rev. Lett. **105**, 022002 (2010).
36. ALICE Collaboration (K. Aamodt *et al.*), Phys. Lett. B **693**, 53 (2010).
37. ALICE Collaboration (K. Aamodt *et al.*), Eur. Phys. J. C **71**, 1594 and 1655 (2011).
38. ATLAS Collaboration (G. Aad *et al.*), New J. Phys. **13**, 053033 (2011).
39. Ming Shao, Li Yi, Zebo Tang, Hongfang Chen, Cheng Li, Zhangbu Xu, J. Phys. G **37**, 085104 (2010).
40. J. Cleymans, D. Worku, J. Phys. G **39**, 025006 (2012).
41. T. Osada, G. Wilk, Phys. Rev. C **77**, 044903 (2008) Phys. Rev. C **78**, 069903(E) (2008).
42. T. Osada, G. Wilk, Prog. Theor. Phys. Suppl. **174**, 168 (2008).
43. T. Osada, G. Wilk, Indian J. Phys. **86**, 941 (2011).
44. J. Rożynek, G. Wilk, J. Phys. G **36**, 125108 (2009).
45. J. Rożynek, G. Wilk, Acta Phys. Polon. B **41**, 351 (2010).
46. J. Rożynek, G. Wilk, Eur. Phys. J. Web Conf. **13**, 5002 (2011).
47. G. Wilk, Z. Włodarczyk, Physica A **390**, 3566 (2011).
48. A.K. Dash, B.M. Mohanty, J. Phys. G **37**, 025102 (2010).

49. C. Geich-Gimbel, Int. J. Mod. Phys. A **4**, 1527 (1989).
50. J.F. Grosse-Oetringhaus, K. Reygers, J. Phys. G **37**, 083001 (2010).
51. Z. Koba, H.B. Nielsen, P. Olesen, Nucl. Phys. B **40**, 319 (1972).
52. M. Rybczyński, Z. Włodarczyk, G. Wilk, Nucl. Phys. B Proc. Suppl. **122**, 325 (2003).
53. F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, Phys. Rev. D **67**, 114002 (2003).
54. P. Carruthers, C.S. Shih, Int. J. Mod. Phys. A **2**, 1447 (1986).
55. C. Vignat, A. Plastino, Phys. Lett. A **360**, 415 (2007).
56. R.A. Fisher, Ann. Eugenics **11**, 182 (1941).
57. V.V. Begun, M. Gaździcki, M. Gorenstein, Phys. Rev. C **78**, 024904 (2008).
58. UA5 Collaboration (G.J. Alner *et al.*), Z. Phys. C **33**, 1 (1986).
59. S. Hegyi, Phys. Lett. B **467**, 126 (1999) and in Proc. ISMD2000, Tihany, Lake Balaton, Hungary (2000).
60. F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, Physica A **340**, 467 (2004).
61. F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, Physica A **344**, 568 (2004).
62. M. Biyajima, M. Kaneyama, T. Mizoguchi, G. Wilk, Eur. Phys. J. C **40**, 243 (2005).
63. M. Biyajima, T. Mizoguchi, N. Nakajima, N. Suzuki, G. Wilk, Eur. Phys. J. C **48**, 593 (2006).
64. G. Wilk, Z. Włodarczyk, W. Wolak, Acta Phys. Polon. B **42**, 1277 (2011).
65. G. Wilk, Z. Włodarczyk, Physica A **390**, 3566 (2011).
66. N. Bohr, *Collected Works*, edited by J. Kalekar, Vol. **6** (North-Holland, Amsterdam, 1985) pp. 316-330 and 376-377.
67. J. Lindhard, *"Complementarity" between energy and temperature*, in *The Lesson of Quantum Theory*, edited by J. de Boer, E. Dal, O. Ulfbeck (North-Holland, Amsterdam, 1986).
68. J. Uffink, J. van Lith, Found. Phys. **29**, 655 (1999).
69. J. Uffink, J. van Lith, *Thermodynamic uncertainty relations*, cond-mat/9806102.
70. B.H. Lavenda, Found. Phys. Lett. **13**, 487 (2000).
71. Kulesh Chandra Kar, Phys. Rev. **21**, 672 (1923).
72. T. Wibig, J. Phys. G **37**, 115009 (2010).
73. G. Wilk, Z. Włodarczyk, PoS **WPCF2011**, 041 (2011) cf. also arXiv:1202.2711[hep-ph].
74. G. Wilk, Z. Włodarczyk, Physica A **376**, 279 (2007).
75. J.L. Lebowitz, J.K. Percus, Phys. Rev. **124**, 1673 (1961).
76. J.L. Lebowitz, J.K. Percus, L. Verlet, Phys. Rev. **153**, 259 (1967).
77. R. Guo, J.L. Du, Physica A **391**, 2853 (2012).
78. L.Y. Liu, J.L. Du, Physica A **387**, 5417 (2008).
79. K. Ürmössy, G.G. Barnaföldi, T.S. Biró, Phys. Lett. B **701**, 111 (2011).
80. K. Ürmössy, G.G. Barnaföldi, T.S. Biró, Acta Phys. Polon. B Proc. Suppl. **5**, 363 (2012).
81. G.G. Barnaföldi, T.S. Biró, K. Ürmössy, G. Kalmár, in *Proceedings of the Workshop Gribov '80* (World Scientific, Singapore, 2011).
82. I.V. Karlin, M. Grmela, N. Gorban, Phys. Rev. E **65**, 036128 (2002).
83. P.N. Rathie, S. Da Silva, Appl. Math. Sci. **2**, 1359 (2008).
84. A. Białas, M. Błeszynski, W. Czyż, Nucl. Phys. B **111**, 461 (1976).
85. K. Fiałkowski, R. Wit, Acta Phys. Polon. B **41**, 1317 (2010).
86. K. Fiałkowski, R. Wit, Eur. Phys. J. A **45**, 51 (2010).
87. PHOBOS Collaboration (B.B. Back *et al.*), Phys. Rev. C **74**, 021902(R) (2006).
88. T. Oikonomou, G.B. Bagci, Phys. Lett. A **374**, 2225 (2010).
89. C. Tsallis, Physica D **193**, 3 (2004).
90. G.L. Ferri, S. Martinez, A. Plastino, J. Stat. Mech. (2005) P04009.
91. C. Beck, F. Schlögl, *Thermodynamics of Chaotic Systems* (Cambridge University Press, Cambridge, 1993).
92. Yu.L. Klimontovich, *Statistical Theory of Open Systems* (Kluwer, Dordrecht, 1995).
93. C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A **261**, 534 (1998).
94. S. Abe, Phys. Rev. E **66**, 046134 (2002).
95. S. Abe, Astrophys. Space Sci. **305**, 241 (2006).
96. S. Abe, G.B. Bagci, Phys. Rev. E **71**, 016139 (2005).
97. S. Abe, EPL **84**, 60006 (2008).
98. S. Abe, Phys. Rev. E **79**, 041116 (2009).
99. R. Hanel, S. Thurner, C. Tsallis, EPL **85**, 20005 (2009).
100. J.F. Lutsko, J.P. Boon, P. Grosfils, EPL **86**, 40005 (2009).
101. G. Wilk, Z. Włodarczyk, AIP Conf. Proc. **965**, 76 (2007) arXiv:0708.2660.
102. H.A. David, *Order Statistics* (Wiley-InterScience, New York, 1981), or H.A. David, H.N. Nagaraja, *Order Statistics*, in *Wiley Series in Probability and Statistics* (Wiley-InterScience, New York, 2003).
103. P. Castorina, D. Kharzeev, H. Satz, Eur. Phys. J. C **52**, 187 (2007).
104. G. Wilk, Z. Włodarczyk, Acta Phys. Polon. B **35**, 871 (2004).
105. G. Wilk, Z. Włodarczyk, Acta Phys. Polon. B **36**, 2513 (2005).
106. C. Tsallis, Eur. Phys. J. ST **161**, 175 (2008).
107. D.J.B. Soares, C. Tsallis, A.M. Mariz, L.R. da Silva, Europhys. Lett. **70**, 70 (2005).
108. G. Wilk, Z. Włodarczyk, Acta Phys. Polon. B **35**, 2141 (2004).
109. M. Gaździcki, M.I. Gorenstein, Phys. Lett. B **517**, 250 (2001).
110. S.K. Biswas, G. Sau, A.C. Das Ghosh, S. Bhattacharyya, Open. J. Microphys. **2**, 1 (2012).
111. A.A. Bylinkin, A.A. Rostovtsev, *Comparative Analysis of Pion, Kaon and Proton Spectra Produced at PHENIX*, arXiv:1203.2840.
112. C. Michael, L. Vanryckeghem, J. Phys. G **3**, L151 (1977).
113. C. Michael, Prog. Part. Nucl. Phys. **2**, 1 (1979).
114. UA1 Collaboration (G. Arnison *et al.*), Phys. Lett. B **118**, 167 (1982).
115. R. Hagedorn, Riv. Nuovo Cimento **6**, 1 (1983).
116. P. Erdös, A. Rényi, Pub. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960).
117. A.L. Barabási, R. Albert, H. Jeong, Physica A **272**, 173 (1999).
118. R. Albert, H. Jeong, A.L. Barabási, Nature **406**, 378 (2000).
119. R. Albert, A.L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
120. T.C.P. Chui, D.R. Swanson, M.J. Adriaans, J.A. Nissen, J.A. Lipa, Phys. Rev. Lett. **69**, 3005 (1992).
121. C. Kittel, Phys. Today **5**, 93 (1988).
122. B.B. Mandelbrot, Phys. Today **42**, 71 (1989).
123. H.B. Prosper, Am. J. Phys. **61**, 54 (1993).
124. G.D.J. Phillies, Am. J. Phys. **52**, 629 (1984).