

# A new energy-dependent quark interaction from a Tamm-Dancoff reduction of an effective field theory quark model

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Received: 23 October 2008 / Revised: 3 December 2008

Published online: 21 January 2009 – © Società Italiana di Fisica / Springer-Verlag 2009

Communicated by G. Orlandini

**Abstract.** A new quark interaction is derived, by means of a Tamm-Dancoff reduction, from an effective field theory constituent quark model. In contrast to the standard Coulombic potential, the obtained interaction is nonlocal and energy dependent. Furthermore, it becomes positive and rises up to a maximum value when the interquark distance increases, partially resembling some aspects of the phenomenological Cornell potential.

**PACS.** 03.65.Pm Relativistic wave equations – 12.39.Ki Relativistic quark model

## 1 Introduction

A successful phenomenological way for studying hadronic systems is represented by the constituent quasi-potential quark models (QPQM). They make use of nonrelativistic or, more recently, relativistic Schrödinger-like equations in which the  $q\bar{q}$  (for mesons) or the  $qqq$  (for baryons) interact by means of suitable quasi-potential. The short-range part of this interaction, inspired to the one-gluon exchange at tree level, is constructed by means of a Fermi-Breit reduction [1] of the corresponding diagram that gives, in the nonrelativistic limit, the usual Coulombic potential. This procedure can be considered correct in the case of the nonrelativistic dynamics of heavy quark systems but, in general, only represents a phenomenological (useful) choice, given that, at extremely short distances, quarks become essentially massless and, in consequence, QPQM cannot be justified at a fundamental level. On the other hand, the long-range part of the interaction must be added *by hand* to reproduce the quark confinement and is usually taken as a linear (with respect to the interquark distance  $r$ ) function. The sum of these two contributions (Coulombic + linear term), denoted as Cornell potential [2] and given in eq. (14), has been adopted for the study of quarkonium [3]. It has been also used with the hypercentral approximation [4] for studying the baryonic systems made up of light quarks [5].

The linear confining behavior of the quark interaction has been given by lattice simulations of QCD [6] but no analytic derivation has been obtained. In general, the quark

confinement is considered an open problem in contemporary physics [7].

Within the standard (perturbative) understanding of field theories, the potential is given by the Fourier transform of the bosonic Feynman propagator, taken in the static limit. In consequence, if the boson is a massless particle one obtains a Coulombic potential and, if the boson mass is nonvanishing, one has the exponentially damped Yukawa potential, widely used in nuclear physics but completely useless for describing confined hadronic systems.

We also emphasize that, in the context of the quark models (in the nonrelativistic regime), a simple inspection of the well-known Fermi-Breit equation [1] shows that the relativistic effects only affect the *short-range* part of the interaction. Also the negative energy states are expected to give short-range contributions [8].

The aim of the present work is to construct a quark interaction model based on a Tamm-Dancoff (TD) reduction [9] of an effective, constituent quark, field Hamiltonian.

We previously point out that, in QCD, the nonperturbative effects produce a nontrivial vacuum structure, that is, with an infinite number of quark-antiquarks in the chiral condensate and an infinite number of gluons in the gluon condensate. In this context, that is drastically different from that of QED, the TD reduction cannot be directly applied.

In our model we *simulate* the action of the QCD quark and gluon fields on the physical vacuum with *effective* quark and gluon field operators acting on the *trivial* vacuum of the model field theory. Also, we take the effective Hamiltonian of the same form as that of QCD.

In general, this procedure can allow to study some field and relativistic effects that are usually ignored when stan-

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dard QPQM are used. In particular, applying the TD reduction to the effective field Hamiltonian, we find that the interaction Kernel of the obtained integral equation is strongly dependent on the mass  $M$  (*i.e.* total rest energy) of the bound system. More precisely, for a  $q\bar{q}$  system, denoting as  $m_q$  the quark (antiquark) mass, our model gives the following result: if  $M > 2m_q$ , as is the case of confined quarks, the interaction Kernel has a *pole* for a definite value of the momentum transfer  $|\mathbf{q}|$ . As a consequence, one obtains an unexpected behavior of the effective potential in the coordinate space. An energy dependence similar to that of our model has been shown to be beneficial for the study of some properties of quarkonium [10]. On the other hand, for  $M < 2m$  that is the standard case of no confined constituents, as  $e^+ e^-$  in positronium and  $e^-$  nucleus in atomic physics (with different constituent masses), no pole is found in the kernel and very small corrections should be added to the standard Coulomb potential.

The remainder of the paper is organized as follows. In sect. 2 we derive by means of the TD reduction the integral equation for the bound system with the mass-dependent potential factor of the model. In sect. 3 we analyze, by means of some approximations, the physical meaning of that potential factor. Finally, in sect. 4 some general conclusions are drawn.

## 2 The TD effective equation

The TD reduction procedure essentially consists in projecting the (field-theoretical) Hamiltonian eigenvalue equation onto a subspace with a fixed number of interacting particles. We adapt a standard derivation [11] to the case of an interaction mediated by a vector massless boson. The derivation is summarized as follows.

For the bound state  $|\Psi\rangle$  we only consider quark-antiquark and gluon quark-antiquark amplitudes:

$$|\Psi\rangle = \varphi|q\bar{q}\rangle + \xi|g q\bar{q}\rangle + \dots \quad (1)$$

For the reasons explained in the introduction, this expansion only makes sense for a constituent quark, field-theoretical model, but not for QCD. The eigenvalue equation is of the standard form

$$(H_0 + H_I)|\Psi\rangle = M|\Psi\rangle. \quad (2)$$

The rest reference frame is used for the calculations, so that the the total mass  $M$  represents the rest energy of the system;  $H_0$  and  $H_I$ , respectively, represent the free and interaction term of the field theory Hamiltonian. Projecting eq. (2) onto the states  $|q\bar{q}\rangle$  and  $|g q\bar{q}\rangle$  the following set of two coupled equations is obtained:

$$[M - 2\epsilon(\mathbf{p})]\varphi(\mathbf{p}) = \int d^3p'_1 d^3p'_2 \bar{h}(\mathbf{p}, \mathbf{p}'_1, \mathbf{p}'_2) \xi(\mathbf{p}'_1, \mathbf{p}'_2), \quad (3)$$

$$[M - \epsilon(\mathbf{p}'_1) - \epsilon(\mathbf{p}'_2) - \omega(\mathbf{k}')] \xi(\mathbf{p}'_1, \mathbf{p}'_2) = \int d^3p'' h(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'') \varphi(\mathbf{p}''), \quad (4)$$

where  $\bar{h}(\mathbf{p}, \mathbf{p}'_1, \mathbf{p}'_2)$  and  $h(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'')$  represent the matrix elements of the gluon absorption and emission, respectively. Also, we have introduced  $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$  that is the relative momentum of the  $q\bar{q}$  system (an analogous relation holds for  $\mathbf{p}''$ ) and  $\mathbf{k}'$  that represents the gluon momentum in the *intermediate* state. Finally,  $\epsilon(\mathbf{p}) = (m_q^2 + \mathbf{p}^2)^{1/2}$  and  $\omega(\mathbf{k}') = |\mathbf{k}'|$ , respectively, represent the quark and gluon energy.

Replacing  $\xi(\mathbf{p}'_1, \mathbf{p}'_2)$  of eq. (4) in eq. (3) and using the standard vector form for the interaction Hamiltonian matrix elements, one finds

$$[M - 2\epsilon(\mathbf{p})]\varphi(\mathbf{p}) = \int d^3p'' \frac{m_q}{\epsilon(\mathbf{p})} \frac{m_q}{\epsilon(\mathbf{p}'')} V(\mathbf{p}, \mathbf{p}'', M) \times \bar{u}_1(\mathbf{p}) \gamma_1^\mu u_1(\mathbf{p}'') \bar{u}_2(-\mathbf{p}) \gamma_2^\mu u_2(-\mathbf{p}'') g_{\mu\nu} \varphi(\mathbf{p}''), \quad (5)$$

where the standard Dirac spinors  $u_i(\mathbf{p}_i)$  and the gamma matrices  $\gamma_i^\mu$  have been used. Possibly divergent terms [11] have been discarded in this preliminar work. The relevant point of this model is the structure of the term that, for brevity, we denote as *potential factor*:

$$V(\mathbf{p}, \mathbf{p}'', M) = -\frac{4}{3} \alpha_s \frac{4\pi}{|\mathbf{p} - \mathbf{p}''| [|\mathbf{p} - \mathbf{p}''| + \epsilon(\mathbf{p}) + \epsilon(\mathbf{p}'') - M]}, \quad (6)$$

where the factor  $-\frac{4}{3}$  is given by the color matrix element in the meson (colorless) state and  $\alpha_s$  is the strong coupling constant. On the other hand, the standard Fermi-Breit reduction gives the Coulombic potential factor

$$V_C(\mathbf{p} - \mathbf{p}'') = -\frac{4}{3} \alpha_s \frac{4\pi}{|\mathbf{p} - \mathbf{p}''|^2}. \quad (7)$$

Note that the potential factor of eq. (6), in contrast to the standard one of eq. (7), is strongly nonlocal and energy dependent. Moreover, for confined systems in which  $M > 2m_q$ , it contains a *pole*, drastically changing the character of the interaction. As anticipated in the introduction, in the case of positronium (or of atomic physics) one has  $M < 2m$  and no pole is found in eq. (6). In this last case the quantity  $\epsilon(\mathbf{p}) + \epsilon(\mathbf{p}'') - M$  is always positive and small and, in the nonrelativistic limit, approximatively corresponds to the absolute value of the (negative) binding energy of the system. In consequence  $V(\mathbf{p}, \mathbf{p}'', M) \simeq V_C(\mathbf{p} - \mathbf{p}'')$ , so that our TD reduction does not give significantly new information about these systems.

In this regard, we recall that a relativistic equation with an energy-dependent factor of the same kind as that of our eq. (6) was derived from the Bethe-Salpeter equation to study Lorentz contraction of bound states in field theories [12]. However, that study was applied only to hydrogen atom and positronium and, in consequence, the energy dependence was correctly neglected. For similar reasons, in the work [11], devoted to the study of deuteron, the energy dependence of the potential factor was not analyzed in detail. The possibility of an energy-dependent potential factor was explored when analyzing the contributions of the off-mass-shell Feynman graphs to the Green

function for a quasi-potential equation with retardation effects [13]. This equation was applied to the study of nuclear bound states and, as before, the energy dependence of the interaction gave no significant result. With respect to the last approach, we observe that the TD reduction procedure allows to avoid the ambiguities encountered when fixing the time component of the momentum transfer  $q^0$ .

Finally, as a test of consistency, we point out that the present model gives back the standard perturbative amplitude for scattering processes simply requiring the standard on-shell condition for the interacting particles, that is by setting  $M = 2\epsilon(\mathbf{p}) = 2\epsilon(\mathbf{p}'')$  in eq. (6).

We conclude this section noting that the TD reduction, represented in the present work by eqs. (3) and (4), follows the standard rules of quantum mechanics for orthogonal states with different content of particles. On the other hand, the final result of eq. (5), due to its energy dependence, does not have the same formal properties of the Schrödinger equation [14].

### 3 The potential factor

To get some insight about the spatial behaviour of our interaction we make, with some approximations, the Fourier transform of the potential factor  $V(\mathbf{p}, \mathbf{p}'', M)$ . Preliminarily, we recall that in the standard case of eq. (7), the Fourier transform of that term gives the local Coulombic potential

$$V_C(\mathbf{r}) = -\frac{4}{3}\alpha_s \frac{1}{r}, \quad (8)$$

where  $r = |\mathbf{r}|$  represents the  $q\bar{q}$  distance. To obtain the previous equation and throughout the present (preliminary) work we consider  $\alpha_s$  as a constant. We also note that the product of the vector currents in the second line of eq. (5) gives *one* in the nonrelativistic limit, allowing to consider, in the same limit,  $V_C(\mathbf{r})$  and, in general the approximated Fourier transform of  $V(\mathbf{p}, \mathbf{p}'', M)$ , as the *potential* in the coordinate space. As discussed in the introduction, the relativistic effects related to the product of the vector currents, only modify the short-range part of the interaction.

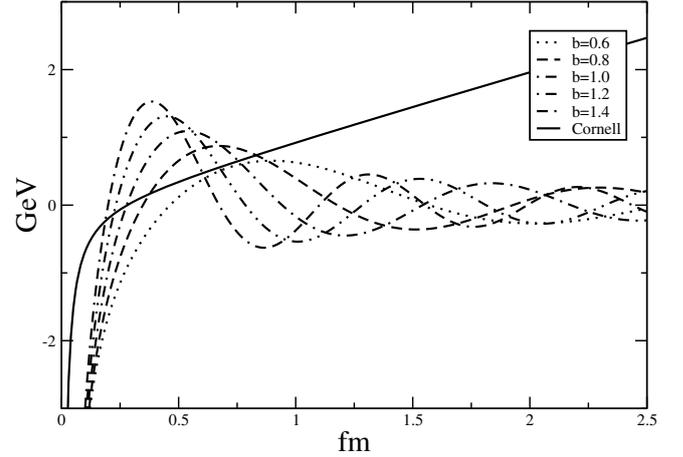
To eliminate nonlocality and make the Fourier transform of eq. (6), as first approximation we consider the extreme nonrelativistic limit, by taking  $\epsilon(\mathbf{p}) \simeq \epsilon(\mathbf{p}'') \simeq m_q$ . Introducing the momentum transfer  $q = |\mathbf{q}| = |\mathbf{p} - \mathbf{p}''|$  one finds

$$V(\mathbf{r}; b) = -\frac{4}{3}\alpha_s \frac{1}{r} \left(\frac{2}{\pi}\right) \int_0^\infty dq \sin(qr) \frac{1}{q-b} \quad (9)$$

being  $b = M - 2m_q$ . In this case the integral can be performed analytically, giving

$$V(\mathbf{r}; b) = -\frac{4}{3}\alpha_s \frac{1}{r} \left(\frac{2}{\pi}\right) \times [\cos(rb)[si(rb) + \pi] - \sin(rb)ci(r|b)]. \quad (10)$$

For the integration [15] we have introduced the sine-integral and cosine-integral functions, respectively, of the



**Fig. 1.** Potential for different values of the parameter  $b$  in the extreme nonrelativistic case.

form

$$si(x) = -\frac{\pi}{2} + \int_0^x dt \frac{\sin(t)}{t}, \quad (11)$$

$$ci(x) = C + \ln(x) + \int_0^x dt \frac{\cos(t) - 1}{t}, \quad (12)$$

where  $C = 0.577216$  is the Euler constant.

To illustrate the character of our interaction we give its expansion obtained by keeping in the parenthesis of eq. (10) the powers up to order  $(rb)^3$  and the logarithmic term:

$$V(\mathbf{r}; b) \simeq \frac{4}{3}\alpha_s \left[ -\frac{1}{r} + b\frac{2}{\pi}(C-1) + b^2\frac{1}{2}r + b^3\frac{2}{\pi} \left( -\frac{C}{6} + \frac{29}{36} \right) r^2 + b\frac{2}{\pi} \ln(r|b|) \right]. \quad (13)$$

Apart from the Coulombic term, all the other contributions of the previous equation are energy dependent through  $b$ . In particular, in the second line we find an attractive term, linear in  $r$ , and an attractive (being  $b > 0$  in confined systems) term, quadratic in  $r$ . A similar potential was used to study heavy quark systems showing that the energy dependence of the confining potential can provide a natural mechanism for the saturation of the spectra [10].

On the other hand, if  $b \simeq 0$  (and negative), eq. (13) only gives very small corrections that should be considered together with all the other field and relativistic effects.

A plot of the energy-dependent potential of eq. (10) is shown in fig. 1 for different values of the energy parameter  $b$ . The Cornell potential

$$V_{Cor}(\mathbf{r}) = -\frac{4}{3}\alpha_s \frac{1}{r} + \sigma r \quad (14)$$

with the standard values  $\alpha_s = 0.3$  and  $\sigma = 1.0 \text{ GeV/fm}$  is also displayed. Just as an example we take  $\alpha_s = 1.1$  for our interaction. One can observe that our potential, in contrast to the Coulombic one, for  $r > \bar{r}$  becomes *positive*, that is  $V(\bar{r}) = 0$ , and for  $r = r_{max}$  reaches the maximum value  $V(r_{max}) = V_{max}$ , linearly increasing with  $b$

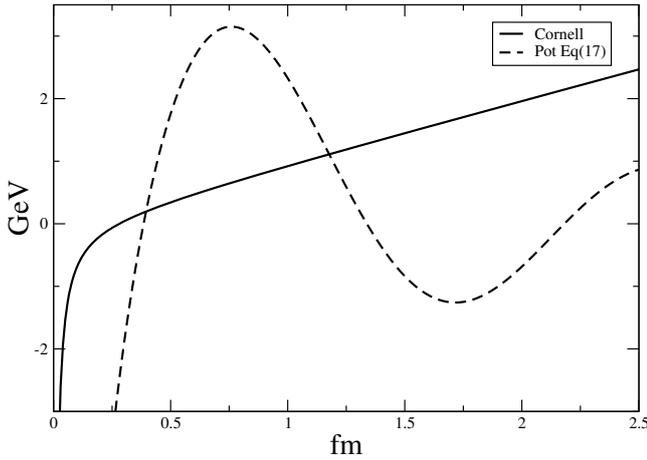


Fig. 2. Potential for the relativistic case.

( $V_{max} \simeq \alpha_s b$ ). This behavior suggests that if the system is excited to a higher-energy state, it finds a higher effective potential that prevents disintegration. Finally, we see that our interaction has an oscillating character, requiring, in this concern, a careful analysis.

In order to show that the main properties of the previous result are not related to the adopted nonrelativistic approximation, we also make a numerical Fourier transform of eq. (6) by means of a different kind of approximation, that is by taking into account some relativistic terms that have been neglected above. To this aim we use the previously defined momentum transfer  $\mathbf{q}$  and introduce  $\mathbf{k} = \mathbf{p} + \mathbf{p}'$ . We express the quantities  $\epsilon(\mathbf{p})$  and  $\epsilon(\mathbf{p}'')$  by means of  $\mathbf{q}$  and  $\mathbf{k}$  and, to avoid angular dependences when making the Fourier transform, we neglect the terms proportional to  $\mathbf{q}\mathbf{k}$ . Note that in eq. (6) these terms exactly cancel at the order  $\mathbf{p}^2/2m_q$ ,  $\mathbf{p}''^2/2m_q$ . One has

$$\epsilon(\mathbf{p}) \simeq \epsilon(\mathbf{p}'') \simeq [q^2/4 + s^2]^{1/2} \quad (15)$$

with

$$s^2 = \mathbf{k}^2/4 + m_q^2 \quad (16)$$

where the nonlocal character of the interaction is taken into account by the dependence on  $\mathbf{k}^2$ . With standard handlings one finally finds

$$V(\mathbf{r}; b; s^2) = -\frac{4}{3}\alpha_s \frac{2}{\pi} \int_0^\infty dq \sin(qr) \times \frac{M - q + 2[q^2/4 + s^2]^{1/2}}{2Mq - M^2 + 4s^2}. \quad (17)$$

The plot of the numerical integration of the previous equation is shown in fig. 2, for the possibly typical (quarkonium) value  $s^2 = 1.69 \text{ GeV}^2$ , given by  $m_q = 1.2 \text{ GeV}$ ,  $\mathbf{k}^2 = 1.0 \text{ GeV}^2$  in eq. (16); we take  $\alpha_s = 1.1$  as before. We see that the general behavior of eq. (10) is confirmed also by the previous expression that contains higher-order relativistic effects. However, the strong nonlocal dependence on  $\mathbf{k}^2$  does not allow to use the potential of eq. (17) in the coordinate space.

## 4 Conclusions

In this work we have shown that, applying a TD reduction to a one gluon exchange effective interaction, an energy-dependent, nonlocal potential factor is obtained for confined quarks. Furthermore, the maximum value of this potential increases linearly with the energy parameter  $b$  of the system. To bring forward the present investigation it is necessary to include in the TD reduction, higher-order states of the Fock space of the effective field theory. On the other hand, one has to study reliable techniques to solve the energy-dependent equation of the model, both in the coordinate space, with the potential of eq. (10) or directly in the momentum space, without further approximations, with eq. (6). A new effective description for confined hadronic systems, with *less* phenomenological *ad hoc* assumptions than QPQM can emerge from this study.

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