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Static analysis of C-shape SMA middle ear prosthesis

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Abstract. Shape memory alloys are a family of metals with the ability to change specimen shape depending on their temperature. This unique property is useful in many areas of mechanical and biomechanical engineering. A new half-ring middle ear prosthesis design made of a shape memory alloy, that is undergoing initial clinical tests, is investigated in this research paper. The analytical model of the studied structure made of nonlinear constitutive material is solved to identify the temperature-dependent stiffness characteristics of the proposed design on the basis of the Crotti-Engesser theorem. The final integral expression for the element deflection is highly complex, thus the solution has to be computed numerically. The final results show the proposed shape memory C-shape element to behave linearly in the analysed range of loadings and temperatures. This is an important observation that significantly simplifies the analysis of the prototype structure and opens wide perspectives for further possible applications of shape memory alloys.

1 Introduction

Shape memory alloys (SMAs) are a family of metals with the ability of changing shape depending on their temperature [1]. Basically, one- or two-way shape memory effects (SME) can be observed. The fundamental difference between the one-way and the two-way SME is that there is no reverse shape change after specimen cooling in the case of the one-way SME, whereas, for the two-way SME, this phenomenon is present. The observed deformations of SMA structural elements result from the martensite and austenite material phases transitions induced by the temperature. Moreover, material phase changes may also be induced, if the SMA specimen is subjected to mechanical loading. This behaviour is known as pseudoelasticity effect.

The thermo-mechanical properties of SMAs can be modelled at different scales. In microscopic and mesoscopic approaches [2], the material behaviour is modelled starting from the molecular and lattice levels, respectively. Another class of models is based on a macroscopic approach, where only phenomenological features of the SMAs are taken into account [3]. Most often these models are based on an assumed phase transformation kinetics and consider certain mathematical functions to describe the phase transformation behaviour of the material. This approach was first proposed by Tanaka and Nagaki [4], and it provided a stimulus for the scientific community to present other modified transformation kinetics laws, see *e.g.* papers by Liang and Rogers [5] and Brinson [6]. These models are very popular in the literature, and play an important role in SMAs structures modelling and analysis [3].

Another group of phenomenological models is based on Devonshire's theory, which postulates a free energy potential function expressed as a polynomial in material strain. Initially proposed for a one-dimensional stress state by Falk [7], it was later extended for a three-dimensional context by Falk and Konopka [8]. Afterwards, a similar model was proposed by Fremond [9] and many others [10–14], also in a simplified form. These free energy potential models can reproduce both the pseudoelastic and shape memory effects depending on the temperature and the stress-strain state. Therefore, Falk's nonlinear SMA model is used in the present study.

The main task of the present paper is to identify and examine the thermo-mechanical properties of a SMA C-shape micro-actuator representing a prototype design of a middle-ear prosthesis made of the shape memory alloy. In particular, the analytical calculations are aimed at stating whether the nonlinear SMAs material behaviour is significant for the mechanical properties of the considered structure or not. Moreover, the stiffness of the prosthesis at the temperature of the human body is to be evaluated. Expected results are important from the practical perspective,

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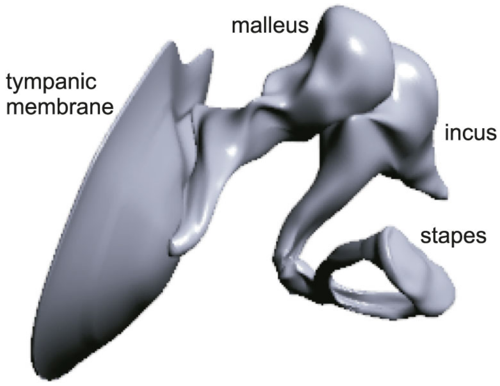

Fig. 1. Intact human middle ear.

Fig. 2. Classical titanium human middle ear prosthesis [15].

since the analysed micro-actuator design is to be implemented in the medical treatment of patients suffering from certain types of the hearing-organ damages. This comes from an innovative concept of C-shape memory-material prosthesis (CSMP) used to reconstruct a damaged ossicular chain. For more information on this idea please refer to the next section.

The paper is organised as follows: first, an exemplary classical human middle ear prostheses is presented followed by a brief description of a new design. Next, a nonlinear model of the SMA material elaborated by Falk is discussed in detail. Then, this formulation is adopted in the analytical model of the SMA material C-shape prosthesis structure. Finally, the temperature-dependent stiffness characteristics of the studied system are plotted by combining a closed form analytical solution and a numerical computation method. In the last section several remarks and practical conclusions are presented.

2 Middle ear prosthesis

The intact human middle ear consists of three ossicles: the malleus, the incus, and the stapes (fig. 1). These ossicles form a sound conduction system that transmits sound waves from the external ear to the fluids of the inner ear. Ossicles are connected to each other while the whole ossicular chain is fixed to the temporal bone. Inflammatory diseases, such as chronic suppurative otitis media or cholesteatoma, may lead to the ossicular chain damages and subsequently to reduced hearing or even irreversible hearing loss. The principle surgical treatment is ossicles reconstruction or even their full replacement by a prosthesis. In the specific case of an incus destruction a common procedure is to connect the stapes directly to the malleus and the tympanic membrane by an inserted implant.

A variety of middle ear prostheses are commercially available, for example, the titanium Kurz prostheses presented in fig. 2 [15]. Despite its numerous advantages the major drawback of this type of design is the fact that the total length of this prosthesis requires adjustment according to individual patient needs. This has to be done by the medical

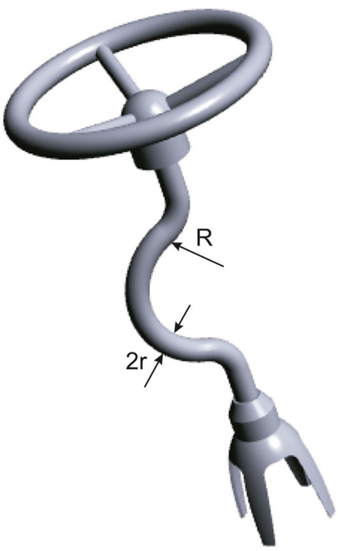

Fig. 3. Conceptional view of the C-shape memory prosthesis of the middle ear made of SMA.

staff during a two-stage surgery operation. After a prior *in situ* check regarding an individual size, the implant has to be removed, trimmed and next placed back and finally anchored. This extends the overall anesthesia time and is more harmful for the patients. Therefore, a new type of prosthesis made of SMA (Nitinol) is proposed that can be easily adjusted to match individual patients needs.

The discussed concept of the prosthesis in the form of a small SMA C-shape structure is presented in fig. 3. The fundamental benefit of this design is the fact that the length of the prosthesis can be tailored to the individual requirements right after its placement. This can be done by a surgeon with any point heat source, like a laser beam, dipole electrode, etc. By setting the right amount of input thermal energy, the material of the prosthesis (a nickel-titanium metal alloy) undergoes deformation to meet the required implant length. After the adjustment —due to the one-way shape memory effect exhibited by the Nitinol— the length remains unchanged, and the prosthesis operates in the constant ambient temperature of the human body. Therefore the new idea of CSMP offers an innovative treatment in medical care reducing the average time of surgery by eliminating the prosthesis-trimming stage and decreasing patient burden.

To successfully implement the discussed innovative concept of the C-shape memory-material prosthesis the stiffness characteristics of the proposed structural design have to be known. Moreover, the importance of the reported nonlinear material behaviour for the mechanical properties of the considered structure needs to be studied.

3 Model definition

As discussed in the first section, there are many different constitutive relations used in stress-strain analysis of specimens made of shape memory alloys. In this research the polynomial model proposed by Falk [7] is adopted. The model is based on the Devonshire's theory, and considers a polynomial formulation of the material free energy. This approach was originally proposed for one-dimensional media and later extended to the three-dimensional context [8]. According to this model, neither internal variables nor dissipation potential are necessary to describe the pseudoelasticity and shape memory effect [3]. The only state variables incorporated by this model are the specimen strain ϵ and its temperature T. Thus, the free energy is formulated as a function of ϵ and T, so that its local minimum and maximum extremes represent a stability and instability in each phase of the SMA.

According to Paiva [3], one-dimensional models usually consider three main alloy phases: austenite (A) and two variants of martensite (M+, M−). Hence, the free energy is defined such that, for the austenite phase, that is present at high temperatures ($T > T_{\mathrm{A}}$), there is only one minimum (global) at the vanishing strain —note the red line in fig. 4. This minimum represents the equilibrium of the austenitic phase. On the other hand, at low temperatures ($T < T_{\mathrm{M}}$), the martensite gets stable and the free energy has two minima (global ones) at non-vanishing strains —note the blue line in fig. 4. At intermediate temperatures ($T_{\mathrm{M}} < T < T_{\mathrm{A}}$), the free energy has equilibrium points represented by local extremes and corresponding to constituents of both phases (the black line in fig. 4).

Considering the above discussion, the free energy should be an even function with respect to the strain variable and has to the defined by a sixth-order polynomial, so that its local minima and maxima represent the stability and

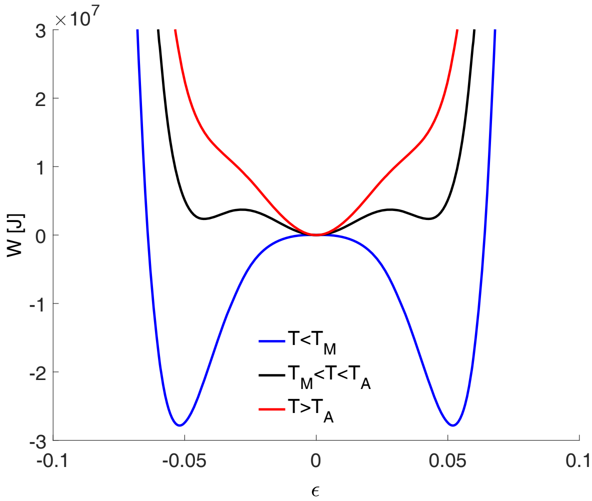

Fig. 4. Free energy of SMA spring at different temperatures according to Falk's model.

instability of SMA phases, respectively. Thus, the free energy is defined as [3]

$$W(\epsilon, T) = \frac{a}{2}(T - T_{\mathrm{M}})\epsilon^2 - \frac{b}{4}\epsilon^4 + \frac{b^2}{24a(T_{\mathrm{A}} - T_{\mathrm{M}})}\epsilon^6, \tag{1}$$

where a and b are positive material constants. Then, the stress in the SMA specimen is given by

$$\sigma = \frac{\partial W}{\partial \epsilon} = a(T - T_{\mathrm{M}})\epsilon - b\epsilon^3 + \frac{b^2}{4a(T_{\mathrm{A}} - T_{\mathrm{M}})}\epsilon^5. \tag{2}$$

The original eq. (2) can be simplified to the form

$$\sigma = \frac{\partial W}{\partial \epsilon} = \alpha_1(T - T_{\mathrm{M}})\epsilon + \alpha_2\epsilon^3 + \alpha_3\epsilon^5. \tag{3}$$

Note that for for the sake of formulation clarity, $\alpha_2 = -b$ is substituted. Usually, ϵ is relatively small and thus the term $\alpha_3\epsilon^5$ is often neglected.

4 Problem analysis

4.1 Analytical solution

To analyse the C-shape element, let us consider a half-ring of radius R loaded by a system of external compressing forces P in equilibrium as represented in fig. 5. The structure is made of a SMA wire material, which exhibits the nonlinear constitutive characteristic

$$\sigma = \alpha_1(T - T_{\mathrm{M}})\epsilon + \alpha_2\epsilon^3, \tag{4}$$

where T is ambient temperature, T_{M} is martensite transition temperature and α_1, α_2 are stiffness coefficients as given in eq. (3); meanwhile, the highest-order strain term present in eq. (3) is skipped for the purpose of further analysis. This can be done since the expected magnitude of structural deflections for the middle ear prosthesis is small, as reported by Volandri *et al.* [16] The wire radius is denoted by r and it is assumed that the ratio R/r is high enough so that the member curvature due to bending is not very pronounced with respect to its neutral state. Therefore the regular (straight beam) flexure formula may be used to compute the stresses distribution. As reported in [17], this simplification is justified if the radius of the curvature-to-depth ratio of the ring is more then 5.

To solve analytically this class of structural equilibrium problems the classical work-energy principle for elastic materials is commonly used. Displacement f of an arbitrary system point can be found following the Castigliano's second theorem. This states that, provided the body is in equilibrium, the derivative of the structure strain energy U with respect to the load applied at a given point is equal to the deflection of this point projected on the direction of that force.

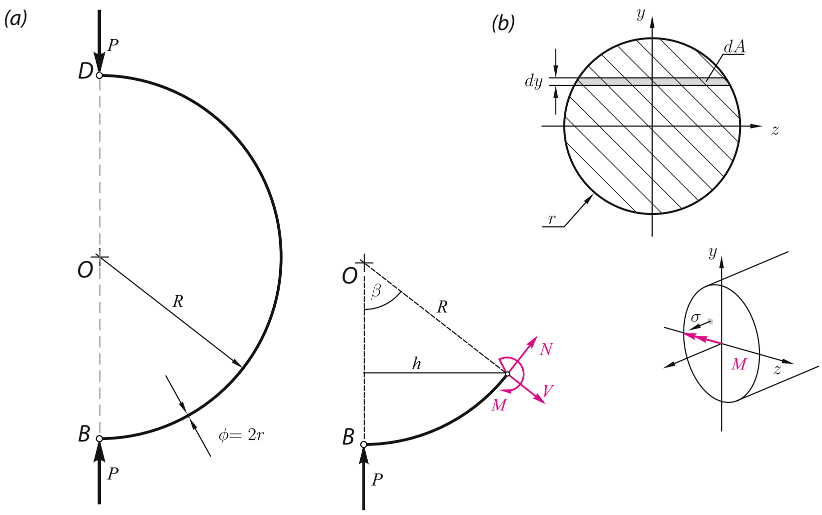

Fig. 5. Analytical model of the half-ring middle ear system prosthesis.

By force and displacement one means generalised force and generalised displacement, that is, a force/displacement pair, or a moment/bending (twisting) angle pair. For problems involving k multiple forces, this implies

$$f_k = \frac{\partial U}{\partial P_k}\,, \tag{5}$$

where P_k is the loading imposed at the point of deflection.

Structure strain energy U and strain energy density u are defined as

$$U = \int_{V_0} u\,\mathrm{d}V_0 \quad \text{and} \quad u = \int_0^{\epsilon_\mathrm{F}} \sigma\,\mathrm{d}\epsilon, \tag{6}$$

where V_0 is the body volume and ϵ_F is the magnitude of the strain at system final deformation.

However, this theory is not applicable to materials exhibiting the nonlinear strain-stress relationship. For the non-Hookean materials, *e.g.* if the elasticity is stress-dependent, the similar theorem by Crotti-Engesser holds. According to this theorem the prescribed deflection at a given point of the structure is equal to the partial derivative of the complementary energy U_c with respect to the driving force acting at this point

$$f_k = \frac{\partial U_c}{\partial P_k}\,. \tag{7}$$

The strain energy density and its complementary one are shown in fig. 6 by shaded and unshaded areas, respectively. The complementary energy density u_c is defined by

$$u_c = \int_0^{\sigma_\mathrm{F}} \epsilon\,\mathrm{d}\sigma, \tag{8}$$

where σ_F is the magnitude of the final stress. In the specific case of linear material law, there is no difference between U, being a function of strain components, and U_c, being a function of stress components, and Crotti-Engesser's theorem is nothing but the statement of Castigliano's theorem in terms of complementary strain energy.

To use the Engesser-Crotti's theorem, the closed form expression for strain with respect to stress is necessary. However, if the calculation of strain is cumbersome it is more convenient to calculate stresses and use them for further analysis. To this aim the complementary strain energy density u_c may be expressed as a difference of a load potential density l and the strain energy density u (see also fig. 6)

$$u_c(\sigma) = \sigma_\mathrm{F}\epsilon_\mathrm{F} - u(\epsilon), \tag{9}$$

where $l = \sigma_\mathrm{F}\epsilon_\mathrm{F}$ is the load potential density. This relationship, which is sometimes referred to as Friedrichs' transformation [18], implies that if the strain energy density exists, then the existence of the complementary strain energy is assured.

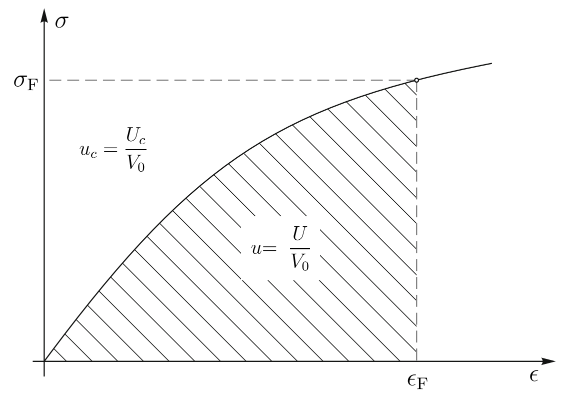

Fig. 6. Strain energy and complementary strain energy densities for an axial strain.

In the case of nonlinear constitutive material law given by relation (4), one observes the calculation of strain for the complementary energy formula (8) to be much complicated. Thus, it is more convenient to follow the Friedrichs' transformation approach. According to the definition (6) the strain energy density u may be expressed as

$$u = \int_0^{\epsilon} \left[\alpha_1(T - T_{\mathrm{M}})\epsilon + \alpha_2\epsilon^3\right] \mathrm{d}\epsilon = \frac{\alpha_1\epsilon^2}{2}(T - T_{\mathrm{M}}) + \frac{\alpha_2\epsilon^4}{4} = \frac{\alpha_1 y^2}{2\rho^2}(T - T_{\mathrm{M}}) + \frac{\alpha_2 y^4}{4\rho^4}\,, \tag{10}$$

where the strain distribution is given by the flexure formula for a straight specimen $\epsilon = y/\rho$; ρ and y are the bending curvature and distance from cross-section neutral axis (see fig. 5(b)), respectively. In further analysis of the discussed structure only the bending effect and its associated energy is taken into account. Contributions coming from normal N and shear forces V can be neglected for slender elements [18]. It can be easily shown that, for the studied geometry, compression and shear energies are about two orders of magnitude smaller than the bending component. Thus the strain energy per unit wire length is

$$\tilde{u} = \int_A u\,\mathrm{d}A = 2\int_0^r 2\sqrt{r^2 - y^2}\left[\frac{\alpha_1(T - T_{\mathrm{M}})y^2}{2\rho^2} + \frac{\alpha_2 y^4}{4\rho^4}\right]\mathrm{d}y = \frac{\pi\alpha_1 r^4}{8\rho^2}(T - T_{\mathrm{M}}) + \frac{\pi\alpha_2 r^6}{32\rho^4}\,, \tag{11}$$

where the expression for an infinitesimal cross-section area element $\mathrm{d}A = 2\sqrt{R^2 - y^2}\mathrm{d}y$ has been used.

The load potential density per unit length is

$$\tilde{l} = \int_A \sigma\epsilon\,\mathrm{d}A = 4\int_0^r \frac{y}{\rho}\left[\frac{\alpha_1(T - T_{\mathrm{M}})y}{\rho} + \frac{\alpha_2 y^3}{\rho^3}\right]\sqrt{r^2 - y^2}\,\mathrm{d}y = \frac{\pi\alpha_1 r^4}{4\rho^2}(T - T_{\mathrm{M}}) + \frac{\pi\alpha_2 r^6}{8\rho^4}\,. \tag{12}$$

Regarding (11), (12) and Friedrichs' transformation the complementary strain energy per unit length is

$$\tilde{u}_c = \frac{\pi\alpha_1 r^4}{8\rho^2}(T - T_{\mathrm{M}}) + \frac{3\pi\alpha_2 r^6}{32\rho^4}\,. \tag{13}$$

To express the complementary strain energy with respect to the loading force P, let us consider the equilibrium condition of the specimen element about the Oz-axis —see also fig. 5(b)—

$$\int_A y\sigma\,\mathrm{d}A + M = 0. \tag{14}$$

Substituting (4) for the stress, $\epsilon = y/\rho$ and for $\mathrm{d}A$ as above results in the general formula for the specimen bending curvature ρ,

$$\rho = \frac{\pi^2\alpha_1^2 r^6}{12M\gamma}(T - T_{\mathrm{M}})^2 + \frac{\pi\alpha_1 r^4}{12M}(T - T_{\mathrm{M}}) + \frac{r^2}{12M}\gamma, \tag{15}$$

where

$$\gamma = \sqrt[3]{\pi^3\alpha_1^3(T - T_{\mathrm{M}})^3 r^6 + 6\pi M\sqrt{6\pi^2\alpha_1^3(T - T_{\mathrm{M}})^3\alpha_2 r^6 + 324\alpha_2^2 M^2} + 108\pi\alpha_2 M^2}. \tag{16}$$

The above formula (15) is the nonlinear expression for the bending curvature ρ with respect to the bending moment M and ambient temperature T. If the linear and isotropic material constitutive law is adopted ($\alpha_2 = 0$ and $\alpha_1 = E$) eq. (15) reduces to the well-known relation

$$\frac{1}{\rho} = \frac{M}{EI_z}\,, \quad \text{where } I_z = \frac{\pi r^4}{4}\,. \tag{17}$$

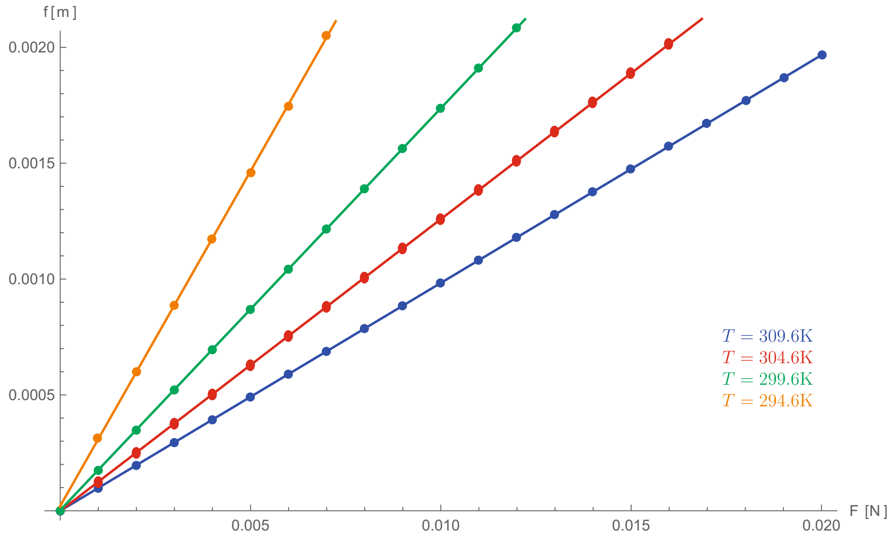

Fig. 7. Stiffness characteristics of shape memory CMA for different ambient temperatures.

Setting for the curvature (15), (16) and expression for the bending moment $M = PR\sin\beta$ into (13), one arrives at the relation for the complementary energy $\tilde{u}_c$ per unit length. Next, integrated over the BD arc length

$$U_c = 2\int_0^{\frac{\pi}{2}} \tilde{u}_c R\,\mathrm{d}\beta, \tag{18}$$

one arrives at the final expressions for the complementary energy. Inserting into Engesser-Crotti theorem (7) the final expression for the specimen deflection is

$$f = \int_0^{\pi/2} \frac{36\pi R^2 A_2 P \sin^2\beta}{A_1^3}\left[\alpha_1(T - T_\mathrm{M}) + \frac{216\alpha_2 R^2 A_2^2 P^2 \sin^2\beta}{r^2 A_1^2}\right] \cdot \left[A_1 A_2 + A_1 A_3 P - \pi\alpha_1 r^2 A_2 A_3 (T - T_\mathrm{M})P - 2A_2^2 A_3 P\right]\mathrm{d}\beta, \tag{19}$$

where subsidiary functions A_i, $i = 1, 2, 3$ are given as

$$\begin{aligned}
A_1 &= A_1(\beta, P, T) = \pi^2\alpha_1^2 r^4 (T - T_\mathrm{M})^2 + \pi\alpha_1 r^2 A_2 (T - T_\mathrm{M}) + A_2^2\\
A_2 &= A_2(\beta, P, T) = \sqrt[3]{\pi^3 r^6 \alpha_1^3 (T - T_\mathrm{M})^3 + 6\sqrt{6}\pi A_4 PR\sin\beta + 108\pi\alpha_2 P^2 R^2 \sin^2\beta}\\
A_3 &= A_3(\beta, P, T) = \frac{2\sqrt{6}\alpha_2 R\sin\beta}{A_4} A_2\\
A_4 &= A_4(\beta, P, T) = \sqrt{\pi^2\alpha_1^3\alpha_2 r^6 (T - T_\mathrm{M})^3 + 54\alpha_2^2 P^2 R^2 \sin^2\beta}.
\end{aligned} \tag{20}$$

The above expression is a highly nonlinear function of the element loading force P and ambient temperature T. The integral variable β is involved in radicals, as well as in trigonometric and rational functions (20), so it is not possible to evaluate indefinite integral to get the closed form solution for the specimen deflection. However, it is still possible to evaluate the definite integral numerically and then plot the stiffness characteristic for a prescribed temperature.

4.2 Results

Numerical calculations were performed for the experimental design of the C-shape middle ear prosthesis. The geometry and material data used in the analysis are as follows: $r = 0.3\times10^{-3}$ m; $R = 3\times10^{-3}$ m; $T_\mathrm{M} = 287$ K; $\alpha_1 = 1\times10^9$ Pa/K; $\alpha_2 = -40\times10^{12}$ Pa. Thermo-mechanical properties of the SMA come from [3].

The stiffness characteristics of the C-shape element for different temperatures are presented in fig. 7. It may be observed that the shape memory actuator is linear in the analysed range of loadings, despite material properties are nonlinear. This is an important observation that simplifies structural analysis of such elements and enables its easy application to various mechanical and biomechanical systems. The element with a linear characteristic is always much more predictable and simpler to control than a nonlinear one.

Moreover, the stiffness of the discussed C-shape structure is strongly dependent on the temperature. Generally, the higher the ambient temperature the higher the effective stiffness of the C-shape memory-material prosthesis. In the presented analysis the temperature $T = 309.6\,\mathrm{K}$ ($36.6\,^{\circ}\mathrm{C}$) is engaged because of a potential application of this C-shape memory-material structure as a middle ear prosthesis. The given above properties, combined with the linear structural stiffness covering the range of implant operating deformations being below 0.5 mm [16] guarantee a regular response of the reconstructed ossicular chain under periodic excitations.

5 Final remarks

The proposed model of C-shape element made of SMA has the linear stiffness characteristic despite the fact that a nonlinear material model has been adopted. This result significantly simplifies the problem and opens a wide perspective of possible applications in mechanical and biomechanical engineering. In the specific context of the postulated C-shape memory prosthesis design several promising observations are made. These are the following:

- Linear characteristic of the proposed implant assures an easy control and good reconstructed ossicular chain predictability as opposed to nonlinear structures that can demonstrate even chaotic response in dynamic states.
- The prosthesis stiffness is strongly influenced by the ambient temperature. Thus, the final operating conditions and temperature of the human body $36.6\,^{\circ}\mathrm{C}$ has to be taken into account at the design stage.
- The CSMP can change its shape and total length during temperature activation. These unique properties of the material allow the surgeon to easily tailor the implant length to fit the needs of an individual patient. This adjustment can be done right after prosthesis placement, and there is no need to remove it for trimming. Comparing to other prosthesis designs and related surgical procedures, the proposed treatment reduces anesthesia time and is less harmful for the patients.
- The studied middle ear prosthesis prototype is a new application of a shape memory structure that extends the list of existing bioengineering SMA designs, like reinforcements of arteries and veins, dental wires, etc. However, the proposed actuator design can also be used in other fields of engineering and industrial sectors.

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