Eur. Phys. J. E (2017) **40**: 100
DOI 10.1140/epje/i2017-11591-7

THE EUROPEAN PHYSICAL JOURNAL E

Regular Article

Single soliton solution to the extended KdV equation over uneven depth

George Rowlands[1], Piotr Rozmej[2,a], Eryk Infeld[3], and Anna Karczewska[4]

[1] Department of Physics, University of Warwick, Coventry, CV4 7A, UK
[2] Institute of Physics, Faculty of Physics and Astronomy, University of Zielona Góra, Szafrana 4a, 65-246 Zielona Góra, Poland
[3] National Centre for Nuclear Research, Hoża 69, 00-681 Warszawa, Poland
[4] Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Szafrana 4a, 65-246 Zielona Góra, Poland

Received 4 August 2017 and Received in final form 28 October 2017
Published online: 20 November 2017

Abstract. In this note we look at the influence of a shallow, uneven riverbed on a soliton. The idea consists in an approximate transformation of the equation governing wave motion over an uneven bottom to an equation for a flat one for which the exact solution exists. The calculation is one space dimensional, and so corresponds to long trenches or banks under wide rivers or oceans.

1 Introduction

Recently, we have found exact solitonic [1] and periodic [2] wave solutions for water waves moving over a smooth riverbed. Amazingly they were simple, though governed by a more exact expansion of the Euler equations with several new terms added when compared to the Korteweg–de Vries (KdV) equation [1,3–5]. Our next step is to consider how these results are modified by a rough river or ocean bottom. We start with a simple case. The geometry is one space dimensional and the wave a soliton. Even so, approximations rear their head! The consideration of a two-dimensional bump on the bottom, as well as periodic waves propagating overhead, are planned for a later effort.

Here we consider the following equation governing the elevation of the water surface η/H above a flat equlibrium at the surface (written in dimensionless variables):

$$\begin{aligned}&\eta_t+\eta_x+\alpha\frac{3}{2}\eta\eta_x+\beta\frac{1}{6}\eta_{3x}\\&+\alpha^2\left(-\frac{3}{8}\eta^2\eta_x\right)+\alpha\beta\left(\frac{23}{24}\eta_x\eta_{2x}+\frac{5}{12}\eta\eta_{3x}\right)+\beta^2\frac{19}{360}\eta_{5x}\\&+\beta\delta\frac{1}{4}\left(-\frac{2}{\beta}(h\eta)_x+(h_{2x}\eta)_x-(h\eta_{2x})_x\right)=0.\end{aligned}\tag{1}$$

The first line gives the KdV equation. The first and second line give the KdV2 equation (both corresponding to even bottom). The last three terms are due to a bottom profile. We emphasize, that (1) was derived in [1,5] under the assumption that α, β, δ are small (positive by definition) and of the same order. As usual, $\alpha=A/H$, *i.e.*, the ratio of the wave amplitude A to the mean water depth H and $\beta=(H/L)^2$, where L is the mean wavelength. The parameter $\delta=A_h/H$ is the ratio of the amplitude of the bottom function $h(x)$ to the mean water depth. Up to this point A, H, L, A_h are dimension quantities. Scaling to dimensionless variables allows us to apply the perturbation approach to the set of Euler equations governing the model of an ideal fluid. Assuming a flat bottom, the KdV equation is obtained from the first-order perturbation approach. Applying the second-order perturbation approach Marchant and Smyth [3] derived eq. (1) limited to the first two lines, the so-called *extended KdV equation.* Since it is derived from second-order perturbation with respect to small parameters, we call it KdV2. Taking into account small bottom fluctuations (again using the second-order perturbation approach) led us in [1,5] to the KdV2 equation for an uneven bottom (1). In scaled variables the amplitudes of wave and bottom profiles are equal to one. In [1,2] we derived exact soliton and periodic solutions to KdV2. These solutions are given by the same functions as the corresponding KdV solutions but with different coefficients.

This paper presents an attempt to describe the dynamics of the exact KdV2 soliton when it approaches a finite interval of an uneven bottom. We will use the reductive perturbation method introduced by Taniuti and Wei [6]. Using two space scales allows us to transform the equation for an uneven bottom (1) into the KdV2 equation with some coefficients altered, that is, the equation for

[a] e-mail: `p.rozmej@if.uz.zgora.pl`

the flat bottom. This transformation is approximate but the analytical solution to the resulting equation is known. This approximate analytic description will be compared with exact numerical calculations.

2 KdV2 soliton (even bottom)

In this section we briefy review the exact soliton solution to the KdV2 equation given in [1].

Assume the form of a soliton moving to the right, $\eta(x,t) = \eta(x-vt)$. So, $\eta_t = -v\,\eta_x$ and the KdV2 equation, that is (1) without the last row, becomes an ordinary differential equation (ODE):

$$(1-v)\eta_x + \alpha\,\frac{3}{2}\eta\eta_x + \beta\,\frac{1}{6}\eta_{3x} - \frac{3}{8}\alpha^2\eta^2\eta_x + \alpha\beta\left(\frac{23}{24}\eta_x\eta_{2x} + \frac{5}{12}\eta\eta_{3x}\right) + \beta^2\frac{19}{360}\eta_{5x} = 0. \tag{2}$$

Integration gives

$$(1-v)\eta + \alpha\,\frac{3}{4}\eta^2 + \beta\,\frac{1}{6}\eta_{2x} - \frac{1}{8}\alpha^2\eta^3 + \alpha\beta\left(\frac{13}{48}\eta_x^2 + \frac{5}{12}\eta\eta_{2x}\right) + \beta^2\frac{19}{360}\eta_{4x} = 0. \tag{3}$$

Then the solution is assumed in the same form as the KdV solution

$$\eta(y) = A\,\mathrm{sech}^2(By), \tag{4}$$

where $A=1$, since in dimensionless variables the amplitude is already rescaled. However, for further consideration it is convenient to keep the general notation. The insertion of the postulated form of the solution (4) and the use of properties of hyperbolic functions give (3) in the polynomial form

$$C2\,\mathrm{sech}^2(By) + C4\,\mathrm{sech}^4(By) + C6\,\mathrm{sech}^6(By) = 0, \tag{5}$$

which requires the simultaneous vanishing of all coefficients $C2$, $C4$, $C6$. These three conditions are as follows:

$$(1-v) + \frac{2}{3}B^2\beta + \frac{38}{45}B^4\beta^2 = 0, \tag{6}$$

$$\frac{3A\alpha}{4} - B^2\beta + \frac{11}{4}A\alpha\,B^2\beta - \frac{19}{3}B^4\beta^2 = 0, \tag{7}$$

$$-\left(\frac{1}{8}\right)(A\alpha)^2 - \frac{43}{12}A\alpha\,B^2\beta + \frac{19}{3}B^4\beta^2 = 0. \tag{8}$$

Denoting $z = \frac{\beta B^2}{\alpha A}$ one obtains (8) as a quadratic equation with respect to z with solutions

$$z_1 = \frac{43-\sqrt{2305}}{152} \approx -0.033 < 0 \tag{9}$$

and

$$z_2 = \frac{43+\sqrt{2305}}{152} \approx 0.599 > 0. \tag{10}$$

Since $B = \sqrt{\frac{\alpha}{\beta}zA}$, only z_2 provides a real B value. (In principle sech^2 of imaginary argument can be expressed by a quotient of expressions given by hyperbolic functions of real arguments. However, these expressions are singular for some values of arguments and therefore physically irrelevant.)

Equations (7) and (8) are consistent only when $\alpha = \alpha_s = \frac{3(51-\sqrt{2305})}{37} \approx 0.242399$. Then (6) determines velocity

$$v = 1 + \frac{2}{3}\alpha_s z_2 + \frac{38}{45}(\alpha_s z_2)^2 \approx 1.114546. \tag{11}$$

3 Variable depth

Equation (1) can be written in the form

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}f(\eta,h) = 0, \tag{12}$$

where $f(\eta,h)$ is given by

$$f(\eta,h) = \eta + \frac{3\alpha}{4}\eta^2 - \frac{\alpha^2}{8}\eta^3 + \alpha\beta\left[\frac{13}{48}\left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{5}{12}\eta\frac{\partial^2\eta}{\partial x^2}\right] + \frac{\beta}{6}\frac{\partial^2\eta}{\partial x^2} + \frac{19}{360}\beta^2\frac{\partial^4\eta}{\partial x^4} + \beta\delta\left[-\frac{2}{\beta}h\eta + \frac{\partial^2 h}{\partial x^2}\eta - h\frac{\partial^2\eta}{\partial y^2}\right]. \tag{13}$$

We treat h as slowly varying and introduce two space scales x and $x_1 (=\epsilon x)$ which are treated as independent until the end of the calculation [6]

$$h = h(\epsilon x) = h(x_1), \quad \epsilon \ll 1. \tag{14}$$

We also introduce

$$y = \int_0^x a(\epsilon x)\mathrm{d}x - t, \tag{15}$$

where a is as yet undefined. To first order in ϵ one has

$$\eta = \eta_0(y,x_1) + \epsilon\eta_1(y,x_1) + \dots, \tag{16}$$

$$\frac{\partial\eta}{\partial t} = -\frac{\partial\eta_0}{\partial y} - \epsilon\frac{\partial\eta_1}{\partial y} + \dots, \tag{17}$$

$$\frac{\partial\eta}{\partial x} = a(x_1)\frac{\partial\eta_0}{\partial y} + \epsilon\frac{\partial\eta_0}{\partial x_1} + \epsilon a(x_1)\frac{\partial\eta_1}{\partial y} + \dots, \tag{18}$$

$$\frac{\partial^2\eta}{\partial x^2} = a^2\frac{\partial^2\eta_0}{\partial y^2} + \epsilon\left(\frac{\partial a}{\partial x_1}\frac{\partial\eta_0}{\partial y} + 2a\frac{\partial^2\eta_0}{\partial y\partial x_1} + a^2\frac{\partial^2\eta_0}{\partial y^2}\right) + \dots. \tag{19}$$

Now

$$\frac{\partial^n\eta}{\partial x^n} = a^n\frac{\partial^n\eta_0}{\partial y^n} + O(\epsilon). \tag{20}$$

We have

$$
\begin{aligned}
f(\eta,h) &= \eta_0 + \frac{3\alpha}{4}\eta_0^2 - \frac{\alpha^2}{8}\eta_0^3 \\
&+\alpha\beta\left[\frac{13}{48}a^2\left(\frac{\partial\eta_0}{\partial y}\right)^2 + \frac{5}{12}\eta_0 a^2\frac{\partial^2\eta_0}{\partial y^2}\right] \\
&+\frac{\beta}{6}a^2\frac{\partial^2\eta_0}{\partial y^2} + \frac{19}{360}\beta^2 a^4\frac{\partial^4\eta_0}{\partial y^4} \\
&+\beta\delta\left[-\frac{h(x_1)\eta_0}{2\beta} - a^2 h(x_1)\frac{\partial^2\eta_0}{\partial y^2}\right] + O(\epsilon) \\
&= f_0(\eta_0,h) + O(\epsilon).
\end{aligned} \tag{21}
$$

From (12), (17) and (21) to lowest order we have

$$
-\frac{\partial\eta_0}{\partial y} + a\frac{\partial}{\partial y}\left[f_0(\eta_0,h)\right] = 0 \tag{22}
$$

and, since $a = a(x_1)$, we obtain

$$
\frac{\partial}{\partial y}(\eta_0 - a f_0) = 0. \tag{23}
$$

We restrict the consideration to a single soliton, so $\eta_0 \to 0$ as $y \to \pm\infty$ and so does f_0. The integration of (23) yields

$$
\eta_0 - a(x_1) f_0 = 0 \tag{24}
$$

to lowest order. Let us introduce $\zeta = y/a(x_1)$ which remains constant in our approximation. Now,

$$
\frac{\partial\eta_0}{\partial y} = \frac{1}{a}\frac{\partial\eta_0}{\partial\zeta}, \tag{25}
$$

and from (24), (21), (25) we obtain

$$
\begin{aligned}
&(1-a(x_1))\eta_0 - \frac{3\alpha}{4}\eta_0^2\, a + \frac{\alpha^2}{8}\eta_0^3\, a \\
&-\alpha\beta\left[\frac{13}{48}\left(\frac{\partial\eta_0}{\partial\zeta}\right)^2 + \frac{5}{12}\eta_0\frac{\partial^2\eta_0}{\partial\zeta^2}\right] a - \frac{19}{360}\beta^2\frac{\partial^4\eta_0}{\partial\zeta^4}\, a \\
&+\beta\delta\, h(x_1)\left[\frac{\eta_0}{2\beta} + \frac{\partial^2\eta_0}{\partial\zeta^2}\right] a - \frac{\beta}{6}\frac{\partial^2\eta_0}{\partial\zeta^2}\, a = 0.
\end{aligned} \tag{26}
$$

Dividing by $(-a)$ yields

$$
\begin{aligned}
&\left(1 - \frac{\delta h}{2} - \frac{1}{a}\right)\eta_0 + \frac{3\alpha}{4}\eta_0^2 - \frac{\alpha^2}{8}\eta_0^3 \\
&+\alpha\beta\left[\frac{13}{48}\left(\frac{\partial\eta_0}{\partial\zeta}\right)^2 + \frac{5}{12}\eta_0\frac{\partial^2\eta_0}{\partial\zeta^2}\right] \\
&+\frac{19}{360}\beta^2\frac{\partial^4\eta_0}{\partial\zeta^4} + \frac{\beta}{6}(1-6\,\delta h)\frac{\partial^2\eta_0}{\partial\zeta^2} = 0.
\end{aligned} \tag{27}
$$

This should be compared to (3) or to eq. (22) of [1]. Remember that at this stage $\delta h(x_1)$ is to be treated as a constant with respect to integration over ζ. The only difference is that the coefficient $(1 - \frac{\delta h}{2} + \frac{1}{a})$ appears instead of $(1-v)$ in the first term and the coefficient $(1 - 6\,\delta h)$ appears instead of 1 in the last term.

Following [1] we obtain

$$
\eta_0 = A\,\mathrm{sech}^2(B\,\zeta), \quad \zeta = \frac{1}{a(x_1)}\left[\int_0^x a(x_1)\mathrm{d}x - t\right]. \tag{28}
$$

In eqs. (6), (7) ((24), (25) and (20) of [1]) we replace $\beta\,B^2$ (but not $\beta^2 B^4$ or $\alpha\beta A B^2$ since we modify only first-order terms) by

$$
\beta\,(1 - 6\,\delta h)\,B^2. \tag{29}
$$

Now $z = z_2 = \frac{43+\sqrt{2305}}{152}$ is as in (9). Hence

$$
\eta_0 = \bar{A}\,\mathrm{sech}^2\left[\frac{\bar{B}}{a(x_1)}\left(\int_0^x a(x_1)\mathrm{d}x - t\right)\right] \tag{30}
$$

with

$$
\frac{1}{a} + \frac{\delta h}{2} = v - \beta\delta h, \qquad q = \frac{b}{B^2}, \quad b = \frac{3z}{\frac{76}{3}z - 11} \tag{31}
$$

and

$$
\bar{A} = A(1 + q\delta h), \qquad \bar{B} = B(1 + q\delta\, h/2),
$$

where A, B, v are given by eqs. (30)–(32) in [1]. Thus

$$
\frac{1}{a} = v - \left(\frac{1}{2} + \beta\right)\delta h. \tag{32}
$$

At this stage we take $x_1 = \epsilon\, x$ and $\delta h = \delta h(x)$. So

$$
\int_0^x a(x)\,\mathrm{d}x = \int_0^x \frac{\mathrm{d}x}{v - (\frac{1}{2}+\beta)\delta h(x)}\,. \tag{33}
$$

Assume $\delta h(x)$ is nonzero only for $x \in [L_1, L_2]$.
For $x < L_1$, $\eta_0 = A\,\mathrm{sech}^2(B(x - vt))$, $\delta h \equiv 0$, $\frac{1}{a} = v$.
For $x > L_2$, $\delta h \equiv 0$, $\frac{1}{a} = v$ and

$$
\eta_0 = A\,\mathrm{sech}^2\left[B\left(v\int_{L_1}^{L_2} a(x)\mathrm{d}x + (x - vt)\right)\right]. \tag{34}
$$

There is a change of phase as the pulse passes through the region where $\delta h \neq 0$. The alteration in the phase is given by

$$
\int_{L_1}^{L_2}\mathrm{d}x\left[\frac{1}{1 - \frac{(1/2+\beta)\delta h}{v}} - 1\right] \approx \frac{\beta + 1/2}{v}\int_{L_1}^{L_2}\delta\, h(x)\,\mathrm{d}x. \tag{35}
$$

If this integral is zero the phase is unaltered. This can happen if a deeper region is followed by a shallower region of appropriate shape or vice versa.

3.1 Examples

In the following figures we present the time evolution of the approximate analytic solution (30) to the KdV2 equation with uneven bottom (1) for several parameter values of the system. These evolutions are compared with "exact" numerical solutions of (1). In both cases the initial conditions were the exact solutions of the KdV2 equation.

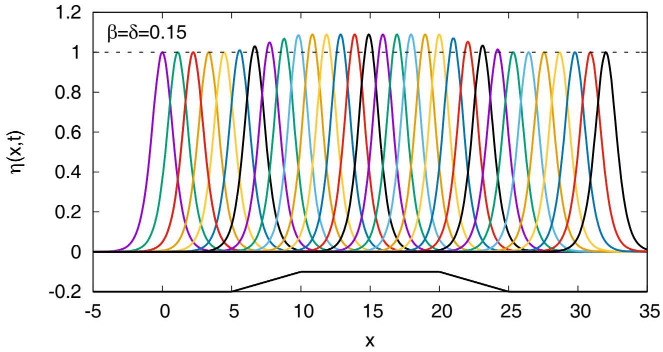

Fig. 1. Profiles of the soliton as given by (30) with $\beta = \delta = 0.15$. The shape of the trapezoidal bottom is shown (not in scale). Consecutive times are $t_n = n$, $n = 0, 1, 2, 3, \ldots, 32$.

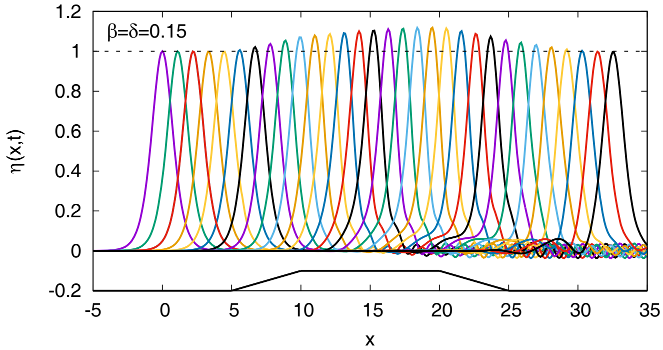

Fig. 2. Profiles of the numerical solution of eq. (1) obtained with the same initial condition. Time instants are the same as in fig. 1.

Therefore in all the presented examples $\alpha = \alpha_s$ and the amplitude of the initial soliton is equal to 1.

In fig. 1 we present the approximate solution (30) for the case when the soliton moves over a trapezoidal elevation with $L_1 = 5$ and $L_2 = 25$. We took $\beta = \delta = 0.15$. For smaller δ the effects of an uneven bottom are very small, for larger δ second-order effects (not present in the analytic approximation) cause stronger overlaps of different profiles.

We compare this approximate solution of (1) to a numerical simulation obtained with the same initial condition. The evolution is shown in fig. 2. We see that the approximate solution has the main properties of the soliton motion as governed by eq. (1). However, since the numerical solution contains higher-order terms depending on the shape of h the exact motion as obtained from numerics shows additional small amplitude structures known from earlier papers, for example [1,5]. This is clearly seen in fig. 3 where profiles obtained in analytic and numeric calculations are compared at time instants $t = 0, 5, 10, 15, 20, 25, 30$ on a wider interval of x. All numerical results were obtained with calculations performed on a wider interval $x \in [-30, 70]$ with periodic boundary conditions. Details of numerics are described in [1,2,5].

In figs. 4–6 we present results analogous to those presented in figs. 1–3 but with a different shape of the bot-

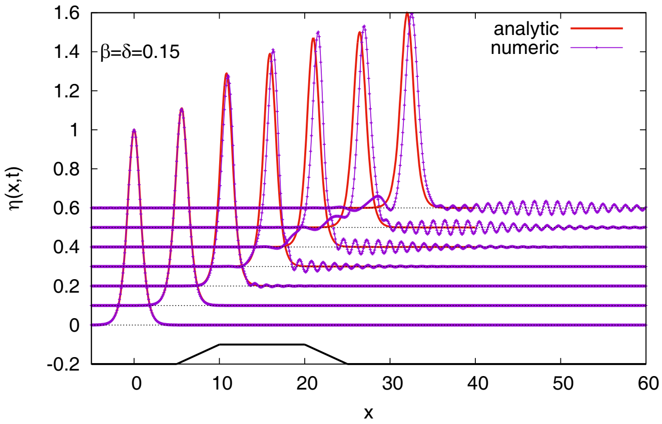

Fig. 3. Comparison of the wave profiles shown in figs. 1 and 2 for time instants $t = 0, 5, 10, 15, 20, 25, 30$. Consecutive profiles are vertically shifted by 0.1.

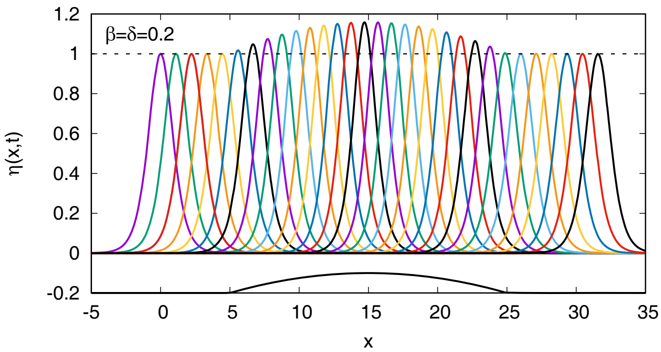

Fig. 4. Profiles of the soliton as given by (30) with $\beta = \delta = 0.2$. The shape of the parabolic bottom is shown (not in scale). Consecutive times are $t_n = n$, $n = 0, 1, 2, 3, \ldots, 32$.

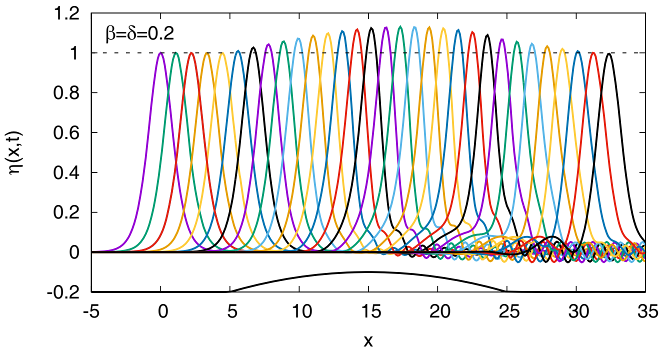

Fig. 5. Profiles of the numerical solution of eq. (1) obtained with the same initial condition. Time instants are the same as in fig. 4.

tom bump and larger values of $\beta = \delta = 0.2$. In this case the bump is chosen as an arc of parabola $h(x) = 1 - (x - 15)^2/100$ between the same $L_1 = 5$ and $L_2 = 24$ as in the trapezoidal case.

In the approximate analytic solution, the KdV2 soliton changes its amplitude and velocity only over bottom fluctuations. When the bottom bump is passed it comes back to its initial shape (only the phase may be changed).

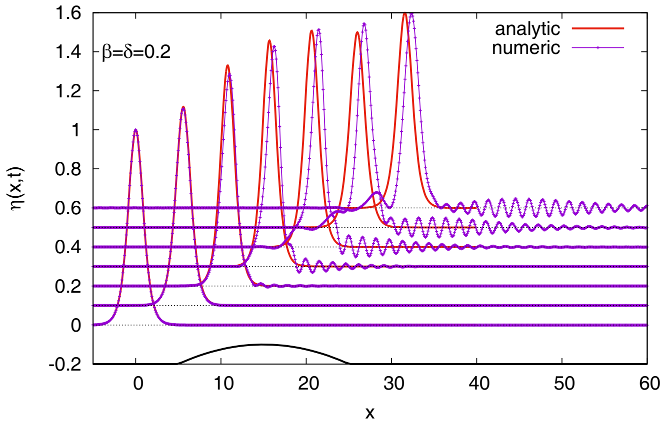

Fig. 6. Comparison of the wave profiles shown in figs. 4 and 5 for time instants $t = 0, 5, 10, 15, 20, 25, 30$.

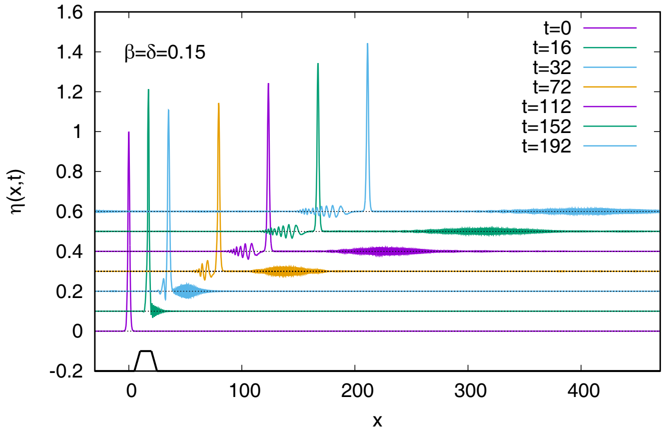

Fig. 7. Long-time numerical evolution with a trapezoidal bottom bump for $\beta = \delta = 0.15$.

This is not the case for the "exact" numerical evolution of the same initial KdV2 soliton when it evolves according to the second-order equation (1). This is clearly visible in figs. 3 and 6. What is this motion for much longer periods? In order to answer this question one has to perform numerical calculations on a much wider interval of x. Such results are presented in fig. 7. The interaction of a soliton with the bottom bump creates two wave packets of small amplitudes. The first moves with a higher frequency faster than the soliton and is created when the soliton enters the bump, while second moves slower with a lower frequency and appears when the soliton leaves it. After some time both are separated from the main wave. Since periodic boundary conditions were used in the numerical algorithm, the head of the wave packet radiated forward travelled for $t > 170$ larger distance than the interval chosen for the calculation and is seen on the left side of the wave profile.

We have to emphasise that this behaviour is generic, and looks similar for different shapes of bottom bumps and different values of β, δ parameters. This was observed in our earlier papers [1, 7, 8] in which initial conditions were in the form of the KdV soliton.

4 Conclusions

We have derived a simple formula which gives an approximate description of a soliton encountering an uneven riverbed. The model reproduces the known increase in amplitude when passing over a shallower region, as well as the change in phase. However, the full dynamics of the soliton motion is much richer, with the uneven bottom causing low amplitude soliton radiation both ahead and after the main wave. This behaviour was observed in our earlier papers [1, 7, 8] in which initial conditions were in the form of the KdV soliton, whereas in the present cases the KdV2 soliton, that is, the exact solution of the KdV2 equation has been used.

Author contribution statement

All the authors were involved in the preparation of the manuscript. All the authors have read and approved the final manuscript.

References

1. A. Karczewska, P. Rozmej, E. Infeld, Phys. Rev. E **90**, 012907 (2014).
2. E. Infeld, A. Karczewska, G. Rowlands, P. Rozmej, *Exact solitonic and periodic solutions of the extended KdV equation*, arXiv:1612.03847.
3. T.R. Marchant, N.F. Smyth, J. Fluid Mech. **221**, 263 (1990).
4. G.I. Burde, A. Sergyeyev, J. Phys. A: Math. Theor. **46**, 075501 (2013).
5. A. Karczewska, P. Rozmej, Ł. Rutkowski, Phys. Scr. **89**, 054026 (2014).
6. T. Taniuti C.C. Wei, J. Phys. Soc. Jpn. **24**, 941 (1968).
7. A. Karczewska, P. Rozmej, E. Infeld, Phys. Rev. E **92**, 053202 (2015).
8. A. Karczewska, P. Rozmej, E. Infeld, G. Rowlands, Phys. Lett. A **381**, 270 (2017).