Galaxy rotation curves via conformal factors

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Abstract We propose a new formula to explain circular velocity profiles of spiral galaxies obtained from the Starobinsky model in the Palatini formalism. It is based on the assumption that the gravity can be described by two conformally related metrics: one of them is responsible for the measurement of distances, while the other, the so-called dark metric, is responsible for a geodesic equation and therefore can be used for the description of the velocity profile. The formula is tested against a subset of galaxies taken from the HI Nearby Galaxy Survey (THINGS).

1 Introduction and motivation

Since there exist many issues that recently appeared in fundamental physics, astrophysics and cosmology which cannot be explained by General Relativity (GR) [1–3], one looks for other approaches which allow one to understand their mechanism. Classical GR is a well-posed theory. Many astronomical observations tested GR and have confirmed that it is the best matching theory that we have had so far for explaining gravitational phenomena. Unfortunately, GR is not enough to describe many unsolved problems such as late-time cosmic acceleration [4, 5] (which one explains by existing an exotic fluid called Dark Energy introduced to the standard Einstein’s field equations as cosmological constant), the Dark Matter puzzle [6–13, 16], inflation [17, 18], and the renormalization problem [19].

There are two main ideas competing for an explanation of the Dark Matter problem: geometric modification of the gravitational field equations (see e.g. [13–15]) or going beyond the Standard Model of elementary particles and introducing weakly interacting particles, which have failed to be detected [20]. In fact, these two ideas do not contradict each other and can be combined in some future successful theory. The existence of Dark Matter is mainly indicated by anomalies in

the observed galactic rotation curves. It interacts only gravitationally with visible matter and radiation, and also it has effects on the large-scale structure of the universe [21, 22].

Many interesting and promising models have faced the Dark Matter problem. The most famous one is Modified Newtonian Dynamics (MOND) [23–30]. It has predicted many galactic phenomena and hence it is widely used by astrophysicists. Closely related is the so-called Tensor/Vector/Scalar (TeVeS) theory of gravity [31, 32] which is, roughly speaking, the relativistic version of MOND. Another approach is to consider Extended Theories of Gravity (ETGs)—one modifies the geometric part of the field equations [33, 34, 36]. There are also attempts to obtain a MOND result from ETGs; see for example [37–42]. Another interesting proposal for explaining rotation curves is by using Weyl conformal gravity [43–45]. It should be noted that there is a model based on a quantum effective action and large-scale renormalization group effects [46, 47], later on constrained by Solar System tests [48]. It provides, up to the first nontrivial order, a conformal transformation of the spacetime metric and a logarithmic term in the modified Newtonian potential.

In the following paper we would like to show how the so-called “Dark Metric” $h_{\mu\nu}$ [49, 50], that is, the metric which is conformally related to the physical metric $g_{\mu\nu}$ appearing in an action of a theory under consideration, may explain the flatness problem of the galaxies’ rotation curves. To this aim we employ the Ehlers–Pirani–Schild approach (EPS) [51] in the way it is presented and considered in Refs. [49, 50]. The formalism assumes that the geometry of spacetime can be described by two structures, that is, conformal and projective ones. The first one is a class of Lorentzian metrics related to each other by the conformal transformation

$$h = \Omega g, \quad (1)$$

where Ω is a positive defined function (such a transformation is often interpreted as a change of frame, e.g. in scalar–tensor

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theories). The projective structure instead is a class of connections such that

$$\tilde{\Gamma}_{\beta\mu}^{\alpha} = \Gamma_{\beta\mu}^{\alpha} + A_{(\mu}\delta_{\beta)}^{\alpha}, \quad (2)$$

with A_{μ} being a 1-form. Because of the positivity of the conformal function Ω , the conformal structure defines light cones as well as timelike, lightlike and spacelike directions in the spacetime considered. It should be noticed that it does not determine lengths of timelike and spacelike curves unless one chooses a representative of the conformal class. Geodesics in spacetime are defined by a connection. The different connections belonging to a considered projective structure define the same geodesics which are parameterized in two different ways [49]. The choice of a parametrization is related to a choice of clock, that is, a metric. We will say that the two discussed structures are EPS-compatible if the following holds (cf. (2)):

$$\tilde{\nabla}_{\mu}g_{\alpha\beta} = 2A_{\mu}g_{\alpha\beta}. \quad (3)$$

A triple consisting of the spacetime manifold M and EPS-compatible structures is called an EPS geometry [49].

We would like to emphasize that GR is a very special case of the formalism described. One assumes at the very beginning that the connection $\tilde{\Gamma}$ is a Levi-Civita connection of the metric g (the 1-form A is zero) which as a result means to treat the action of the theory as just metric-dependent. One may also treat the Einstein–Hilbert action as the one depending on two independent objects, that is, the metric g and the connection $\tilde{\Gamma}$. This approach is called the Palatini formalism. Considering the simplest gravitational Lagrangian, linear in the scalar curvature R , the Palatini approach leads to the dynamical result that $\tilde{\Gamma}$ is a Levi-Civita connection of the metric g . It is not so in the case of more complicated Lagrangians appearing in ETGs. Moreover, as shown in [52], all Palatini connections of the form (2) are singled out by the variational principle.

Our aim is to show how important the EPS interpretation can be for an explanation of the galaxy rotation curves. It turns out, as expected, that at the end we deal with an expression which consists of the Newtonian part and some modification which depends on the theory one wants to study in the EPS approach. As the simplest example which we want to examine is the Starobinsky quadratic Lagrangian [17] in the Palatini formalism, which currently reaches very good results in the cosmological applications [53–55]. The starting point will be the standard geodesic equation from which we will derive the rotational velocity. It will be shown that the velocity can be written as GR plus extra terms coming from the conformal factor. Its usefulness is tested on a sample of six HSB galaxies. The conclusions and future ideas will be presented in the last part of the paper.

The metric signature convention is $(-, +, +, +)$.

2 Velocities via conformal factors

Let us now derive a formula for the velocity of a star moving on a periodical trajectory in a given galaxy. For simplicity (and in good agreement with astronomical observations [56]) we will assume the orbit to be circular. In this case the centripetal acceleration and the velocity are related by

$$a = -\frac{v^2}{r}. \quad (4)$$

On the other hand the Einstein equivalence principle remains valid for a theory of gravity that is conformally related with standard GR. This implies that a test particle (a star in our considerations) will satisfy the geodesic equation

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma_{\nu\sigma}^{\mu} \frac{dx^{\nu}}{ds} \frac{dx^{\sigma}}{ds} = 0. \quad (5)$$

Stars can move around the galactic center at very high velocities. However, compared with the speed of light, the velocities are still very small; the condition $v \ll c$ is always satisfied. Using the coordinate parametrization $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi)$ it follows immediately that if $v \ll c$ then also $v^i = (dr/dt, r d\theta/dt, r \sin\theta d\varphi/dt) \ll dx^0/dt$. Under these conditions, together with the weak field limit of the geodesic equation (5) for a static spacetime ($\Gamma_{00}^0 = 0$), one obtains for the radial component

$$\frac{d^2r}{dt^2} = -c^2 \Gamma_{00}^r. \quad (6)$$

Inserting now Eq. (6) into (4) we simply get

$$v^2(r) = rc^2 \Gamma_{00}^r. \quad (7)$$

We have already discussed the idea of projective structures, that is, the class of connections related to each other by Eq. (2). As already mentioned, connections belonging to the same projective structure describe the same geodesics but are differently parameterized. One needs to choose which metric from the conformal structure is connected to the geodesic motion. Let us consider the case when one deals with the Weyl geometry: the connection appearing above is a Levi-Civita connection of the conformal metric $\tilde{g} \equiv h$:

$$h_{\mu\nu} = \Omega g_{\mu\nu}. \quad (8)$$

One sees that Γ_{00}^r entering into Eq. (7) is

$$\Gamma_{00}^r = \frac{1}{2} h^{r\sigma} (h_{\sigma 0,0} + h_{0\sigma,0} - h_{00,\sigma}) = -\frac{1}{2} h^{rj} \partial_j h_{00}, \quad (9)$$

which for a spherically symmetric metric

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu = -c^2 B(r) dt^2 + A(r) dr^2 + r^2 d\tilde{\Omega}^2 \quad (10)$$

takes the form

$$\Gamma_{00}^r = \frac{1}{2} \frac{B'(r)}{A(r)} \quad (11)$$

and needs to be computed for a chosen model of gravity. If we consider any modified Einstein field equations (using the convention of $\kappa = -8\pi G c^{-4}$ from [57]) of the form

$$\sigma(G_{\mu\nu} - W_{\mu\nu}) = \kappa T_{\mu\nu}, \quad (12)$$

where σ represents a coupling to gravity (for example a scalar field), one can write [58]

$$\Gamma_{00}^r = \frac{B(r)}{2A(r)} \left(\frac{A(r) - 1}{r} - A(r)r W_r^r \right). \quad (13)$$

We assume that the functions $\tilde{B}(r) = B(r)/\Omega$ and $\tilde{A}(r) = A(r)/\Omega$ will take the Schwarzschild form in the weak field limit, that is, when we consider distances much smaller than the core size of a galaxy. In that case, the term $g_{00}^{(1)} \equiv B^{(1)}$ in the approximation

$$h_{\mu\nu} = \Omega(\eta_{\mu\nu} + g_{\mu\nu}^{(1)}) \quad (14)$$

represents the first order correction coming from the weak field limit of GR (see for example [57])

$$g_{00}^{(1)} = -\frac{2\phi_N}{c^2} = \frac{2GM}{c^2 r}. \quad (15)$$

In the next section we will derive the exact form of (13) for a particular model of gravity which admits EPS interpretation.

2.1 An example: Starobinsky model

In principle there exists an entire class of gravity theories [49, 51] that are conformally related with Einstein general relativity via Eq. (8). Our aim is to explain the observed galaxy rotation curves using Eqs. (13) and (7) without assuming the existence of Dark Matter.

In what follows we would like to propose a model that fits well the astronomical observed data on galaxy rotation curves. Our analysis is performed on a subset of galaxies obtained from THINGS: the HI Nearby Galaxy Survey catalogue [61, 62], which is a high spectral and spatial resolution survey of HI emission lines from 34 nearby galaxies.

Any new model must take into account and reproduce the observed flatness of the galaxy rotation curves. At short

distances (at least of the order of the size of the solar system) the velocity should have as a limit the Newtonian result $v^2(r) = GM/r$. This imposes some constraints on the functions $A(r)$ and $B(r)$.

Before we move on to the Starobinsky model, let us briefly recall the Palatini formalism. The action is

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(\hat{R}) + S_m, \quad (16)$$

where $f(\hat{R})$ is a function of the Ricci scalar $\hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu}(\hat{\Gamma})$, while S_m is a matter action independent of the connection. The Ricci scalar is constructed by the metric-independent torsion-free connection $\hat{\Gamma}$. Varying the action with respect to the metric gives

$$f'(\hat{R}) \hat{R}_{(\mu\nu)} - \frac{1}{2} f(\hat{R}) g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (17)$$

where the prime means differentiation with respect to $\hat{R}$ and as usually $T_{\mu\nu}$ is the standard (symmetric) energy-momentum tensor given by the variation of the matter action S_m with respect to $g_{\mu\nu}$. The g -trace of (17) arises as the structural equation of the spacetime controlling (17),

$$f'(\hat{R}) \hat{R} - 2f(\hat{R}) = \kappa T. \quad (18)$$

Assuming that we are able to solve (18) as $\hat{R}(T)$ we see that $f(\hat{R})$ is a function of the energy-momentum tensor trace T , that is, $T = g^{\mu\nu} T_{\mu\nu}$. The variation of (16) with respect to the connection is

$$\hat{\nabla}_\lambda \left(\sqrt{-g} f'(\hat{R}) g^{\mu\nu} \right) = 0. \quad (19)$$

From the above it immediately follows that the connection $\hat{\nabla}$ is the Levi-Civita connection for the conformally related metric $f'(\hat{R}) g_{\mu\nu}$. For a more detailed discussion we suggest to refer for example to [63–65].

The matter part of the modified Einstein equations will be considered as the perfect fluid energy-momentum tensor with the trace

$$T = 3p - \rho. \quad (20)$$

In the following work we will consider the pressureless case, that is, for dust $p = 0$. Moreover, we will assume only the radial dependence of the energy density, that is, $\rho = \rho(r)$.

Let us notice [66] that Eq. (17) can be transformed into a so-called Einstein frame,

$$\tilde{G}_{\mu\nu} = \kappa \tilde{T}_{\mu\nu} - \frac{1}{2} h_{\mu\nu} \tilde{U}, \quad (21)$$

where the Einstein tensor is constructed with the conformal metric $h_{\mu\nu}$, while $\tilde{T}_{\mu\nu} = T_{\mu\nu}/f'(\hat{R})$ and the effective potential $\tilde{U} = f'(\hat{R})^{-2}(\hat{R}f'(\hat{R}) - f(\hat{R}))$. Thus, we see that

$$f'(\hat{R})\left(\tilde{G}_{\mu\nu} + \frac{1}{2}h_{\mu\nu}\tilde{U}\right) = \kappa T_{\mu\nu} \quad (22)$$

can be treated in a similar manner to Eq. (12) with $W_{\mu\nu} = -\frac{1}{2}h_{\mu\nu}\tilde{U}$.

Now, we are equipped with all the tools needed in order to examine the model due to our assumption as regards the dynamics. As was already mentioned, the easiest modification of the Einstein–Hilbert action is adding the Starobinsky quadratic term [17]:

$$f(\hat{R}) = \hat{R} + \gamma \hat{R}^2, \quad (23)$$

where γ is a very small parameter having significance in the case of a strong gravitational field. From the cosmological consideration of the model in Ref. [55], with cosmological constant added additionally to the matter part, one finds that γ is of order 10^{-11} . Since in the considered case the conformal factor is simply $\Omega = f'(\hat{R}) = 1 + 2\gamma \hat{R}$ and since the structural equation (18) for the dust matter gives rise to

$$\hat{R} = -\kappa T = c^2\kappa\rho, \quad (24)$$

we get particularly for the Starobinsky model

$$\Omega = 1 + 2\kappa c^2\gamma\rho. \quad (25)$$

Using the results from [58] one finds that

$$A(r) = \left(1 - \frac{2GM(r)}{c^2r}\right)^{-1} \quad (26)$$

where the modified mass distribution has the following form:

$$M(r) = \int_0^r \frac{4\pi\tilde{r}^2\rho(\tilde{r})}{1 + 2c^2\kappa\gamma\rho(\tilde{r})} \left(1 + \frac{c^2\kappa\gamma\rho(\tilde{r})}{2(1 + 2c^2\kappa\gamma\rho(\tilde{r}))}\right) d\tilde{r}. \quad (27)$$

Let us now assume a simple galaxy model obtained from the following matter distribution:

$$\tilde{M}(r) = M_0 \left(\sqrt{\frac{R_0}{r_c}} \frac{r}{r + r_c}\right)^{3\beta} \quad (28)$$

where r_c can be interpreted as the “core radius”, M_0 is the total mass of the galaxy and R_0 is the scale length of the galaxy to be matched with the observed one. Like in Ref. [59], the values of the parameter β are 1 for high surface brightness galaxies (HSBs) and $\beta = 2$ in the case of low

surface brightness galaxies (LSBs). This matter distribution is a slightly modified version of the model used in Ref. [59] in order to be closer to the actual mass profile as inferred from the observed photometric profile. Although Eq. (28) is still a very rough approximation it is good enough for the purposes of this paper.

It can be noticed that the mass distributions (27) and (28) can be identified for a suitable choice of energy density $\rho(r)$ in (27); more exactly, comparing the derivatives $\tilde{M}'(r) = M'(r)$ one gets an algebraic equation for $\rho(r)$.

Moreover, with the solution (26) we are able to see that Eq. (13) is now

$$\Gamma_{00}^r = \frac{B(r)}{2} \left(\frac{2GM(r)}{c^2r^2} + \frac{\kappa^2\gamma c^4r\rho^2}{2(1 + 2\kappa\gamma c^2\rho)^2} \right) \quad (29)$$

and therefore we write the quadratic velocity as

$$v^2 = \frac{GB(r)M(r)}{r} \left(1 - \frac{2\pi\kappa\gamma c^2r^3\rho^2}{M(r)(1 + 2\kappa\gamma c^2\rho)^2} \right). \quad (30)$$

Since the exact form of $B(r)$ is very complex, let us take the approximated value, that is,

$$B(r) = \Omega(1 + g_{00}^{(1)}) \approx 1 + g_{00}^{(1)} = 1 + \frac{2GM(r)}{c^2r}. \quad (31)$$

It follows immediately that the circular velocity of a star around the galactic center can be approximated by

$$v^2 \approx \frac{GM(r)}{r} \left(1 + \frac{2GM(r)}{c^2r} - \frac{2\pi\kappa\gamma c^2r^3\rho^2}{M(r)(1 + 2\kappa\gamma c^2\rho)^2} \right). \quad (32)$$

Table 1 Best-fit results according to Eq. (32) using the parametric mass distribution (28). These numerical values correspond to rotation curves presented in Fig. 2. Col. (1) galaxy name; Col. (2) total gas mass, in units of $10^{10} M_\odot$, given by $M_{\text{gas}} = 4/3 M_{\text{HI}}$, with the M_{HI} data taken from [61]; Col. (3) measured scale length of the galaxy in kpc; Col. (4) galaxy luminosity in the B-band, in units of $10^{10} L_\odot$, calculated from [61]; Col. (5) presents the best-fit results for the predicted total mass of the galaxy M_0 (in $10^{10} M_\odot$ units); col. (6) gives the predicted core radius r_c in kpc; Col. (7) reduced χ_r^2 ; and Col. (8) the stellar mass-to-light ratio in units of $M_\odot/L_\odot$. Note: all six galaxies are of type HSB

Galaxy (1)	M_{gas} (2)	R_0 (3)	L_B (4)	M_0 (5)	r_c (6)	χ_r^2 (7)	M/L (8)
NGC 3031	0.48	2.6	3.049	14.86	2.10	4.88	4.71
NGC 3521	1.07	3.3	3.698	38.45	3.69	1.84	10.10
NGC 3627	0.11	3.1	3.076	8.68	2.25	0.45	2.78
NGC 4736	0.05	2.1	1.294	0.53	0.59	2.41	0.37
NGC 6946	0.55	2.9	2.729	78.19	5.09	2.18	28.44
NGC 7793	0.12	1.7	0.511	18.24	3.36	4.82	35.45

Fig. 1 The difference $v - v_{\text{newt}}$ (vertical axis in Km/s) as a function of distance (horizontal axis in kpc), where v is given by Eq. (32) and v_{newt} is the Newtonian velocity $GM(r)/r$. The galaxies from top to bottom are as follows: NGC3031, NGC3627, NGC3521, NGC6946 and NGC7793. The galaxy NGC4736 is not shown, but for it always $v - v_{\text{newt}} < 0.7$

Equation (32) is the main result of this paper: the circular velocity obtained from the Starobinsky Lagrangian in the Palatini formalism. It is now used to obtain plots for a sample of six HSB galaxies after determining the parameters M_0 and r_c using the `NonlinearModelFit` function in Wolfram Mathematica. They are presented in Fig. 2.

The corresponding values for M_0 and r_c are presented in Table 1. We observe very good agreement between the data points and the fitted continuous black curve. Unfortunately, after plotting the Newtonian curves using the same values of M_0 and r_c from Table 1 we have found that there is almost no difference (see Fig. 1) between the Newtonian curve and the one derived using Eq. (32). Furthermore, the mass-to-light M/L values inferred from the fit in Table 1 are also too large

compared to what is expected based on stellar population synthesis models [60].

3 Conclusions

In this paper we have considered the possible explanation of the observed galaxy rotation curves by the assumption that the metric and the connection are independent objects in the spirit of EPS formalism. We studied the case when the connection is a Levi-Civita connection of a metric conformally related to the metric which is responsible for the measurement of distances and angles. Due to that interpretation, masses moving in a gravitational field should follow geodesics determined by the connection, providing different equations of motion. It turns out that the rotational velocity formula obtained under this formalism differs from the Newtonian one by the presence of extra terms coming indirectly from the conformal factor of the metrics. This term is treated as a deviation from the Newtonian limit of General Relativity.

In Sect. 2.1 we used Palatini gravity, which is a representation of the EPS formalism, and as a working example we took the Starobinsky Lagrangian $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2$ in order to derive a rotational velocity formula given by the expression in Eq. (32) for a star moving in a circular trajectory around the galactic center. Our results are presented in Table 1 together with Figs. 1 and 2. Although the galaxy masses resulting from the fitting of the data sub-sample proved to be too high, giving thus rise to unsatisfactory values for the mass-to-light ratio, nonetheless we have showed that the approach of obtaining galaxy rotation curves via conformal factors can be valid. It should also be noticed that we have used a very simple mat-

Fig. 2 (color online) Rotational velocities (in km/s) as a function of distance (in kpc). The black curve represents the best parametric fit of galaxy rotation curves using Eq. (32) for the sub-sample of six THINGS galaxies. The values of the best-fit parameters can be found in Table 1

ter distribution (28) in order to be able to obtain an expression for the energy density $\rho(r)$. More complex distributions could possibly give corrections which would provide different mass-to-light M/L values. This is one of the tasks of our future work.

Thus, we would like to briefly conclude by saying that the approach of obtaining galaxy rotation curves using two conformally related metrics can be valid and deserves further investigations. Furthermore, by trying other Lagrangians and other gravity models in the future, it is definitely possible to improve the findings reported here.

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References

1. A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1915**(part 2), 844–847 (1915)
2. A. Einstein, Ann. Phys. **49**, 769–822 (1916)
3. A. Einstein, Ann. Phys. **14**, 517 (2005)
4. D. Huterer, M.S. Turner, Phys. Rev. D **60**, 081301 (1999)
5. E.J. Sami, M.S. Tsujikawa, Int. J. Mod. Phys. D **15**, 1753–1935 (2006)
6. J.C. Kapteyn, ApJ **55**, 302 (1922)
7. J.H. Oort, Bull. Astron. Inst. Neth **6**, 249 (1932)
8. F. Zwicky, Helv. Phys. Acta **6**, 110–127 (1933)
9. F. Zwicky, ApJ **86**, 217 (1937)
10. H.W. Babcock, Lick Observ. Bull. **19**, 41–51 (1939)
11. V.C. Rubin, W. Kent Ford Jr., ApJ **159**, 379 (1970)
12. V.C. Rubin, W. Kent Ford Jr., N. Thonnard, ApJ **238**, 471–487 (1980)
13. S. Capozziello, M. De Laurentis, Phys. Rep. **509**(4), 167–321 (2011)
14. S. Nojiri, S.D. Odintsov, TSPU Bull. **8**(110), 7–19 (2011)
15. S. Nojiri, S.D. Odintsov, Phys. Rep. **505**(2), 59–144 (2011)
16. S. Capozziello, V. Faraoni, *Beyond Einstein gravity: a survey of gravitational theories for cosmology and astrophysics*, vol. 170 (Springer, Dordrecht, 2011)
17. A.A. Starobinsky, Phys. Lett. B **91**, 99–102 (1980)
18. A.H. Guth, Phys. Rev. D **23**, 347–356 (1981)
19. K.S. Stelle, Phys. Rev. D **16**(4), 953 (1977)
20. G. Bertone, D. Hooper, J. Silk, Phys. Rep. **405**, 279–390 (2005)
21. M. Davis, G. Efstathiou, C.S. Frenk, S.D. White, ApJ **292**, 371–394 (1985)
22. A. Refregier, Annu. Rev. Astron. Astrophys. **41**, 645–668 (2003)
23. M. Milgrom, ApJ **270**, 365 (1983)
24. M. Milgrom, ApJ **270**, 371–389 (1983)
25. R.H. Sanders, S.S. McGaugh, ARAA **40**, 263 (2002)
26. J.D. Bekenstein, Contemp. Phys. **47**, 387 (2006)
27. M. Milgrom, In: Proceedings XIX Rencontres de Blois (2008); [arXiv:0801.3133](https://arxiv.org/abs/0801.3133)
28. C. Skordis, Class. Quant. Grav. **26**(14), 143001 (2009)
29. S.S. McGaugh, W.J.G. De Blok, ApJ **499**, 66 (1998)
30. S.S. McGaugh, Phys. Rev. Lett. **106**, 121303 (2011)
31. J.D. Bekenstein, Phys. Rev. D **70**, 083509 (2004)
32. J.W. Moffat, V.T. Toth, Phys. Rev. D **91**, 043004 (2015)
33. L. Iorio, M.L. Ruggiero, Scholarly Research Exchange, vol. 2008, article ID 968393 (2008)
34. S. Capozziello, V.F. Cardone, A. Troisi, JCAP **08**, 001 (2006)
35. S. Capozziello, V.F. Cardone, A. Troisi, MNRAS **375**, 1423–1440 (2007)
36. L. Sebastiani, S. Vagnozzi, R. Myrzakulov, Class. Quant. Grav. **33**(12), 125005 (2016)
37. A.O. Barvinsky, JCAP **01**, 014 (2014)
38. B. Famaey, S.S. McGaugh, Living Rev. Relat. **15**, 10 (2012)
39. T. Bernal, S. Capozziello, J.C. Hidalgo, S. Mendoza, Eur. Phys. J. C **71**, 1794 (2011)
40. E. Barrientos, S. Mendoza, Eur. Phys. J. Plus **131**, 367 (2016)
41. J.P. Bruneton, G. Esposito-Farese, Phys. Rev. D **76**, 129902 (2007)
42. G. Esposito-Farese, Fund. Theor. Phys. **162**, 461–489 (2011)
43. P.D. Mannheim, J.G. O’Brien, PRL **106**, 121101 (2011)
44. P.D. Mannheim, J.G. O’Brien, Phys. Rev. D **85**, 124020 (2012)
45. P.D. Mannheim, Prog. Part. Nucl. Phys. **56**, 340–445 (2006)
46. D.C. Rodrigues, P.S. Letelier, I.L. Shapiro, JCAP **04**, 020 (2010)
47. D.C. Rodrigues, P.L. de Oliveira, J.C. Fabris, G. Gentile, MNRAS **445**(4), 3823 (2014)
48. D.C. Rodrigues, S. Mauro, A.O.F. de Almeida, Phys. Rev. D **94**, 084036 (2016)
49. L. Fatibene, M. Francaviglia, Int. J. Geom. Meth. Mod. Phys. **11**, 1450008 (2014)
50. S. Capozziello, M. De Laurentis, M. Francaviglia, S. Mercadante, Found. Phys. **39**(10), 1161–1176 (2009)
51. J. Ehlers, F.A.E. Pirani, A. Schild, The geometry of free fall and light propagation, in *General relativity*, ed. by L. O’Raifeartaigh (Clarendon, Oxford, 1972)
52. A.N. Bernal, B. Janssen, A. Jimenez-Cano, J.A. Orejuela, M. Sanchez, P. Sanchez-Moreno, Phys. Lett. B **768**, 280–287 (2017)
53. A. Borowiec, A. Stachowski, M. Szydlowski, A. Wojnar, JCAP **01**, 040 (2016)
54. M. Szydlowski, A. Stachowski, A. Borowiec, A. Wojnar, Eur. Phys. J. C **76**, 567 (2016)
55. A. Stachowski, M. Szydlowski, A. Borowiec, Eur. Phys. J. C **77**, 406 (2017)
56. J. Binney, S. Tremaine, *Galactic dynamics*, 2nd edn. (Princeton University Press, Princeton, 2008)
57. S. Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativity*, 1st edn. (Wiley, Hoboken, 1972)
58. A. Wojnar, H. Velten, Eur. Phys. J. C **76**, 697 (2016)
59. J.R. Brownstein, J.W. Moffat, ApJ **636**, 721–741 (2006)
60. S.S. McGaugh, J.M. Schombert, AJ **148**, 77 (2014)
61. F. Walter et al., ApJ **136**, 2563 (2008)
62. W.J.G. de Blok et al., ApJ **136**, 2648 (2008)
63. G. Allemandi, A. Borowiec, M. Francaviglia, S.D. Odintsov, Phys. Rev. D **72**, 063505 (2005)

- 64. G. Allemandi, A. Borowiec, M. Francaviglia, Phys. Rev. D **70**, 043524 (2004)
- 65. G. Allemandi, A. Borowiec, M. Francaviglia, Phys. Rev. D **70**, 103503 (2004)
- 66. R.H. Dicke, Phys. Rev. **125**, 2163 (1962)
- 67. A. Borowiec, M. Ferraris, M. Francaviglia, I. Volovich, Class. Quant. Grav. **15**, 43–55 (1998)
- 68. A. Borowiec, In: Proceedings of the 24th International Colloquium on Group Theoretical Methods in Physics. IOP Conference Series CS **173**, pp 241–244 (2003)