Goldstone modes in renormalizable supersymmetric SO(10) model

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Abstract We solve the Goldstone modes in the renormalizable SUSY SO(10) model with general couplings. The Goldstones are expressed by the Vacuum Expectation Values and the Clebsch–Gordan coefficients of relevant symmetries without explicit dependence on the parameters of the model.

1 Introduction

Grand Unified Theories (GUTs) are very important candidates for the physics beyond the Standard Model (SM). The Pati–Salam $G_{422} = SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$ [1], the Georgi–Glashow $SU(5)$ [2], the flipped $SU(5)$ of $G_{51} = (SU(5) \otimes U(1))^{\text{Flipped}}$ [3], the SO(10) [4, 5] and the E_6 [6] models have all been studied extensively in the literature. Except $SU(5)$, the GUT symmetries of these models are larger in ranks than the SM symmetry $G_{321} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. Consequently, they have several different routines to break into the SM symmetry, since the SM symmetry G_{321} is not the maximal subgroup of the GUT symmetries.

The supersymmetric (SUSY) GUT models are even more important in realizing gauge coupling unification [7–15]. Again, except in the $SU(5)$ models, these GUT symmetries have different routines in the symmetry breaking chains.

Spontaneously Symmetry Breaking (SSB) generates the massless Goldstone bosons corresponding to the broken generators [16–18]. In gauge theories, these Goldstone bosons act as the longitudinal components of the gauge bosons. In model buildings, we need to check the spectra to verify the existence of these Goldstone bosons. When more than one Higgs multiplets contribute to the SSB, we need to check that there are massless eigenstates in the mass squared matrices with correct representations.

In SUSY models, there are easier ways to get the Goldstone bosons. The Goldstinos, which are the fermionic partners of the Goldstone bosons, are also massless which mix with gauginos, the SUSY partners of the gauge bosons. Then, we can get the Goldstinos by solving the massless eigenstates of the mass matrices. Being the SUSY partners, the Goldstone bosons are determined accordingly.

Usually in building GUT models, the Higgs sectors relevant to SSB are complicated. Hereafter we will focus on SUSY SO(10) [19–36]. Even in the minimal renormalizable SUSY SO(10) model [26], Higgs in $210 + 126 + \overline{126}$ are needed to break SO(10) into G_{321} . There are Goldstinos in $(1, 1, 0)$, $(1, 1, 1)$, $(3, 1, \frac{2}{3})$, $(3, 2, -\frac{5}{6})$, $(3, 2, \frac{1}{6})$ and the conjugates under G_{321} . None of these representations has only one field in the spectrum. To verify the Goldstone modes, usually people need to calculate the eigenvalues of the mass matrices for the Goldstinos [26, 32]. In more general models with also $45 + 54$ in the SSB of the GUT symmetry, the Goldstino check is done numerically [23, 29].

In this work, we will study the Goldstone modes within SUSY. In [37] it has been realized that for the SSB of $U(1)$ symmetries in SUSY models, in the Goldstone mode a component is proportional to the charge and to the Vacuum Expectation Value (VEV). Furthermore, it has been shown in the Non-Abelian cases [38], that in any Goldstone mode, the component from a representation is proportional to the VEV from the same representation and the CGCs determine the remainder dependence. Examples on the simplest $SU(2)$ SSB are given in [38]. Here we will show test explicitly in the SUSY SO(10) model. We will give the Goldstone modes for the SSB of SO(10) into G_{321} in the general renormalizable model.

In Sect. 2 we will give a general discussion on the method to solve the Goldstone modes. Then, in Sect. 3, we associate in $SO(10)$ the Goldstones and the corresponding broken symmetries. In Sects. 4 and 5, Goldstones associated with the SSB of G_{422} and $SU(5)$ are solved, respectively, and relevant identities among the CGCs are examined. In Sect. 6,

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Goldstones associated with the SSB of the flipped $SU(5)$ are solved. The relevant CGCs, which are not available in the literature, are presented and the identities are examined. The mass matrix for the Goldstones is given in the Appendix. Finally we summarize in Sect. 7.

2 General SSB in SUSY models

Following [38], we consider the SSB of $G_1 \rightarrow G_2$ in SUSY models. For models with only real fields, the general Higgs superpotential can be written as

$$W = \frac{1}{2} \sum_{I,i} m_I (I_i)^2 + \sum_{IJK,ijk} \lambda^{IJK} I_i J_j K_k, \quad (1)$$

where I, J, K denotes the superfields, i, j, k denote the representations of group G_1 . Under G_2 , the couplings in (1) are of the forms

$$\lambda^{IJK} I_{ia} J_{jb} K_{kc} C_{iajbkc}^{IJK}, \quad (2)$$

where a, b, c are the representations under G_2 , and C_{iajbkc}^{IJK} is the Clebsch–Gordan coefficient (CGC). A singlet of G_2 , denoted by I_{i_1} etc., can have a VEV $\hat{I}_{i_1}$ in the SSB. SUSY requires the F-flatness conditions for the singlets,

$$0 \equiv F_{I_{i_1}} = \left\langle \frac{\partial W}{\partial I_{i_1}} \right\rangle = m_I \hat{I}_{i_1} + \sum_{JK,jk} \lambda^{IJK} \hat{J}_{j_1} \hat{K}_{k_1} C_{i_1 j_1 k_1}^{IJK}. \quad (3)$$

The Goldstinos in the SSB will be denoted by $\alpha, \bar{\alpha}$ under G_2 . A mass matrix element for the Goldstinos is

$$M_{i\alpha j\bar{\alpha}}^{IJ} = \left\langle \frac{\partial^2 W}{\partial I_{i\alpha} \partial J_{j\bar{\alpha}}} \right\rangle = m_I \delta_{IJ} \delta_{ij} + \sum_{K,k} \lambda^{IJK} \hat{K}_{k_1} C_{i\alpha j\bar{\alpha} k_1}^{IJK}.$$

For $G_{i\alpha}^I$ denotes the element of the Goldstino corresponding to $\bar{\alpha}$, the zero-eigenvalue equation is

$$0 \equiv \sum_{J,j} M_{i\alpha j\bar{\alpha}}^{IJ} G_{j\bar{\alpha}}^J = m_I G_{i\alpha}^I + \sum_{JK,jk} \lambda^{IJK} G_{j\bar{\alpha}}^J \hat{K}_{k_1} C_{i\alpha j\bar{\alpha} k_1}^{IJK}. \quad (4)$$

Eliminating M_I in (3) and (4) gives

$$0 = G_{i\alpha}^I \sum_{JK,jk} \lambda^{IJK} \hat{J}_{j_1} \hat{K}_{k_1} C_{i_1 j_1 k_1}^{IJK} - \hat{I}_{i_1} \sum_{JK,jk} \lambda^{IJK} G_{j\bar{\alpha}}^J \hat{K}_{k_1} C_{i\alpha j\bar{\alpha} k_1}^{IJK}. \quad (5)$$

It follows that if a representations of G_1 does not contain a G_2 singlet, even if it may contain representations of the Goldstones, it does not contribute to the Goldstino modes since the M_I term in (4) cannot be eliminated unless multiplied by zero.

The superpotential parameters can be arbitrary so that we can focus on a specified coupling λ^{IJK} , and the summation over different λ s is unnecessary. Furthermore, denoting

$$T_{\bar{\alpha}}^{Ii} = \frac{G_{i\alpha}^I}{\hat{I}_{i_1}}, \text{ etc.}, \quad (6)$$

(5) becomes

$$0 = T_{\bar{\alpha}}^{Ii} \lambda^{IJK} \hat{J}_{j_1} \hat{K}_{k_1} C_{i_1 j_1 k_1}^{IJK} - T_{\bar{\alpha}}^{Jj} \lambda^{IJK} \hat{J}_{j_1} \hat{K}_{k_1} C_{i\alpha j\bar{\alpha} k_1}^{IJK} - T_{\bar{\alpha}}^{Kk} \lambda^{IJK} \hat{J}_{j_1} \hat{K}_{k_1} C_{i\alpha j\bar{\alpha} k_1}^{IJK}.$$

The nonzero $\lambda^{IJK} \hat{J}_{j_1} \hat{K}_{k_1}$ can be eliminated and we can reiterate the same operation for J, K , then

$$0 = \begin{pmatrix} -C_{i_1 j_1 k_1}^{IJK} & C_{i\alpha j\bar{\alpha} k_1}^{IJK} & C_{i\alpha k\bar{\alpha} j_1}^{IJK} \\ C_{j\alpha i\bar{\alpha} k_1}^{IJK} & -C_{j_1 i_1 k_1}^{IJK} & C_{j\alpha k\bar{\alpha} i_1}^{IJK} \\ C_{k\alpha i\bar{\alpha} j_1}^{IJK} & C_{k\alpha j\bar{\alpha} i_1}^{IJK} & -C_{k_1 j_1 i_1}^{IJK} \end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}}^{Ii} \\ T_{\bar{\alpha}}^{Jj} \\ T_{\bar{\alpha}}^{Kk} \end{pmatrix}. \quad (7)$$

Similar result is given for the Goldstino in α . (7) is followed by an identity that the determinant of the square matrix in (7) is zero, which must hold for any SSB.

Some special cases need to be clarified now.

(1) If k does not contain $\alpha, \bar{\alpha}$, (7) is simplified into

$$0 = \begin{pmatrix} -C_{i_1 j_1 k_1}^{IJK} & C_{i\alpha j\bar{\alpha} k_1}^{IJK} \\ C_{j\alpha i\bar{\alpha} k_1}^{IJK} & -C_{j_1 i_1 k_1}^{IJK} \end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}}^{Ii} \\ T_{\bar{\alpha}}^{Jj} \end{pmatrix}, \quad (8)$$

which gives an identity that the determinant of the square matrix is zero, and a simple ratio between the two components. Taking all different couplings λ s setup all relations among the Goldstino components. This determines the Goldstino content up to an overall normalization.

(2) For $J = K$ and $j = k$, we have

$$0 = \begin{pmatrix} -C_{i_1 j_1 j_1}^{IJJ} & 2C_{i\alpha j\bar{\alpha} j_1}^{IJJ} \\ C_{i\alpha j\bar{\alpha} j_1}^{IJJ} & C_{i_1 j\bar{\alpha} j\bar{\alpha}}^{IJJ} - C_{i_1 j_1 j_1}^{IJJ} \end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}}^{Ii} \\ T_{\bar{\alpha}}^{Ij} \end{pmatrix}. \quad (9)$$

(3) For $I = J = K$ and $i = j = k$, the result is

$$C_{i_1 i_1 i_1}^{III} = 2C_{i\alpha i\bar{\alpha} i\bar{\alpha}}^{III}, \quad (10)$$

without information on T .

Generalization to the models with complex fields is straightforward with the general results given in [38]. We have also identities among the CGCs and equations for the Goldstinos involving the CGCs. Two types of special superpotentials might be relevant in the $SO(10)$ study.

(4) For $I, \bar{I}, K$ contain Goldstones and K is real, if

$$W = m_I I \bar{I} + \frac{1}{2} m_K^2 + \lambda I^2 K + \bar{\lambda} \bar{I}^2 K,$$

which gives

$$0 = \begin{pmatrix} -C_{i_1 i_1 k_1}^{IIK} & C_{i_{\bar{\alpha}} i_1 k_{\bar{\alpha}}}^{IIK} & C_{i_{\bar{\alpha}} i_{\bar{\alpha}} k_1}^{IIK} \\ X C_{i_{\bar{\alpha}} i_1 k_{\alpha}}^{\bar{I} \bar{I} K} & -C_{i_1 i_1 k_1}^{IIK} & C_{i_{\bar{\alpha}} i_1 k_{\alpha}}^{IIK} \\ C_{i_{\bar{\alpha}} i_{\alpha} k_1}^{\bar{I} \bar{I} K} & C_{i_{\bar{\alpha}} i_1 k_{\alpha}}^{\bar{I} \bar{I} K} & -C_{i_1 i_1 k_1}^{\bar{I} \bar{I} K} \end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}}^{\bar{I} i} \\ T_{\bar{\alpha}}^{K k} \\ T_{\bar{\alpha}}^{I i} \end{pmatrix}, \quad (11)$$

where $X = C_{i_1 i_1 k_1}^{\bar{I} \bar{I} K} / C_{i_1 i_1 k_1}^{IIK}$.

(5) For $I, \bar{I}, K$ contain Goldstones and K is real, if

$$W = m_I I \bar{I} + \frac{1}{2} m_K^2 + \lambda I \bar{I} K,$$

which gives

$$0 = \begin{pmatrix} -C_{i_1 i_1 k_1}^{IIK} & C_{i_{\bar{\alpha}} i_1 k_{\bar{\alpha}}}^{\bar{I} \bar{I} K} & 0 \\ C_{i_1 i_{\bar{\alpha}} k_{\alpha}}^{I \bar{I} K} & -C_{i_1 i_1 k_1}^{\bar{I} \bar{I} K} & C_{i_{\bar{\alpha}} i_1 k_{\alpha}}^{\bar{I} \bar{I} K} \\ 0 & C_{i_1 i_{\alpha} k_{\bar{\alpha}}}^{I \bar{I} K} & -C_{i_1 i_1 k_1}^{\bar{I} \bar{I} K} \end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}}^{\bar{I} i} \\ T_{\bar{\alpha}}^{K k} \\ T_{\bar{\alpha}}^{I i} \end{pmatrix}. \quad (12)$$

Furthermore, in case that I_i ($\bar{I}$) contains only α ($\bar{\alpha}$) but not $\bar{\alpha}$ (α), we have

$$0 = \begin{pmatrix} -C_{i_1 i_1 k_1}^{IIK} & C_{i_{\bar{\alpha}} i_1 k_{\bar{\alpha}}}^{\bar{I} \bar{I} K} \\ C_{i_1 i_{\bar{\alpha}} k_{\alpha}}^{I \bar{I} K} & -C_{i_1 i_1 k_1}^{\bar{I} \bar{I} K} \end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}}^{\bar{I} i} \\ T_{\bar{\alpha}}^{K k} \end{pmatrix}. \quad (13)$$

We summarize in this section that the Goldstino and thus the Goldstone components are proportional to the VEVs, and the remaining determinations of these components need only the calculations of the CGCs. This is the approach which we will use to determine the Goldstones in the following sections, and the identities of the CGC determinants will be used to check the consistencies of these calculations.

3 SSB and Goldstones in SO(10)

We study the most general renormalizable couplings containing Higgs $H(10)$, $D(120)$, $\bar{\Delta}(\bar{1}26) + \Delta(126)$, $A(45)$, $E(54)$ and $\Phi(210)$ in the SUSY $SO(10)$ models. The most general renormalizable Higgs superpotential is [29]

$$\begin{aligned} W = & \frac{1}{2} m_1 \Phi^2 + m_2 \bar{\Delta} \Delta + \frac{1}{2} m_3 H^2 + \frac{1}{2} m_4 A^2 \\ & + \frac{1}{2} m_5 E^2 + \frac{1}{2} m_6 D^2 \\ & + \lambda_1 \Phi^3 + \lambda_2 \Phi \bar{\Delta} \Delta + (\lambda_3 \Delta + \lambda_4 \bar{\Delta}) H \Phi + \lambda_5 A^2 \Phi \\ & - i \lambda_6 A \bar{\Delta} \Delta + \frac{\lambda_7}{120} \varepsilon A \Phi^2 \\ & + E(\lambda_8 E^2 + \lambda_9 A^2 + \lambda_{10} \Phi^2 + \lambda_{11} \Delta^2 \\ & + \lambda_{12} \bar{\Delta}^2 + \lambda_{13} H^2) \end{aligned}$$

$$\begin{aligned} & + D^2 (\lambda_{14} E + \lambda_{15} \Phi) \\ & + D \{ \lambda_{16} H A + \lambda_{17} H \Phi + (\lambda_{18} \Delta + \lambda_{19} \bar{\Delta}) A \\ & + (\lambda_{20} \Delta + \lambda_{21} \bar{\Delta}) \Phi \}, \end{aligned} \quad (14)$$

When we study the SSB of a $SO(10)$ subgroup G_1 into G_2 , we firstly decompose the $SO(10)$ representations into G_1 representations, and the couplings are now of the form

$$\lambda^{IJK} I J K = \lambda^{IJK} \sum_{ijk} C_{ijk}^{IJK} I_i J_j K_k, \quad (15)$$

where, for example, I_i stands for the representations i of G_1 from $SO(10)$ superfield I . Then, under G_2 , i, j, k are further decomposed, (15) is

$$\begin{aligned} \lambda^{IJK} I J K &= \lambda^{IJK} \sum_{ijk} \sum_{abc} C_{ijk}^{IJK} C_{abc}^{ijk} I_{ia} J_{jb} K_{kc} \\ &\equiv \lambda^{IJK} \sum_{ijk} \sum_{abc} C_{abc}^{I_j J_k K_k} I_{ia} J_{jb} K_{kc}, \end{aligned} \quad (16)$$

where I_{ia} is a representations a of G_2 coming from I_i , and C_{iajbkc}^{IJK} is the $SO(10)$ CGC. Most of these CGCs have been given [29,36] so that no separate calculations of C_{ijk}^{IJK} and C_{abc}^{ijk} are needed. A special kind of CGCs relevant for the SSB of the flipped $SU(5)$ symmetry will be given later.

To study SSB of $SO(10)$ into G_{321} , it is necessary to decompose the Higgs representations of $SO(10)$ under the G_{321} subgroup. There are $45 - 12 = 33$ Goldstone modes. We link the Goldstones of a SM representation with the SSB of a specific subgroup of $SO(10)$, as is summarized in Table 1. In Table 1, the first three modes can be studied by the $SO(10)$ CGCs using G_{422} as the maximal subgroup[29], the $(3, 2, -\frac{5}{6}) + \text{c.c.}$ modes need the CGCs using $SU(5)$ [36], and $(3, 2, \frac{1}{6}) + \text{c.c.}$ using G_{51} CGCs which will be given below.

4 Goldstone modes for SSB of G_{422}

Under G_{422} , the following representations

$$\begin{aligned} A_1 &\equiv \hat{A}_{(1,1,3)}, \quad A_2 \equiv \hat{A}_{(15,1,1)}, \quad E \equiv \hat{E}_{(1,1,1)}, \\ v_R &\equiv \hat{\Delta}_{(\bar{10},1,3)}, \quad \bar{v}_R \equiv \hat{\bar{\Delta}}_{(10,1,3)}, \\ \Phi_1 &\equiv \hat{\Phi}_{(1,1,1)}, \quad \Phi_2 \equiv \hat{\Phi}_{(15,1,1)}, \quad \Phi_3 \equiv \hat{\Phi}_{(15,1,3)}, \end{aligned} \quad (17)$$

contain the SM singlets $(1, 1, 0)$ whose VEVs will be denoted in the same symbols. Obviously, E and Φ_1 contain no Goldstones relevant for the G_{422} breaking, A_1 contains Goldstone components relevant for the $SU(2)_R$ breaking, A_2, Φ_2 for the $SU(4)_C$ breaking, and $v_R, \bar{v}_R, \Phi_3$ for both. The needed CGCs are taken from [29] and are summarized in Tables 2, 3 and 4.

Table 1 The Goldstone modes and the corresponding SSB

Goldstones	G_1	G_2
$(1, 1, 0)$	$U(1)_{I_{3R}} \otimes U(1)_{B-L}$	$U(1)_Y$
$(1, 1, 1) + \text{c.c.}$	$SU(2)_R$	$U(1)_{I_{3R}}$
$(3, 1, \frac{2}{3}) + \text{c.c.}$	$SU(4)_C$	$SU(3)_C \otimes U(1)_{B-L}$
$(3, 2, -\frac{5}{6}) + \text{c.c.}$	$SU(5)$	G_{321}
$(3, 2, \frac{1}{6}) + \text{c.c.}$	$G_{51} = (SU(5) \otimes U(1))^{Flipped}$	$SU(3)_C \otimes SU(2)_L \otimes U(1)' \otimes U(1)_X$

Table 2 CGCs for the SM singlets $(1, 1, 0)$ under G_{422} SSB. These SM singlets also correspond to Goldstones $(1, 1, 0)$ in the SSB of $U(1)_{I_{3R}} \otimes U(1)_{B-L}$ into $U(1)_Y$. Hereafter, the CGCs for $A\Phi\Phi$ are understood as those for $\frac{1}{120} \epsilon A\Phi\Phi$ [29]. Here, for example, A_1 stands for $\hat{A}_{(1,1,3)}^{(1,1,0)}$. For $X = \hat{\Delta}_{(10,1,3)}^{(1,1,0)}$ and $Y = \hat{\Delta}_{(10,1,3)}^{(1,1,0)}$, XYA_1 corresponds to the CGC $C_{111}^{\bar{v}_R v_R A_1}$

X	Y	XYA_1	XYA_2	XYE	$XY\Phi_1$	$XY\Phi_2$	$XY\Phi_3$
$\hat{A}_{(1,1,3)}^{(1,1,0)}$	$\hat{A}_{(1,1,3)}^{(1,1,0)}$	0	0	$\frac{\sqrt{3}}{2\sqrt{5}}$	$\frac{1}{\sqrt{6}}$	0	0
$\hat{A}_{(15,1,1)}^{(1,1,0)}$	$\hat{A}_{(15,1,1)}^{(1,1,0)}$	0	0	$-\frac{1}{\sqrt{15}}$	0	$\frac{\sqrt{2}}{3}$	0
$\hat{A}_{(1,1,3)}^{(1,1,0)}$	$\hat{A}_{(15,1,1)}^{(1,1,0)}$	0	0	0	0	0	$\frac{1}{\sqrt{6}}$
$\hat{E}_{(1,1,1)}^{(1,1,0)}$	$\hat{E}_{(1,1,1)}^{(1,1,0)}$	0	0	$\frac{1}{2\sqrt{15}}$	0	0	0
$\hat{\Delta}_{(10,1,3)}^{(1,1,0)}$	$\hat{\Delta}_{(10,1,3)}^{(1,1,0)}$	$-\frac{i}{5}$	$-\frac{i\sqrt{3}}{5\sqrt{2}}$	0	$\frac{1}{10\sqrt{6}}$	$\frac{1}{10\sqrt{2}}$	$\frac{1}{10}$
$\hat{\Phi}_{(1,1,1)}^{(1,1,0)}$	$\hat{\Phi}_{(1,1,1)}^{(1,1,0)}$	0	0	$\frac{\sqrt{3}}{2\sqrt{5}}$	0	0	0
$\hat{\Phi}_{(1,1,1)}^{(1,1,0)}$	$\hat{\Phi}_{(15,1,1)}^{(1,1,0)}$	0	$\frac{\sqrt{2}}{5}$	0	0	0	0
$\hat{\Phi}_{(1,1,1)}^{(1,1,0)}$	$\hat{\Phi}_{(15,1,3)}^{(1,1,0)}$	0	0	0	0	0	$\frac{1}{6\sqrt{6}}$
$\hat{\Phi}_{(15,1,1)}^{(1,1,0)}$	$\hat{\Phi}_{(15,1,1)}^{(1,1,0)}$	0	0	$-\frac{1}{\sqrt{15}}$	0	$\frac{1}{9\sqrt{2}}$	0
$\hat{\Phi}_{(15,1,1)}^{(1,1,0)}$	$\hat{\Phi}_{(15,1,3)}^{(1,1,0)}$	$\frac{\sqrt{2}}{5}$	0	0	0	0	$\frac{1}{9\sqrt{2}}$
$\hat{\Phi}_{(15,1,3)}^{(1,1,0)}$	$\hat{\Phi}_{(15,1,3)}^{(1,1,0)}$	0	$\frac{2\sqrt{2}}{5\sqrt{3}}$	$\frac{1}{4\sqrt{15}}$	$\frac{1}{6\sqrt{6}}$	$\frac{1}{9\sqrt{2}}$	0

Table 3 CGCs for the Goldstones $(1, 1, 1) + \text{c.c.}$ of G_{422} SSB. Here, for example, XYA_1 for $X = \hat{\Delta}_{(10,1,3)}^{(1,1,1)}$ and $Y = \hat{\Delta}_{(10,1,3)}^{(1,1,-1)}$ corresponds to the CGC $C_{\bar{\alpha}\alpha 1}^{\bar{v}_R v_R A_2}$ where $\bar{\alpha} = (1, 1, 1)$ under G_{321}

X	Y	XYA_1	XYA_2	XYE	$XY\Delta$	$XY\bar{\Delta}$	$XY\Phi_1$	$XY\Phi_2$	$XY\Phi_3$
$\hat{A}_{(1,1,3)}^{(1,1,1)}$	$\hat{A}_{(1,1,3)}^{(1,1,-1)}$	0	0	$\frac{\sqrt{3}}{2\sqrt{5}}$	0	0	$\frac{1}{\sqrt{6}}$	0	0
$\hat{A}_{(1,1,3)}^{(1,1,1)}$	$\hat{\Delta}_{(10,1,3)}^{(1,1,-1)}$	0	0	0	$-\frac{1}{5}$	0	0	0	0
$\hat{A}_{(1,1,3)}^{(1,1,1)}$	$\hat{\Phi}_{(15,1,3)}^{(1,1,-1)}$	0	$-\frac{1}{\sqrt{6}}$	0	0	0	0	$-\frac{\sqrt{2}}{5}$	0
$\hat{\Delta}_{(10,1,3)}^{(1,1,1)}$	$\hat{\Delta}_{(10,1,3)}^{(1,1,-1)}$	0	$-\frac{i\sqrt{3}}{5\sqrt{2}}$	0	0	0	$\frac{1}{10\sqrt{6}}$	$\frac{1}{10\sqrt{2}}$	0
$\hat{\Delta}_{(10,1,3)}^{(1,1,1)}$	$\hat{A}_{(1,1,3)}^{(1,1,-1)}$	0	0	0	0	$-\frac{1}{5}$	0	0	0
$\hat{\Delta}_{(10,1,3)}^{(1,1,1)}$	$\hat{\Phi}_{(15,1,3)}^{(1,1,-1)}$	0	0	0	0	$-\frac{1}{10}$	0	0	0
$\hat{\Phi}_{(15,1,3)}^{(1,1,1)}$	$\hat{\Phi}_{(15,1,3)}^{(1,1,-1)}$	0	$\frac{2\sqrt{2}}{5\sqrt{3}}$	$\frac{1}{4\sqrt{15}}$	0	0	$\frac{1}{6\sqrt{6}}$	$\frac{1}{9\sqrt{2}}$	0
$\hat{\Phi}_{(15,1,3)}^{(1,1,1)}$	$\hat{\Delta}_{(10,1,3)}^{(1,1,-1)}$	0	0	0	$-\frac{1}{10}$	0	0	0	0

4.1 Goldstone $(1, 1, 0)$ under G_{321}

This Goldstone is the result of the SSB of $U(1)_{I_{3R}} \otimes U(1)_{B-L}$ into $U(1)_Y$ and has been studied in [37]. In the renormalizable models, only $\Delta(\mathbf{126}) + \bar{\Delta}(\mathbf{126})$ have the SM singlets with nonzero $U(1)_{I_{3R}}, U(1)_{B-L}$ charges. Furthermore, if we include also fields in the spinor representations $\Psi(\mathbf{16}) + \bar{\Psi}(\mathbf{16})$ of $SO(10)$, which are usually used in building non-renormalizable models and have halves of the charges

of $\Delta(\mathbf{126}) + \bar{\Delta}(\mathbf{126})$, the Goldstino and thus the Goldstone mode is found to be

$$\begin{aligned} & \frac{\langle \Delta \rangle}{N} \hat{\Delta}_{(10,1,3)}^{(1,1,0)} - \frac{\langle \bar{\Delta} \rangle}{N} \hat{\Delta}_{(10,1,3)}^{(1,1,0)} + \frac{1}{2} \frac{\langle \Psi \rangle}{N} \hat{\Psi}_{(4,1,2)}^{(1,1,0)} \\ & - \frac{1}{2} \frac{\langle \bar{\Psi} \rangle}{N} \hat{\Psi}_{(4,1,2)}^{(1,1,0)}, \end{aligned} \quad (18)$$

where N is a simple normalization factor. Equation (18) tell us two important conclusions in the $U(1)$ symmetry break-

Table 4 CGCs for the Goldstones $(3, 1, \frac{2}{3}) + \text{c.c.}$ of G_{422} SSB

X	Y	XYA_1	XYA_2	XYE	$XY\Delta$	$XY\bar{\Delta}$	$XY\Phi_1$	$XY\Phi_2$	$XY\Phi_3$
$\hat{A}_{(15,1,1)}^{(3,1,\frac{2}{3})}$	$\hat{A}_{(15,1,1)}^{(\bar{3},1,-\frac{2}{3})}$	0	0	$-\frac{1}{\sqrt{15}}$	0	0	0	$\frac{1}{3\sqrt{2}}$	0
$\hat{A}_{(15,1,1)}^{(3,1,\frac{2}{3})}$	$\hat{\Delta}_{(10,1,3)}^{(\bar{3},1,-\frac{2}{3})}$	0	0	0	0	$-\frac{1}{5}$	0	0	0
$\hat{A}_{(15,1,1)}^{(3,1,\frac{2}{3})}$	$\hat{\Phi}_{(15,1,1)}^{(\bar{3},1,-\frac{2}{3})}$	0	$-\frac{1}{3\sqrt{2}}$	0	0	0	$-\frac{\sqrt{2}}{5}$	0	0
$\hat{A}_{(15,1,1)}^{(\bar{3},1,-\frac{2}{3})}$	$\hat{\Phi}_{(15,1,3)}^{(3,1,\frac{2}{3})}$	$-\frac{1}{\sqrt{6}}$	0	0	0	0	0	0	$-\frac{\sqrt{2}}{5\sqrt{3}}$
$\hat{\Delta}_{(10,1,3)}^{(3,1,\frac{2}{3})}$	$\hat{\Delta}_{(10,1,3)}^{(\bar{3},1,-\frac{2}{3})}$	$-\frac{i}{5}$	$-\frac{i}{5\sqrt{6}}$	0	0	0	$\frac{1}{10\sqrt{6}}$	$\frac{1}{30\sqrt{2}}$	$\frac{1}{30}$
$\hat{\Delta}_{(10,1,3)}^{(3,1,\frac{2}{3})}$	$\hat{A}_{(15,1,1)}^{(\bar{3},1,-\frac{2}{3})}$	0	0	0	$-\frac{1}{5}$	0	0	0	0
$\hat{\Delta}_{(10,1,3)}^{(3,1,\frac{2}{3})}$	$\hat{\Phi}_{(15,1,3)}^{(\bar{3},1,-\frac{2}{3})}$	0	0	0	$-\frac{1}{5\sqrt{6}}$	0	0	0	0
$\hat{\Phi}_{(15,1,1)}^{(3,1,\frac{2}{3})}$	$\hat{\Phi}_{(15,1,1)}^{(\bar{3},1,-\frac{2}{3})}$	0	0	$-\frac{1}{\sqrt{15}}$	0	0	0	$\frac{1}{18\sqrt{2}}$	0
$\hat{\Phi}_{(15,1,1)}^{(3,1,\frac{2}{3})}$	$\hat{\Phi}_{(15,1,3)}^{(\bar{3},1,-\frac{2}{3})}$	$\frac{\sqrt{2}}{5}$	0	0	0	0	0	0	$\frac{1}{18\sqrt{2}}$
$\hat{\Phi}_{(15,1,3)}^{(3,1,\frac{2}{3})}$	$\hat{\Phi}_{(15,1,3)}^{(\bar{3},1,-\frac{2}{3})}$	0	$\frac{\sqrt{2}}{5\sqrt{3}}$	$\frac{1}{4\sqrt{15}}$	0	0	$\frac{1}{6\sqrt{6}}$	$\frac{1}{18\sqrt{2}}$	0
$\hat{\Phi}_{(15,1,1)}^{(\bar{3},1,-\frac{2}{3})}$	$\hat{\Delta}_{(10,1,3)}^{(3,1,\frac{2}{3})}$	0	0	0	$-\frac{1}{10\sqrt{3}}$	0	0	0	0
$\hat{\Phi}_{(15,1,3)}^{(\bar{3},1,-\frac{2}{3})}$	$\hat{\Delta}_{(10,1,3)}^{(3,1,\frac{2}{3})}$	0	0	0	$-\frac{1}{5\sqrt{6}}$	0	0	0	0

ing. First, the component of a field in the Goldstone mode is proportional to its charge under the breaking $U(1)$ and to its VEV. Second, there is no dependence on the superpotential parameters of the model, besides through the VEV determinations indirectly.

In the model (14), we have

$$\vec{G}_{(1,1,0)} = v_R \hat{\Delta}_{(10,1,3)}^{(1,1,0)} - \bar{v}_R \hat{\Delta}_{(10,1,3)}^{(1,1,0)}. \quad (19)$$

up to an obvious normalization.

4.2 $[(1, 1, 1) + \text{c.c.}]$

They are relevant for the SSB of $SU(2)_R$ into $U(1)_{I_{3R}}$. The fields contain $\bar{\alpha} = (1, 1, 1)$ are

$$\hat{A}_{(1,1,3)}^{(1,1,1)}, \hat{D}_{(10,1,1)}^{(1,1,1)}, \hat{\Delta}_{(10,1,3)}^{(1,1,1)}, \hat{\Phi}_{(15,1,3)}^{(1,1,1)}.$$

Note that D has no VEV, the Goldstone corresponding to $\bar{\alpha} = (1, 1, 1)$ of G_{321} is written as

$$\vec{G}_{(1,1,1)} = T_{\bar{\alpha}}^{A_1} A_1 \hat{A}_{(1,1,3)}^{(1,1,1)} + T_{\bar{\alpha}}^{v_R} v_R \hat{\Delta}_{(10,1,3)}^{(1,1,1)} + T_{\bar{\alpha}}^{\Phi_3} \Phi_3 \hat{\Phi}_{(15,1,3)}^{(1,1,1)}.$$

Here $A_1 \dots$ are the VEVs, $\hat{A}, \dots$ are the fields and T s are what will be solved. We will first classify the couplings and discuss their results, both on solve the Goldstones and on the identities among the CGCs.

(1) The representations E, A_2, Φ_1, Φ_2 have no $(1, 1, \pm 1)$ under G_{321} , so that couplings of them with two same representations, such as $A_1^2 E$ or $\Phi_3^2 \Phi_1$, do not give relations

among the T s. These couplings lead to relations of CGCs such as

$$0 = C_{111}^{A_1 A_1 E} - C_{\bar{\alpha} \alpha 1}^{A_1 A_1 E} = \frac{\sqrt{3}}{2\sqrt{5}} - \frac{\sqrt{3}}{2\sqrt{5}},$$

$$0 = C_{111}^{\Phi_3 \Phi_3 \Phi_1} - C_{\bar{\alpha} \alpha 1}^{\Phi_3 \Phi_3 \Phi_1} = \frac{1}{6\sqrt{6}} - \frac{1}{6\sqrt{6}},$$

which are trivial.

(2) Any coupling of E, A_2, Φ_1, Φ_2 with two different representations will set up a relation of the two T s. Following (8), the coupling $A_1 \Phi_3 A_2$ gives

$$0 = \begin{pmatrix} -C_{111}^{A_1 \Phi_3 A_2} & C_{\bar{\alpha} \alpha 1}^{A_1 \Phi_3 A_2} \\ C_{\bar{\alpha} \alpha 1}^{A_1 \Phi_3 A_2} & -C_{111}^{A_1 \Phi_3 A_2} \end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}}^{A_1} \\ T_{\bar{\alpha}}^{\Phi_3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}}^{\Phi_2} \\ T_{\bar{\alpha}}^{\Phi_3} \end{pmatrix},$$

where the determinant is zero, so

$$\frac{T_{\bar{\alpha}}^{A_1}}{T_{\bar{\alpha}}^{\Phi_3}} = \frac{C_{\bar{\alpha} \alpha 1}^{A_1 \Phi_3 A_2}}{C_{111}^{A_1 \Phi_3 A_2}} = \left(-\frac{1}{\sqrt{6}}\right) / \left(\frac{1}{\sqrt{6}}\right) = -1.$$

The coupling $A_1 \Phi_3 \Phi_2$ also gives

$$\frac{T_{\bar{\alpha}}^{A_1}}{T_{\bar{\alpha}}^{\Phi_3}} = \frac{C_{\bar{\alpha} \alpha 1}^{A_1 \Phi_3 \Phi_2}}{C_{111}^{A_1 \Phi_3 \Phi_2}} = \left(-\frac{1}{\sqrt{6}}\right) / \left(\frac{1}{\sqrt{6}}\right) = -1.$$

(3) $\Phi_3 v_R \bar{v}_R$ and $A_1 v_R \bar{v}_R$ are couplings with all three representations contain the Goldstone modes. However, v_R contains only $\bar{\alpha}$ while $\bar{v}_R$ contains only α . Following (13), the coupling $\Phi_3 v_R \bar{v}_R$ gives

$$\frac{T_{\bar{\alpha}}^{\Phi_3}}{T_{\bar{\alpha}}^{v_R}} = \frac{C_{\bar{\alpha} \alpha 1}^{v_R \Phi_3 \bar{v}_R}}{C_{1 1 1}^{v_R \Phi_3 \bar{v}_R}} = \left(-\frac{1}{10}\right) / \left(\frac{1}{10}\right) = -1,$$

and $A_1 v_R \bar{v}_R$ gives

$$\frac{T_{\bar{\alpha}}^{A_1}}{T_{\bar{\alpha}}^{v_R}} = \frac{C_{\bar{\alpha} \alpha 1}^{v_R A_1 \bar{v}_R}}{C_{1 1 1}^{v_R A_1 \bar{v}_R}} = \left(-\frac{1}{5}\right) / \left(-\frac{1}{5}\right) = 1.$$

Altogether, we have consistently $T_{\bar{\alpha}}^{A_1} : T_{\bar{\alpha}}^{v_R} : T_{\bar{\alpha}}^{\Phi_3} = 1 : 1 : (-1)$, so the Goldstone mode is

$$\vec{G}_{(1,1,1)} = A_1 \hat{A}_{(1,1,3)}^{(1,1,1)} + v_R \hat{\Delta}_{(10,1,3)}^{(1,1,1)} - \Phi_3 \hat{\Phi}_{(15,1,3)}^{(1,1,1)} \quad (20)$$

with an obvious normalization.

4.3 $[(3, 1, \frac{2}{3}) + \text{c.c.}]$

They are the Goldstones for $SU(4)_C$ SSB into $SU(3)_C \otimes U(1)_{B-L}$. The fields corresponding to $\bar{\alpha} = (3, 1, \frac{2}{3})$ are

$$\hat{A}_{(15,1,1)}^{(3,1,\frac{2}{3})}, \hat{D}_{(6,1,3)}^{(3,1,\frac{2}{3})}, \hat{\Delta}_{(10,1,3)}^{(3,1,\frac{2}{3})}, \hat{\Phi}_{(15,1,1)}^{(3,1,\frac{2}{3})}, \hat{\Phi}_{(15,1,3)}^{(3,1,\frac{2}{3})}.$$

Again, D does not have a VEV, the Goldstone mode is

$$\vec{G}_{(3,1,\frac{2}{3})} = T_{\bar{\alpha}}^{A_2} A_2 \hat{A}_{(15,1,1)}^{(3,1,\frac{2}{3})} + T_{\bar{\alpha}}^{\bar{v}_R} \bar{v}_R \hat{\Delta}_{(10,1,3)}^{(3,1,\frac{2}{3})} + T_{\bar{\alpha}}^{\Phi_2} \Phi_2 \hat{\Phi}_{(15,1,1)}^{(3,1,\frac{2}{3})} + T_{\bar{\alpha}}^{\Phi_3} \Phi_3 \hat{\Phi}_{(15,1,3)}^{(3,1,\frac{2}{3})}.$$

(1) According to (10), Φ_2^3 leads to

$$0 = C_{1 1 1}^{\Phi_2 \Phi_2 \Phi_2} - 2C_{\bar{\alpha} \alpha 1}^{\Phi_2 \Phi_2 \Phi_2} = \frac{1}{9\sqrt{2}} - 2 * \frac{1}{18\sqrt{2}}.$$

(2) A_1, E, Φ_1 does not contain α or $\bar{\alpha}$ and are $SU(4)$ singlets, their couplings with the same fields, $A_2^2 E, \Phi_2^3 E, \Phi_3^2 \Phi_1$ lead to only trivial results. In the couplings of one of them with two different fields, according to (8), they give relations between two Goldstone components. $A_2 \Phi_3 A_1$ gives

$$\frac{T_{\bar{\alpha}}^{A_2}}{T_{\bar{\alpha}}^{\Phi_3}} = \frac{C_{\bar{\alpha} \alpha 1}^{A_2 \Phi_3 A_1}}{C_{1 1 1}^{A_2 \Phi_3 A_1}} = \left(-\frac{1}{\sqrt{6}}\right) / \left(\frac{1}{\sqrt{6}}\right) = -1,$$

$A_2 \Phi_2 \Phi_1$ gives

$$\frac{T_{\bar{\alpha}}^{A_2}}{T_{\bar{\alpha}}^{\Phi_2}} = \frac{C_{\bar{\alpha} \alpha 1}^{A_2 \Phi_2 \Phi_1}}{C_{1 1 1}^{A_2 \Phi_2 \Phi_1}} = \left(-\frac{\sqrt{2}}{5}\right) / \left(\frac{\sqrt{2}}{5}\right) = -1,$$

and $\Phi_2 \Phi_3 A_1$ gives

$$\frac{T_{\bar{\alpha}}^{A_2}}{T_{\bar{\alpha}}^{\Phi_3}} = \frac{C_{\bar{\alpha} \alpha 1}^{A_2 \Phi_3 A_1}}{C_{1 1 1}^{A_2 \Phi_3 A_1}} = \left(-\frac{1}{\sqrt{6}}\right) / \left(\frac{1}{\sqrt{6}}\right) = -1.$$

(3) v_R contains only $\bar{\alpha}$ while $\bar{v}_R$ contains only α . According to (13), $v_R \bar{v}_R \Phi_2$ gives

$$\frac{T_{\bar{\alpha}}^{\Phi_2}}{T_{\bar{\alpha}}^{v_R}} = \frac{C_{\bar{\alpha} \alpha 1}^{\bar{v}_R \Phi_2 v_R}}{C_{1 1 1}^{\bar{v}_R \Phi_2 v_R}} = \left(-\frac{1}{10\sqrt{3}}\right) / \left(\frac{1}{10\sqrt{2}}\right) = -\sqrt{\frac{2}{3}},$$

$v_R \bar{v}_R \Phi_3$ gives

$$\frac{T_{\bar{\alpha}}^{\Phi_3}}{T_{\bar{\alpha}}^{v_R}} = \frac{C_{\bar{\alpha} \alpha 1}^{\bar{v}_R \Phi_3 v_R}}{C_{1 1 1}^{\bar{v}_R \Phi_3 v_R}} = \left(-\frac{1}{5\sqrt{6}}\right) / \left(\frac{1}{10}\right) = -\sqrt{\frac{2}{3}},$$

and $v_R \bar{v}_R A_2$ gives

$$\frac{T_{\bar{\alpha}}^{A_2}}{T_{\bar{\alpha}}^{v_R}} = \frac{C_{\bar{\alpha} \alpha 1}^{\bar{v}_R A_2 v_R}}{C_{1 1 1}^{\bar{v}_R A_2 v_R}} = \left(-\frac{1}{5}\right) / \left(-\frac{\sqrt{3}}{5\sqrt{2}}\right) = \sqrt{\frac{2}{3}}.$$

(4) According to (9), $\Phi_2 \Phi_3^2$ gives

$$0 = \begin{pmatrix} -C_{1 1 1}^{\Phi_2 \Phi_3 \Phi_3} & 2C_{\bar{\alpha} \alpha 1}^{\Phi_2 \Phi_3 \Phi_3} \\ C_{\bar{\alpha} \alpha 1}^{\Phi_3 \Phi_2 \Phi_3} & C_{\bar{\alpha} \alpha 1}^{\Phi_3 \Phi_3 \Phi_2} - C_{1 1 1}^{\Phi_3 \Phi_3 \Phi_2} \end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}}^{\Phi_2} \\ T_{\bar{\alpha}}^{\Phi_3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{9\sqrt{2}} & 2 * \frac{1}{18\sqrt{2}} \\ \frac{1}{18\sqrt{2}} & \frac{1}{9\sqrt{2}} - \frac{1}{18\sqrt{2}} \end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}}^{\Phi_2} \\ T_{\bar{\alpha}}^{\Phi_3} \end{pmatrix},$$

so

$$\frac{T_{\Phi_3}}{T_{\Phi_2}} = \frac{C_{\bar{\alpha} \alpha 1}^{\Phi_3 \Phi_2 \Phi_3}}{C_{1 1 1}^{\Phi_3 \Phi_3 \Phi_2} - C_{\bar{\alpha} \alpha 1}^{\Phi_3 \Phi_3 \Phi_2}} = \left(\frac{1}{18\sqrt{2}}\right) / \left(\frac{1}{9\sqrt{2}} - \frac{1}{18\sqrt{2}}\right) = 1.$$

$$\text{or: } \frac{T_{\Phi_3}}{T_{\Phi_2}} = \frac{C_{1 1 1}^{\Phi_2 \Phi_3 \Phi_3}}{2C_{\bar{\alpha} \alpha 1}^{\Phi_2 \Phi_3 \Phi_3}} = \left(\frac{1}{9\sqrt{2}}\right) / \left(2 * \frac{1}{18\sqrt{2}}\right) = 1.$$

Similarly, $\Phi_2 A_2^2$ gives

$$\frac{T_{\bar{\alpha}}^{A_2}}{T_{\bar{\alpha}}^{\Phi_2}} = \frac{C_{\bar{\alpha} \alpha 1}^{A_2 \Phi_2 A_2}}{C_{1 1 1}^{A_2 A_2 \Phi_2} - C_{\bar{\alpha} \alpha 1}^{A_2 A_2 \Phi_2}} = \left(-\frac{1}{3\sqrt{2}}\right) / \left(\frac{\sqrt{2}}{3} - \frac{1}{3\sqrt{2}}\right) = -1.$$

$$\text{or: } \frac{T_{\bar{\alpha}}^{A_2}}{T_{\bar{\alpha}}^{\Phi_2}} = \frac{C_{1 1 1}^{\Phi_2 A_2 A_2}}{2C_{\bar{\alpha} \alpha 1}^{\Phi_2 A_2 A_2}} = \left(\frac{\sqrt{2}}{3}\right) / \left(2 * \frac{-1}{3\sqrt{2}}\right) = -1.$$

$A_2 \Phi_3^2$ gives

$$\frac{T_{\bar{\alpha}}^{A_2}}{T_{\bar{\alpha}}^{\Phi_3}} = \frac{2C_{\bar{\alpha} \alpha 1}^{A_2 \Phi_3 \Phi_3}}{C_{1 1 1}^{A_2 \Phi_3 \Phi_3}} = \left(-\frac{2 * \sqrt{2}}{5\sqrt{3}}\right) / \left(\frac{2\sqrt{2}}{5\sqrt{3}}\right) = -1,$$

$$\text{or: } \frac{T_{\bar{\alpha}}^{A_2}}{T_{\bar{\alpha}}^{\Phi_3}} = \frac{C_{1 1 1}^{\Phi_3 \Phi_3 A_2} - C_{\bar{\alpha} \alpha 1}^{\Phi_3 \Phi_3 A_2}}{C_{\bar{\alpha} \alpha 1}^{\Phi_3 A_2 \Phi_3}} = \left(\frac{2\sqrt{2}}{5\sqrt{3}} - \frac{\sqrt{2}}{5\sqrt{3}}\right) / \left(-\frac{\sqrt{2}}{5\sqrt{3}}\right) = -1.$$

Table 5 CGCs for the SM singlets (1, 1, 0) under $SU(5)$. Here, for example, a_2 stands for $\widehat{A}_{(24)}^{(1,1,0)}$ etc.

X	Y	XYa_1	XYa_2	$XY S$	$XY\phi_1$	$XY\phi_2$	$XY\phi_3$
$\widehat{A}_{(1)}^{(1,1,0)}$	$\widehat{A}_{(1)}^{(1,1,0)}$	0	0	0	$\frac{4}{\sqrt{15}}$	0	0
$\widehat{A}_{(24)}^{(1,1,0)}$	$\widehat{A}_{(24)}^{(1,1,0)}$	0	0	$\frac{1}{\sqrt{15}}$	$-\frac{1}{\sqrt{15}}$	$-\frac{2}{3\sqrt{15}}$	$\frac{5}{3\sqrt{3}}$
$\widehat{A}_{(1)}^{(1,1,0)}$	$\widehat{A}_{(24)}^{(1,1,0)}$	0	0	$\sqrt{\frac{2}{5}}$	0	$\sqrt{\frac{2}{5}}$	0
$\widehat{E}_{(24)}^{(1,1,0)}$	$\widehat{E}_{(24)}^{(1,1,0)}$	0	0	$\sqrt{\frac{3}{5}}$	0	0	0
$\widehat{\Delta}_{(1)}^{(1,1,0)}$	$\widehat{\Delta}_{(1)}^{(1,1,0)}$	$-\frac{i}{\sqrt{10}}$	0	0	$\frac{1}{2\sqrt{15}}$	0	0
$\widehat{\Phi}_{(1)}^{(1,1,0)}$	$\widehat{\Phi}_{(1)}^{(1,1,0)}$	$\frac{6\sqrt{2}}{5\sqrt{5}}$	0	0	$\sqrt{\frac{3}{5}}$	0	0
$\widehat{\Phi}_{(1)}^{(1,1,0)}$	$\widehat{\Phi}_{(24)}^{(1,1,0)}$	0	$-\frac{2\sqrt{3}}{5\sqrt{5}}$	$\frac{\sqrt{3}}{2\sqrt{5}}$	0	$\frac{1}{2\sqrt{15}}$	0
$\widehat{\Phi}_{(1)}^{(1,1,0)}$	$\widehat{\Phi}_{(75)}^{(1,1,0)}$	0	0	0	0	0	$-\frac{1}{\sqrt{15}}$
$\widehat{\Phi}_{(24)}^{(1,1,0)}$	$\widehat{\Phi}_{(24)}^{(1,1,0)}$	$-\frac{4\sqrt{2}}{5\sqrt{5}}$	$\frac{8}{15\sqrt{15}}$	$\frac{1}{6\sqrt{15}}$	$\frac{1}{2\sqrt{15}}$	$-\frac{7}{9\sqrt{15}}$	$\frac{5}{18\sqrt{3}}$
$\widehat{\Phi}_{(24)}^{(1,1,0)}$	$\widehat{\Phi}_{(75)}^{(1,1,0)}$	0	$\frac{2}{3\sqrt{3}}$	$\frac{5}{6\sqrt{3}}$	0	$\frac{5}{18\sqrt{3}}$	$-\frac{8}{9\sqrt{15}}$
$\widehat{\Phi}_{(75)}^{(1,1,0)}$	$\widehat{\Phi}_{(75)}^{(1,1,0)}$	$\frac{2\sqrt{2}}{5\sqrt{5}}$	$-\frac{32}{15\sqrt{15}}$	$\frac{4}{3\sqrt{15}}$	$-\frac{1}{\sqrt{15}}$	$-\frac{8}{9\sqrt{15}}$	$\frac{8}{9\sqrt{3}}$

All together we have $T_A = \sqrt{\frac{2}{3}} T_{\bar{\alpha}}^{\overline{VR}} = -T_{\bar{\alpha}}^{\Phi_2} = -T_{\bar{\alpha}}^{\Phi_3}$, then the Goldstone is

$$\begin{aligned} \vec{G}_{(3,1,\frac{2}{3})} = & A_2 \widehat{A}_{(15,1,1)}^{(3,1,\frac{2}{3})} + \sqrt{\frac{3}{2}} \overline{v}_R \widehat{\Delta}_{(10,1,3)}^{(3,1,\frac{2}{3})} \\ & - \Phi_2 \widehat{\Phi}_{(15,1,1)}^{(3,1,\frac{2}{3})} - \Phi_3 \widehat{\Phi}_{(15,1,3)}^{(3,1,\frac{2}{3})}. \end{aligned} \quad (21)$$

5 Goldstone modes for SSB of $SU(5)$: $[(3, 2, -\frac{5}{6}) + \text{c.c.}]$

In study SSB of $SU(5)$, the fields need to be decomposed into $SU(5)$ representations and use the CGCs calculated in [36]. Under $SU(5)$, the following representations

$$\begin{aligned} a_1 &\equiv \widehat{A}_{(1)}, \quad a_2 \equiv \widehat{A}_{(24)}, \quad S \equiv \widehat{E}_{(24)}, \\ V_R &\equiv \widehat{\Delta}_{(1)}, \quad \overline{V}_R \equiv \widehat{\Delta}_{(1)}, \\ \phi_1 &\equiv \widehat{\Phi}_{(1)}, \quad \phi_2 \equiv \widehat{\Phi}_{(24)}, \quad \phi_3 \equiv \widehat{\Phi}_{(75)}. \end{aligned} \quad (22)$$

contain the SM singlets whose VEVs will be denoted by the same symbols. Here the subscripts are the representations under $SU(5)$.

Since the $SU(5)$ singlets do not contribute to the Goldstone modes, the fields contains $\bar{\alpha} = (3, 2, -\frac{5}{6})$ are

$$\widehat{A}_{(24)}^{(3,2,-\frac{5}{6})}, \widehat{E}_{(24)}^{(3,2,-\frac{5}{6})}, \widehat{\Phi}_{(24)}^{(3,2,-\frac{5}{6})}, \widehat{\Phi}_{(75)}^{(3,2,-\frac{5}{6})}.$$

The Goldstone corresponding to $(3, 2, -\frac{5}{6})$ is denoted as

$$\begin{aligned} \vec{G}_{(3,2,-\frac{5}{6})} = & T_{\bar{\alpha}}^{a_2} a_2 \widehat{A}_{(24)}^{(3,2,-\frac{5}{6})} + T_{\bar{\alpha}}^S S \widehat{E}_{(24)}^{(3,2,-\frac{5}{6})} \\ & + T_{\bar{\alpha}}^{\phi_2} \phi_2 \widehat{\Phi}_{(24)}^{(3,2,-\frac{5}{6})} + T_{\bar{\alpha}}^{\phi_3} \phi_3 \widehat{\Phi}_{(75)}^{(3,2,-\frac{5}{6})}. \end{aligned}$$

The needed CGCs are taken from [36] and summarized in Tables 5 and 6.

(1) According to (10), the coupling ϕ_2^3 gives

$$0 = C_{111}^{\phi_2\phi_2\phi_2} - 2C_{\bar{\alpha}\alpha 1}^{\phi_2\phi_2\phi_2} = \left(-\frac{7}{9\sqrt{15}}\right) - 2 * \left(-\frac{7}{18\sqrt{15}}\right),$$

ϕ_3^3 gives

$$0 = C_{111}^{\phi_3\phi_3\phi_3} - 2C_{\bar{\alpha}\alpha 1}^{\phi_3\phi_3\phi_3} = \frac{8}{9\sqrt{3}} - 2 * \frac{4}{9\sqrt{3}},$$

and S^3 gives

$$0 = C_{111}^{SSS} - 2C_{\bar{\alpha}\alpha 1}^{SSS} = \sqrt{\frac{3}{5}} - 2 * \frac{\sqrt{3}}{2\sqrt{5}}.$$

(2) ϕ_1, a_1 contain neither Goldstones, the couplings $\phi_2^2\phi_1, \phi_3^2\phi_1, a_2^2\phi_1, \phi_2^2a_1$ and $\phi_3^2a_1$ lead to only trivial results. The coupling $a_2\phi_2a_1$, according to (8), gives

$$\frac{T_{\bar{\alpha}}^{a_2}}{T_{\bar{\alpha}}^{\phi_2}} = \frac{C_{\bar{\alpha}\alpha 1}^{a_2\phi_2a_1}}{C_{111}^{a_2\phi_2a_1}} = \left(-\sqrt{\frac{2}{5}}\right) / \left(\sqrt{\frac{2}{5}}\right) = -1,$$

$a_2\phi_2\phi_1$ gives

$$\frac{T_{\bar{\alpha}}^{a_2}}{T_{\bar{\alpha}}^{\phi_2}} = \frac{C_{\bar{\alpha}\alpha 1}^{a_2\phi_2\phi_1}}{C_{111}^{a_2\phi_2\phi_1}} = \left(\frac{2\sqrt{3}}{5\sqrt{5}}\right) / \left(-\frac{2\sqrt{3}}{5\sqrt{5}}\right) = -1,$$

Sa_2a_1 gives

$$\frac{T_{\bar{\alpha}}^S}{T_{\bar{\alpha}}^{a_2}} = \frac{C_{\bar{\alpha}\alpha 1}^{Sa_2a_1}}{C_{111}^{Sa_2a_1}} = \left(-\sqrt{\frac{2}{5}}\right) / \left(\sqrt{\frac{2}{5}}\right) = -1,$$

and $S\phi_2\phi_1$ gives

$$\frac{T_{\bar{\alpha}}^S}{T_{\bar{\alpha}}^{\phi_2}} = \frac{C_{\bar{\alpha}\alpha 1}^{S\phi_2\phi_1}}{C_{111}^{S\phi_2\phi_1}} = \left(\frac{\sqrt{3}}{2\sqrt{5}}\right) / \left(\frac{\sqrt{3}}{2\sqrt{5}}\right) = 1.$$

Table 6 CGCs for the Goldstones $(3, 2, -\frac{5}{6}) + \text{c.c. of } SU(5) \text{ SSB}$

X	Y	XYa_1	XYa_2	$XY S$	$XY\phi_1$	$XY\phi_2$	$XY\phi_3$
$\hat{A}_{(24)}^{(3,2,-\frac{5}{6})}$	$\hat{A}_{(24)}^{(\bar{3},2,\frac{5}{6})}$	0	0	$\frac{1}{2\sqrt{15}}$	$-\frac{1}{\sqrt{15}}$	$-\frac{1}{3\sqrt{15}}$	$\frac{1}{3\sqrt{3}}$
$\hat{A}_{(24)}^{(3,2,-\frac{5}{6})}$	$\hat{E}_{(24)}^{(\bar{3},2,\frac{5}{6})}$	$-\sqrt{\frac{2}{5}}$	$-\frac{1}{2\sqrt{15}}$	0	0	0	0
$\hat{A}_{(24)}^{(3,2,-\frac{5}{6})}$	$\hat{\Phi}_{(24)}^{(\bar{3},2,\frac{5}{6})}$	$-\sqrt{\frac{2}{5}}$	$\frac{1}{3\sqrt{15}}$	0	$\frac{2\sqrt{3}}{5\sqrt{5}}$	$-\frac{4}{15\sqrt{15}}$	$-\frac{2}{15\sqrt{3}}$
$\hat{A}_{(24)}^{(3,2,-\frac{5}{6})}$	$\hat{\Phi}_{(75)}^{(\bar{3},2,\frac{5}{6})}$	0	$-\frac{\sqrt{10}}{3\sqrt{3}}$	0	0	$-\frac{2\sqrt{2}}{3\sqrt{15}}$	$\frac{4\sqrt{2}}{15\sqrt{3}}$
$\hat{E}_{(24)}^{(3,2,-\frac{5}{6})}$	$\hat{E}_{(24)}^{(\bar{3},2,\frac{5}{6})}$	0	0	$\frac{\sqrt{3}}{2\sqrt{5}}$	0	0	0
$\hat{E}_{(24)}^{(3,2,-\frac{5}{6})}$	$\hat{\Phi}_{(24)}^{(\bar{3},2,\frac{5}{6})}$	0	0	0	$\frac{\sqrt{3}}{2\sqrt{5}}$	$\frac{1}{12\sqrt{15}}$	$\frac{1}{6\sqrt{3}}$
$\hat{E}_{(24)}^{(3,2,-\frac{5}{6})}$	$\hat{\Phi}_{(75)}^{(\bar{3},2,\frac{5}{6})}$	0	0	0	0	$\frac{\sqrt{5}}{3\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$
$\hat{\Phi}_{(24)}^{(3,2,-\frac{5}{6})}$	$\hat{\Phi}_{(24)}^{(\bar{3},2,\frac{5}{6})}$	$-\frac{4\sqrt{2}}{5\sqrt{5}}$	$\frac{4}{15\sqrt{15}}$	$\frac{1}{12\sqrt{15}}$	$\frac{1}{2\sqrt{15}}$	$-\frac{7}{18\sqrt{15}}$	$\frac{1}{18\sqrt{3}}$
$\hat{\Phi}_{(24)}^{(3,2,-\frac{5}{6})}$	$\hat{\Phi}_{(75)}^{(\bar{3},2,\frac{5}{6})}$	0	$\frac{2\sqrt{2}}{3\sqrt{15}}$	$\frac{\sqrt{5}}{3\sqrt{6}}$	0	$\frac{\sqrt{5}}{9\sqrt{6}}$	$-\frac{\sqrt{2}}{9\sqrt{3}}$
$\hat{\Phi}_{(75)}^{(3,2,-\frac{5}{6})}$	$\hat{\Phi}_{(75)}^{(\bar{3},2,\frac{5}{6})}$	$\frac{2\sqrt{2}}{5\sqrt{5}}$	$-\frac{22}{15\sqrt{15}}$	$\frac{11}{12\sqrt{15}}$	$-\frac{1}{\sqrt{15}}$	$-\frac{11}{18\sqrt{15}}$	$\frac{4}{9\sqrt{3}}$

(3) According to (9), $\phi_2\phi_3^2$ gives

$$\frac{T_{\tilde{\alpha}}^{\phi_2}}{T_{\tilde{\alpha}}^{\phi_3}} = \frac{2C_{\tilde{\alpha} \alpha 1}^{\phi_2\phi_3\phi_3}}{C_{111}^{\phi_2\phi_3\phi_3}} = 2 * \left(-\frac{\sqrt{2}}{9\sqrt{3}} \right) / \left(-\frac{8}{9\sqrt{15}} \right) = \sqrt{\frac{5}{8}},$$

$$\text{or: } \frac{T_{\tilde{\alpha}}^{\phi_2}}{T_{\tilde{\alpha}}^{\phi_3}} = \frac{C_{111}^{\phi_3\phi_3\phi_2} - C_{\tilde{\alpha} \alpha 1}^{\phi_3\phi_3\phi_2}}{C_{\tilde{\alpha} \alpha 1}^{\phi_3\phi_3\phi_3}} = \left(\frac{-8}{9\sqrt{15}} - \frac{-11}{18\sqrt{15}} \right) / \left(-\frac{\sqrt{2}}{9\sqrt{3}} \right) = \sqrt{\frac{5}{8}},$$

$\phi_2^2\phi_3$ gives

$$\frac{T_{\tilde{\alpha}}^{\phi_2}}{T_{\tilde{\alpha}}^{\phi_3}} = \frac{C_{\tilde{\alpha} \alpha 1}^{\phi_2\phi_3\phi_2}}{C_{111}^{\phi_2\phi_3\phi_3} - C_{\tilde{\alpha} \alpha 1}^{\phi_2\phi_3\phi_3}} = \left(\frac{\sqrt{5}}{9\sqrt{6}} \right) / \left(\frac{5}{18\sqrt{3}} - \frac{1}{18\sqrt{3}} \right) = \sqrt{\frac{5}{8}},$$

$$\text{or: } \frac{T_{\tilde{\alpha}}^{\phi_2}}{T_{\tilde{\alpha}}^{\phi_3}} = \frac{C_{111}^{\phi_3\phi_2\phi_2}}{2C_{\tilde{\alpha} \alpha 1}^{\phi_3\phi_2\phi_2}} = \left(\frac{5}{18\sqrt{3}} \right) / \left(2 * \frac{\sqrt{5}}{9\sqrt{6}} \right) = \sqrt{\frac{5}{8}},$$

$a_2^2\phi_2$ gives

$$\frac{T_{\tilde{\alpha}}^{a_2}}{T_{\tilde{\alpha}}^{\phi_2}} = \frac{C_{\tilde{\alpha} \alpha 1}^{a_2\phi_2a_2}}{C_{111}^{a_2a_2\phi_2} - C_{\tilde{\alpha} \alpha 1}^{a_2a_2\phi_2}} = \left(\frac{1}{3\sqrt{15}} \right) / \left(\frac{-2}{3\sqrt{15}} - \frac{-1}{3\sqrt{15}} \right) = -1,$$

$$\text{or: } \frac{T_{\tilde{\alpha}}^{a_2}}{T_{\tilde{\alpha}}^{\phi_2}} = \frac{C_{111}^{\phi_2a_2a_2}}{2C_{\tilde{\alpha} \alpha 1}^{\phi_2a_2a_2}} = \left(-\frac{2}{3\sqrt{15}} \right) / 2 * \left(\frac{1}{3\sqrt{15}} \right) = -1,$$

$a_2^2\phi_3$ gives

$$\frac{T_{\tilde{\alpha}}^{a_2}}{T_{\tilde{\alpha}}^{\phi_3}} = \frac{C_{\tilde{\alpha} \alpha 1}^{a_2\phi_3a_2}}{C_{111}^{a_2a_2\phi_3} - C_{\tilde{\alpha} \alpha 1}^{a_2a_2\phi_3}} = \left(-\frac{\sqrt{10}}{3\sqrt{3}} \right) / \left(\frac{5}{3\sqrt{3}} - \frac{1}{3\sqrt{3}} \right) = -\sqrt{\frac{5}{8}},$$

$$\text{or: } \frac{T_{\tilde{\alpha}}^{a_2}}{T_{\tilde{\alpha}}^{\phi_3}} = \frac{C_{111}^{\phi_3a_2a_2}}{2C_{\tilde{\alpha} \alpha 1}^{\phi_3a_2a_2}} = \left(\frac{5}{3\sqrt{3}} \right) / \left(2 * \left(-\frac{\sqrt{10}}{3\sqrt{3}} \right) \right) = -\sqrt{\frac{5}{8}},$$

$\phi_2^2a_2$ gives

$$\frac{T_{\tilde{\alpha}}^{a_2}}{T_{\tilde{\alpha}}^{\phi_2}} = \frac{2C_{\tilde{\alpha} \alpha 1}^{a_2\phi_2\phi_2}}{C_{111}^{a_2\phi_2\phi_2}} = 2 * \left(-\frac{4}{15\sqrt{15}} \right) / \left(\frac{8}{15\sqrt{15}} \right) = -1,$$

$$\text{or: } \frac{T_{\tilde{\alpha}}^{a_2}}{T_{\tilde{\alpha}}^{\phi_2}} = \frac{C_{111}^{\phi_2\phi_2a_2} - C_{\tilde{\alpha} \alpha 1}^{\phi_2\phi_2a_2}}{C_{\tilde{\alpha} \alpha 1}^{\phi_2a_2\phi_2}} = \left(\frac{8}{15\sqrt{15}} - \frac{4}{15\sqrt{15}} \right) / \left(-\frac{4}{15\sqrt{15}} \right) = -1,$$

$\phi_3^2 a_2$ gives

$$\begin{aligned}\frac{T_{\tilde{\alpha}}^{a_2}}{T_{\tilde{\alpha}}^{\phi_3}} &= \frac{2C_{\tilde{\alpha} \alpha 1}^{a_2 \phi_3 \phi_3}}{C_{1111}^{a_2 \phi_3 \phi_3}} \\ &= 2 * \left(\frac{4\sqrt{2}}{15\sqrt{3}} \right) / \left(-\frac{32}{15\sqrt{15}} \right) = -\sqrt{\frac{5}{8}}, \\ \text{or: } \frac{T_{\tilde{\alpha}}^{a_2}}{T_{\tilde{\alpha}}^{\phi_3}} &= \frac{C_{1111}^{\phi_3 \phi_3 a_2} - C_{\tilde{\alpha} \alpha 1}^{\phi_3 \phi_3 a_2}}{C_{\tilde{\alpha} \alpha 1}^{\phi_3 a_2 \phi_3}} \\ &= \left(\frac{-32}{15\sqrt{15}} - \frac{-22}{15\sqrt{15}} \right) / \left(\frac{4\sqrt{2}}{15\sqrt{3}} \right) = -\sqrt{\frac{5}{8}},\end{aligned}$$

Sa_2^2 gives

$$\begin{aligned}\frac{T_{\tilde{\alpha}}^S}{T_{\tilde{\alpha}}^{a_2}} &= \frac{2C_{\tilde{\alpha} \alpha 1}^{Sa_2 a_2}}{C_{1111}^{Sa_2 a_2}} \\ &= 2 * \left(-\frac{1}{2\sqrt{15}} \right) / \left(\frac{1}{\sqrt{15}} \right) = -1, \\ \text{or: } \frac{T_{\tilde{\alpha}}^S}{T_{\tilde{\alpha}}^{a_2}} &= \frac{C_{1111}^{a_2 a_2 S} - C_{\tilde{\alpha} \alpha 1}^{a_2 a_2 S}}{C_{\tilde{\alpha} \alpha 1}^{a_2 S a_2}} \\ &= \left(\frac{1}{\sqrt{15}} - \frac{1}{2\sqrt{15}} \right) / \left(-\frac{1}{2\sqrt{15}} \right) = -1,\end{aligned}$$

$S\phi_2^2$ gives

$$\begin{aligned}\frac{T_{\tilde{\alpha}}^S}{T_{\tilde{\alpha}}^{\phi_2}} &= \frac{2C_{\tilde{\alpha} \alpha 1}^{S\phi_2 \phi_2}}{C_{1111}^{S\phi_2 \phi_2}} \\ &= 2 * \left(\frac{1}{12\sqrt{15}} \right) / \left(\frac{1}{6\sqrt{15}} \right) = 1, \\ \text{or: } \frac{T_{\tilde{\alpha}}^S}{T_{\tilde{\alpha}}^{\phi_2}} &= \frac{C_{1111}^{\phi_2 \phi_2 S} - C_{\tilde{\alpha} \alpha 1}^{\phi_2 \phi_2 S}}{C_{\tilde{\alpha} \alpha 1}^{\phi_2 S \phi_2}} \\ &= \left(\frac{1}{6\sqrt{15}} - \frac{1}{12\sqrt{15}} \right) / \left(\frac{1}{12\sqrt{15}} \right) = 1,\end{aligned}$$

and $S\phi_3^2$ gives

$$\begin{aligned}\frac{T_{\tilde{\alpha}}^S}{T_{\tilde{\alpha}}^{\phi_3}} &= \frac{2C_{\tilde{\alpha} \alpha 1}^{S\phi_3 \phi_3}}{C_{1111}^{S\phi_3 \phi_3}} \\ &= 2 * \left(\frac{1}{3\sqrt{6}} \right) / \left(\frac{4}{3\sqrt{15}} \right) = \sqrt{\frac{5}{8}}, \\ \text{or: } \frac{T_{\tilde{\alpha}}^S}{T_{\tilde{\alpha}}^{\phi_3}} &= \frac{C_{1111}^{\phi_3 \phi_3 S} - C_{\tilde{\alpha} \alpha 1}^{\phi_3 \phi_3 S}}{C_{\tilde{\alpha} \alpha 1}^{\phi_3 S \phi_3}} \\ &= \left(\frac{4}{3\sqrt{15}} - \frac{11}{12\sqrt{15}} \right) / \left(\frac{1}{3\sqrt{6}} \right) = \sqrt{\frac{5}{8}}.\end{aligned}$$

(4) According to (7), $a_2 \phi_2 \phi_3$ gives

$$\begin{aligned}0 &= \begin{pmatrix} -C_{1111}^{a_2 \phi_2 \phi_3} & C_{\tilde{\alpha} \alpha 1}^{a_2 \phi_2 \phi_3} & C_{\tilde{\alpha} \alpha 1}^{a_2 \phi_3 \phi_2} \\ C_{\tilde{\alpha} \alpha 1}^{\phi_2 a_2 \phi_3} & -C_{1111}^{\phi_2 a_2 \phi_3} & C_{\tilde{\alpha} \alpha 1}^{\phi_2 \phi_3 a_2} \\ C_{\tilde{\alpha} \alpha 1}^{\phi_3 a_2 \phi_2} & C_{\tilde{\alpha} \alpha 1}^{\phi_3 \phi_2 a_2} & -C_{1111}^{\phi_3 a_2 \phi_2} \end{pmatrix} \begin{pmatrix} T_{\tilde{\alpha}}^{A_2} \\ T_{\tilde{\alpha}}^{\phi_2} \\ T_{\tilde{\alpha}}^{\phi_3} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{3\sqrt{3}} & -\frac{2}{15\sqrt{3}} & -\frac{2\sqrt{2}}{3\sqrt{15}} \\ -\frac{2}{15\sqrt{3}} & -\frac{2}{3\sqrt{3}} & \frac{2\sqrt{2}}{3\sqrt{15}} \\ -\frac{2\sqrt{2}}{3\sqrt{15}} & \frac{2\sqrt{2}}{3\sqrt{15}} & -\frac{2}{3\sqrt{3}} \end{pmatrix} \begin{pmatrix} T_{\tilde{\alpha}}^{A_2} \\ T_{\tilde{\alpha}}^{\phi_2} \\ T_{\tilde{\alpha}}^{\phi_3} \end{pmatrix},\end{aligned}$$

so that

$$T_{\tilde{\alpha}}^{A_2} : T_{\tilde{\alpha}}^{\phi_2} : T_{\tilde{\alpha}}^{\phi_3} = 1 : -1 : \left(-\sqrt{\frac{8}{5}} \right),$$

and $S\phi_2 \phi_3$ gives

$$\begin{aligned}0 &= \begin{pmatrix} -C_{1111}^{S\phi_2 \phi_3} & C_{\tilde{\alpha} \alpha 1}^{S\phi_2 \phi_3} & C_{\tilde{\alpha} \alpha 1}^{S\phi_3 \phi_2} \\ C_{\tilde{\alpha} \alpha 1}^{\phi_2 S \phi_3} & -C_{1111}^{\phi_2 S \phi_3} & C_{\tilde{\alpha} \alpha 1}^{\phi_2 \phi_3 S} \\ C_{\tilde{\alpha} \alpha 1}^{\phi_3 S \phi_2} & C_{\tilde{\alpha} \alpha 1}^{\phi_3 \phi_2 S} & -C_{1111}^{\phi_3 S \phi_2} \end{pmatrix} \begin{pmatrix} T_{\tilde{\alpha}}^S \\ T_{\tilde{\alpha}}^{\phi_2} \\ T_{\tilde{\alpha}}^{\phi_3} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{5}{6\sqrt{3}} & \frac{1}{6\sqrt{3}} & \frac{\sqrt{5}}{3\sqrt{6}} \\ \frac{1}{6\sqrt{3}} & -\frac{5}{6\sqrt{3}} & \frac{\sqrt{5}}{3\sqrt{6}} \\ \frac{\sqrt{5}}{3\sqrt{6}} & \frac{\sqrt{5}}{3\sqrt{6}} & -\frac{5}{6\sqrt{3}} \end{pmatrix} \begin{pmatrix} T_{\tilde{\alpha}}^S \\ T_{\tilde{\alpha}}^{\phi_2} \\ T_{\tilde{\alpha}}^{\phi_3} \end{pmatrix},\end{aligned}$$

so that

$$T_{\tilde{\alpha}}^S : T_{\tilde{\alpha}}^{\phi_2} : T_{\tilde{\alpha}}^{\phi_3} = 1 : 1 : \sqrt{\frac{8}{5}}.$$

Altogether, the Goldstone mode is

$$\begin{aligned}\vec{G}_{(3,2,-\frac{5}{6})} &= a_2 \hat{A}_{(24)}^{(3,2,-\frac{5}{6})} - S \hat{E}_{(24)}^{(3,2,-\frac{5}{6})} \\ &\quad - \phi_2 \hat{\Phi}_{(24)}^{(3,2,-\frac{5}{6})} - \sqrt{\frac{8}{5}} \phi_3 \hat{\Phi}_{(75)}^{(3,2,-\frac{5}{6})}.\end{aligned}\quad (23)$$

6 Goldstone modes for SSB of $G_{51} : [(3, 2, \frac{1}{6}) + \text{c.c.}]$

In studying the SSB of the flipped $SU(5)$, the fields need to be decomposed into $G_{51} = (SU(5) \otimes U(1))^{Flipped}$ representations and use the CGCs accordingly. Under G_{51} , the following representations

$$\begin{aligned}
\tilde{a}_1 &\equiv \hat{A}_{(1,0)}, \quad \tilde{a}_2 \equiv \hat{A}_{(24,0)}, \quad \tilde{S} \equiv \hat{E}_{(24,0)}, \\
\delta &\equiv \hat{\Delta}_{(\overline{50},-2)}, \quad \bar{\delta} \equiv \hat{\Delta}_{(50,2)}, \\
\tilde{\phi}_1 &\equiv \hat{\Phi}_{(1,0)}, \quad \tilde{\phi}_2 \equiv \hat{\Phi}_{(24,0)}, \quad \tilde{\phi}_3 \equiv \hat{\Phi}_{(75,0)}.
\end{aligned} \quad (24)$$

contain the SM singlets whose VEVs will be denoted by the same symbols. Here the subscripts are the representations under G_{51} . These VEVs are related to those under G_{422} and $SU(5)$ by

$$\tilde{S} = E = S, \quad \delta = v_R = V_R, \quad \bar{\delta} = \overline{v_R} = \overline{V_R}, \quad (25)$$

$$\begin{aligned}
\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} &= \begin{bmatrix} \sqrt{\frac{2}{5}} & -\sqrt{\frac{3}{5}} \\ \sqrt{\frac{3}{5}} & \sqrt{\frac{2}{5}} \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \\
&= \begin{bmatrix} -\frac{1}{5} & \frac{2\sqrt{6}}{5} \\ \frac{2\sqrt{6}}{5} & \frac{1}{5} \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},
\end{aligned} \quad (26)$$

and

$$\begin{aligned}
\begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \end{pmatrix} &= \begin{bmatrix} \sqrt{\frac{1}{10}} & \sqrt{\frac{3}{10}} & -\sqrt{\frac{3}{5}} \\ \sqrt{\frac{2}{5}} & -\sqrt{\frac{8}{15}} & -\sqrt{\frac{1}{15}} \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} \end{bmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \\
&= \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{2\sqrt{5}}{5} \\ -\frac{2}{5} & \frac{13}{15} & \frac{2\sqrt{5}}{15} \\ \frac{2\sqrt{5}}{5} & \frac{2\sqrt{5}}{15} & \frac{1}{3} \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.
\end{aligned} \quad (27)$$

The fields contain to $\tilde{\alpha} = (3, 2, \frac{1}{6})$ are

$$\begin{aligned}
\hat{A}_{(24,0)}^{(3,2,\frac{1}{6})}, \hat{E}_{(24,0)}^{(3,2,\frac{1}{6})}, \hat{D}_{(10,-6)}^{(3,2,\frac{1}{6})}, \hat{\Delta}_{(\overline{50},-2)}^{(3,2,\frac{1}{6})}, \hat{\Delta}_{(45,-2)}^{(3,2,\frac{1}{6})}, \\
\hat{\Phi}_{(24,0)}^{(3,2,\frac{1}{6})}, \hat{\Phi}_{(75,0)}^{(3,2,\frac{1}{6})},
\end{aligned}$$

they are related to the fields under G_{422} and $SU(5)$ by

$$\begin{aligned}
\hat{A}_{(24,0)}^{(3,2,\frac{1}{6})} &= \hat{A}_{(6,2,2)}^{(3,2,\frac{1}{6})} = \hat{A}_{(10,4)}^{(3,2,\frac{1}{6})}, \\
\hat{E}_{(24,0)}^{(3,2,\frac{1}{6})} &= \hat{E}_{(6,2,2)}^{(3,2,\frac{1}{6})} = \hat{E}_{(15,4)}^{(3,2,\frac{1}{6})}, \\
\hat{D}_{(10,-6)}^{(3,2,\frac{1}{6})} &= \hat{D}_{(15,2,2)}^{(3,2,\frac{1}{6})} = \hat{D}_{(10,-6)}^{(3,2,\frac{1}{6})}, \\
\hat{\Delta}_{(\overline{50},-2)}^{(3,2,\frac{1}{6})} &= \hat{\Delta}_{(15,2,2)}^{(3,2,\frac{1}{6})} = \hat{\Delta}_{(15,6)}^{(3,2,\frac{1}{6})}, \\
\hat{\Delta}_{(45,-2)}^{(3,2,\frac{1}{6})} &= \hat{\Delta}_{(15,2,2)}^{(3,2,\frac{1}{6})} = \hat{\Delta}_{(15,-6)}^{(3,2,\frac{1}{6})},
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} \Phi_{(24,0)}^{(3,2,\frac{1}{6})} \\ \Phi_{(75,0)}^{(3,2,\frac{1}{6})} \end{pmatrix} &= \begin{bmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{pmatrix} \Phi_{(6,2,2)}^{(3,2,\frac{1}{6})} \\ \Phi_{(10,2,2)}^{(3,2,\frac{1}{6})} \end{pmatrix} \\
&= \begin{bmatrix} \frac{2\sqrt{2}}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2\sqrt{2}}{3} \end{bmatrix} \begin{pmatrix} \Phi_{(10,0)}^{(3,2,\frac{1}{6})} \\ \Phi_{(40,0)}^{(3,2,\frac{1}{6})} \end{pmatrix}.
\end{aligned} \quad (28)$$

Note that $\hat{D}_{(10,-6)}$ and $\hat{\Delta}_{(45,-2)}$ contain no VEV, so the Goldstone mode can be denoted as

$$\begin{aligned}
\vec{G}_{(3,2,\frac{1}{6})} &= T_{\tilde{\alpha}}^{\tilde{a}_2} \tilde{a}_2 \hat{A}_{(24,0)}^{(3,2,\frac{1}{6})} + T_{\tilde{\alpha}}^{\tilde{S}} \tilde{S} \hat{E}_{(24,0)}^{(3,2,\frac{1}{6})} + T_{\tilde{\alpha}}^{\delta} \delta \hat{\Delta}_{(\overline{50},-2)}^{(3,2,\frac{1}{6})} \\
&\quad + T_{\tilde{\alpha}}^{\bar{\delta}} \bar{\delta} \hat{\Delta}_{(45,-2)}^{(3,2,\frac{1}{6})} + T_{\tilde{\alpha}}^{\tilde{\phi}_2} \tilde{\phi}_2 \hat{\Phi}_{(24,0)}^{(3,2,\frac{1}{6})} + T_{\tilde{\alpha}}^{\tilde{\phi}_3} \tilde{\phi}_3 \hat{\Phi}_{(75,0)}^{(3,2,\frac{1}{6})}.
\end{aligned}$$

The needed CGCs can be transformed from those under G_{422} through (25)–(28), and are summarized in Tables 7 and 8. For completeness, the full mass matrix for $[(3, 2, \frac{1}{6}) + \text{c.c.}]$ can be also transformed through (25)–(28) and is given in the Appendix, for the readers to check the results of this section.

(1) According to (10), the coupling $\tilde{\phi}_2^3$ gives

$$0 = C_{111}^{\tilde{\phi}_2 \tilde{\phi}_2 \tilde{\phi}_2} - 2C_{\tilde{\alpha} \alpha 1}^{\tilde{\phi}_2 \tilde{\phi}_2 \tilde{\phi}_2} = \left(-\frac{7}{9\sqrt{15}}\right) - 2 * \left(-\frac{7}{18\sqrt{15}}\right),$$

$\tilde{\phi}_3^3$ gives

$$0 = C_{111}^{\tilde{\phi}_3 \tilde{\phi}_3 \tilde{\phi}_3} - 2C_{\tilde{\alpha} \alpha 1}^{\tilde{\phi}_3 \tilde{\phi}_3 \tilde{\phi}_3} = \frac{8}{9\sqrt{3}} - 2 * \frac{4}{9\sqrt{3}},$$

and $\tilde{S}^3$ gives

$$0 = C_{111}^{\tilde{S} \tilde{S} \tilde{S}} - 2C_{\tilde{\alpha} \alpha 1}^{\tilde{S} \tilde{S} \tilde{S}} = \sqrt{\frac{3}{5}} - 2 * \frac{\sqrt{3}}{2\sqrt{5}}.$$

(2) $\tilde{\phi}_1, \tilde{a}_1$ contain neither Goldstones, the couplings $\tilde{\phi}_2^2 \tilde{\phi}_1 \tilde{\phi}_3^2 \tilde{\phi}_1 \tilde{a}_2^2 \tilde{\phi}_1 \tilde{\phi}_2^2 \tilde{a}_1 \tilde{\phi}_3^2 \tilde{a}_1$ lead to only trivial results. The coupling $\delta \tilde{\delta} \tilde{\phi}_2$, according to (8), gives

$$\frac{T_{\tilde{\alpha}}^{\tilde{\phi}_2}}{T_{\tilde{\alpha}}^{\delta}} = \frac{C_{\tilde{\alpha} \alpha 1}^{\delta \tilde{\phi}_2 \tilde{\delta}}}{C_{111}^{\delta \tilde{\phi}_2 \tilde{\delta}}} = \left(-\frac{1}{30}\right) / \left(-\frac{1}{5\sqrt{15}}\right) = \frac{\sqrt{15}}{6},$$

$\delta \tilde{\delta} \tilde{\phi}_3$ gives

$$\frac{T_{\tilde{\alpha}}^{\tilde{\phi}_3}}{T_{\tilde{\alpha}}^{\delta}} = \frac{C_{\tilde{\alpha} \alpha 1}^{\delta \tilde{\phi}_3 \tilde{\delta}}}{C_{111}^{\delta \tilde{\phi}_3 \tilde{\delta}}} = \left(-\frac{\sqrt{2}}{15}\right) / \left(\frac{1}{5\sqrt{3}}\right) = -\frac{\sqrt{6}}{3},$$

$\delta \tilde{\delta} \tilde{a}_2$ gives

$$\frac{T_{\tilde{\alpha}}^{\tilde{a}_2}}{T_{\tilde{\alpha}}^{\delta}} = \frac{C_{\tilde{\alpha} \alpha 1}^{\delta \tilde{a}_2 \tilde{\delta}}}{C_{111}^{\delta \tilde{a}_2 \tilde{\delta}}} = \left(-\frac{1}{5}\right) / \left(-\frac{2\sqrt{3}}{5\sqrt{5}}\right) = \frac{\sqrt{15}}{6},$$

Table 7 CGCs for the SM singlets (1, 1, 0) through G_{51} . Here, for example, $\tilde{\phi}_3$ stands for $\hat{\Phi}_{(75,0)}^{(1,1,0)}$

X	Y	$XY\tilde{a}_1$	$XY\tilde{a}_2$	$XY\tilde{S}$	$XY\tilde{\phi}_1$	$XY\tilde{\phi}_2$	$XY\tilde{\phi}_3$
$\hat{A}_{(1,0)}^{(1,1,0)}$	$\hat{A}_{(1,0)}^{(1,1,0)}$	0	0	0	$\frac{4}{\sqrt{15}}$	0	0
$\hat{A}_{(24,0)}^{(1,1,0)}$	$\hat{A}_{(24,0)}^{(1,1,0)}$	0	0	$\frac{1}{\sqrt{15}}$	$-\frac{1}{\sqrt{15}}$	$-\frac{2}{3\sqrt{15}}$	$\frac{5}{3\sqrt{3}}$
$\hat{A}_{(1,0)}^{(1,1,0)}$	$\hat{A}_{(24,0)}^{(1,1,0)}$	0	0	$\sqrt{\frac{2}{5}}$	0	$\sqrt{\frac{2}{5}}$	0
$\hat{E}_{(24,0)}^{(1,1,0)}$	$\hat{E}_{(24,0)}^{(1,1,0)}$	0	0	$\sqrt{\frac{3}{5}}$	0	0	0
$\hat{\Delta}_{(50,-2)}^{(1,1,0)}$	$\hat{\Delta}_{(50,2)}^{(1,1,0)}$	$\frac{i}{5\sqrt{10}}$	$-\frac{i2\sqrt{3}}{5\sqrt{5}}$	0	$-\frac{1}{10\sqrt{15}}$	$-\frac{1}{5\sqrt{15}}$	$\frac{1}{5\sqrt{3}}$
$\hat{\Phi}_{(1,0)}^{(1,1,0)}$	$\hat{\Phi}_{(1,0)}^{(1,1,0)}$	$-\frac{6\sqrt{2}}{5\sqrt{5}}$	0	0	$\sqrt{\frac{3}{5}}$	0	0
$\hat{\Phi}_{(1,0)}^{(1,1,0)}$	$\hat{\Phi}_{(24,0)}^{(1,1,0)}$	0	$\frac{2\sqrt{3}}{5\sqrt{5}}$	$\frac{\sqrt{3}}{2\sqrt{5}}$	0	$\frac{1}{2\sqrt{15}}$	0
$\hat{\Phi}_{(1,0)}^{(1,1,0)}$	$\hat{\Phi}_{(75,0)}^{(1,1,0)}$	0	0	0	0	0	$-\frac{1}{\sqrt{15}}$
$\hat{\Phi}_{(24,0)}^{(1,1,0)}$	$\hat{\Phi}_{(24,0)}^{(1,1,0)}$	$\frac{4\sqrt{2}}{5\sqrt{5}}$	$-\frac{8}{15\sqrt{15}}$	$\frac{1}{6\sqrt{15}}$	$\frac{1}{2\sqrt{15}}$	$-\frac{7}{9\sqrt{15}}$	$\frac{5}{18\sqrt{3}}$
$\hat{\Phi}_{(24,0)}^{(1,1,0)}$	$\hat{\Phi}_{(75,0)}^{(1,1,0)}$	0	$-\frac{2}{3\sqrt{3}}$	$\frac{5}{6\sqrt{3}}$	0	$\frac{5}{18\sqrt{3}}$	$-\frac{8}{9\sqrt{3}}$
$\hat{\Phi}_{(75,0)}^{(1,1,0)}$	$\hat{\Phi}_{(75,0)}^{(1,1,0)}$	$-\frac{2\sqrt{2}}{5\sqrt{5}}$	$\frac{32}{15\sqrt{15}}$	$\frac{4}{3\sqrt{15}}$	$-\frac{1}{15}$	$-\frac{8}{9\sqrt{15}}$	$\frac{8}{9\sqrt{3}}$

Table 8 CGCs for the Goldstone $(3, 2, \frac{1}{6}) + \text{c.c.}$ of G_{51} SSB

X	Y	$XY\tilde{a}_1$	$XY\tilde{a}_2$	$XY\tilde{S}$	$XY\tilde{\delta}$	$XY\tilde{\phi}_1$	$XY\tilde{\phi}_2$	$XY\tilde{\phi}_3$
$\hat{A}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\hat{A}_{(24,0)}^{(3,2,-\frac{1}{6})}$	0	0	$\frac{1}{2\sqrt{15}}$	0	0	$-\frac{1}{\sqrt{15}}$	$-\frac{1}{3\sqrt{15}}$
$\hat{A}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\hat{E}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\sqrt{\frac{2}{5}}$	$\frac{1}{2\sqrt{15}}$	0	0	0	0	0
$\hat{A}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\hat{\Delta}_{(50,2)}^{(3,2,-\frac{1}{6})}$	0	0	0	$-\frac{1}{5}$	0	0	0
$\hat{A}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\hat{\Phi}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\sqrt{\frac{2}{5}}$	$-\frac{1}{3\sqrt{15}}$	0	0	$\frac{2\sqrt{3}}{5\sqrt{5}}$	$-\frac{4}{15\sqrt{15}}$	$-\frac{2}{15\sqrt{3}}$
$\hat{A}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\hat{\Phi}_{(75,0)}^{(3,2,-\frac{1}{6})}$	0	$-\frac{\sqrt{10}}{3\sqrt{3}}$	0	0	0	$\frac{2\sqrt{2}}{3\sqrt{15}}$	$-\frac{4\sqrt{2}}{15\sqrt{3}}$
$\hat{E}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\hat{E}_{(24,0)}^{(3,2,-\frac{1}{6})}$	0	0	$\frac{\sqrt{3}}{2\sqrt{5}}$	0	0	0	0
$\hat{E}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\hat{\Phi}_{(24,0)}^{(3,2,-\frac{1}{6})}$	0	0	0	0	$\frac{\sqrt{3}}{2\sqrt{5}}$	$\frac{1}{12\sqrt{15}}$	$\frac{1}{6\sqrt{3}}$
$\hat{E}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\hat{\Phi}_{(75,0)}^{(3,2,-\frac{1}{6})}$	0	0	0	0	0	$-\frac{\sqrt{5}}{3\sqrt{6}}$	$-\frac{1}{3\sqrt{6}}$
$\hat{\Delta}_{(50,-2)}^{(3,2,-\frac{1}{6})}$	$\hat{\Delta}_{(50,2)}^{(3,2,-\frac{1}{6})}$	$\frac{i}{5\sqrt{10}}$	$-\frac{i7}{10\sqrt{15}}$	0	0	0	$-\frac{1}{10\sqrt{15}}$	$-\frac{7}{60\sqrt{15}}$
$\hat{\Delta}_{(50,-2)}^{(3,2,-\frac{1}{6})}$	$\hat{A}_{(24,0)}^{(3,2,-\frac{1}{6})}$	0	0	0	0	$-\frac{1}{5}$	0	0
$\hat{\Delta}_{(50,-2)}^{(3,2,-\frac{1}{6})}$	$\hat{\Phi}_{(24,0)}^{(3,2,-\frac{1}{6})}$	0	0	0	0	$-\frac{1}{30}$	0	0
$\hat{\Delta}_{(50,-2)}^{(3,2,-\frac{1}{6})}$	$\hat{\Phi}_{(75,0)}^{(3,2,-\frac{1}{6})}$	0	0	0	0	$-\frac{\sqrt{2}}{15}$	0	0
$\hat{\Phi}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\hat{\Phi}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\frac{4\sqrt{2}}{5\sqrt{5}}$	$-\frac{4}{15\sqrt{15}}$	$\frac{1}{12\sqrt{15}}$	0	$\frac{1}{2\sqrt{15}}$	$-\frac{7}{18\sqrt{15}}$	$\frac{1}{18\sqrt{3}}$
$\hat{\Phi}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\hat{\Phi}_{(75,0)}^{(3,2,-\frac{1}{6})}$	0	$\frac{2\sqrt{2}}{3\sqrt{15}}$	$-\frac{\sqrt{5}}{3\sqrt{6}}$	0	0	$-\frac{\sqrt{5}}{9\sqrt{6}}$	$\frac{\sqrt{2}}{9\sqrt{3}}$
$\hat{\Phi}_{(75,0)}^{(3,2,-\frac{1}{6})}$	$\hat{\Phi}_{(75,0)}^{(3,2,-\frac{1}{6})}$	$-\frac{2\sqrt{2}}{5\sqrt{5}}$	$\frac{22}{15\sqrt{15}}$	$\frac{11}{12\sqrt{15}}$	0	$-\frac{1}{\sqrt{15}}$	$-\frac{11}{18\sqrt{15}}$	$\frac{4}{9\sqrt{3}}$
$\hat{\Phi}_{(24,0)}^{(3,2,-\frac{1}{6})}$	$\hat{\Delta}_{(50,2)}^{(3,2,-\frac{1}{6})}$	0	0	0	$-\frac{1}{30}$	0	0	0
$\hat{\Phi}_{(75,0)}^{(3,2,-\frac{1}{6})}$	$\hat{\Delta}_{(50,2)}^{(3,2,-\frac{1}{6})}$	0	0	0	$-\frac{\sqrt{2}}{15}$	0	0	0

$\tilde{a}_2\tilde{\phi}_2\tilde{a}_1$ gives

$$\frac{T_{\tilde{\alpha}}^{\tilde{a}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_2}} = \frac{C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{\phi}_2\tilde{a}_1}}{C_{111}^{\tilde{a}_2\tilde{\phi}_2\tilde{a}_1}} = \left(\sqrt{\frac{2}{5}}\right) / \left(\sqrt{\frac{2}{5}}\right) = 1,$$

$\tilde{a}_2\tilde{\phi}_2\tilde{\phi}_1$ gives

$$\frac{T_{\tilde{\alpha}}^{\tilde{a}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_2}} = \frac{C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{\phi}_2\tilde{\phi}_1}}{C_{111}^{\tilde{a}_2\tilde{\phi}_2\tilde{\phi}_1}} = \left(\frac{2\sqrt{3}}{5\sqrt{5}}\right) / \left(\frac{2\sqrt{3}}{5\sqrt{5}}\right) = 1,$$

$\tilde{S}\tilde{a}_2\tilde{a}_1$ gives

$$\frac{T_{\tilde{\alpha}}^{\tilde{S}}}{T_{\tilde{\alpha}}^{\tilde{a}_2}} = \frac{C_{\tilde{\alpha}\alpha 1}^{\tilde{S}\tilde{a}_2\tilde{a}_1}}{C_{111}^{\tilde{S}\tilde{a}_2\tilde{a}_1}} = \left(\sqrt{\frac{2}{5}}\right) / \left(\sqrt{\frac{2}{5}}\right) = 1,$$

and $\tilde{S}\tilde{\phi}_2\tilde{\phi}_1$ gives

$$\frac{T_{\tilde{\alpha}}^{\tilde{S}}}{T_{\tilde{\alpha}}^{\tilde{\phi}_2}} = \frac{C_{\tilde{\alpha}\alpha 1}^{\tilde{S}\tilde{\phi}_2\tilde{\phi}_1}}{C_{111}^{\tilde{S}\tilde{\phi}_2\tilde{\phi}_1}} = \left(\frac{\sqrt{3}}{2\sqrt{5}}\right) / \left(\frac{\sqrt{3}}{2\sqrt{5}}\right) = 1.$$

(3) According to (9), $\tilde{\phi}_2\tilde{\phi}_3^2$ gives

$$\begin{aligned} \frac{T_{\tilde{\alpha}}^{\tilde{\phi}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_3}} &= \frac{2C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{\phi}_3\tilde{\phi}_3}}{C_{111}^{\tilde{\phi}_2\tilde{\phi}_3\tilde{\phi}_3}} \\ &= 2 * \left(\frac{\sqrt{2}}{9\sqrt{3}}\right) / \left(-\frac{8}{9\sqrt{15}}\right) = -\sqrt{\frac{5}{8}}, \\ \text{or: } \frac{T_{\tilde{\alpha}}^{\tilde{\phi}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_3}} &= \frac{C_{111}^{\tilde{\phi}_2\tilde{\phi}_3\tilde{\phi}_2} - C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{\phi}_3\tilde{\phi}_2}}{C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{\phi}_3\tilde{\phi}_3}} \\ &= \left(\frac{-8}{9\sqrt{15}} - \frac{-11}{18\sqrt{15}}\right) / \left(\frac{\sqrt{2}}{9\sqrt{3}}\right) = -\sqrt{\frac{5}{8}}, \end{aligned}$$

$\tilde{\phi}_2^2\tilde{\phi}_3$ gives

$$\begin{aligned} \frac{T_{\tilde{\alpha}}^{\tilde{\phi}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_3}} &= \frac{C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{\phi}_3\tilde{\phi}_2}}{C_{111}^{\tilde{\phi}_2\tilde{\phi}_3\tilde{\phi}_3} - C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{\phi}_3\tilde{\phi}_3}} \\ &= \left(-\frac{\sqrt{5}}{9\sqrt{6}}\right) / \left(\frac{5}{18\sqrt{3}} - \frac{1}{18\sqrt{3}}\right) = -\sqrt{\frac{5}{8}}, \\ \text{or: } \frac{T_{\tilde{\alpha}}^{\tilde{\phi}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_3}} &= \frac{C_{111}^{\tilde{\phi}_3\tilde{\phi}_2\tilde{\phi}_2}}{2C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_3\tilde{\phi}_2\tilde{\phi}_2}} \\ &= \left(\frac{5}{18\sqrt{3}}\right) / 2 * \left(-\frac{\sqrt{5}}{9\sqrt{6}}\right) = -\sqrt{\frac{5}{8}}, \end{aligned}$$

$\tilde{a}_2^2\tilde{\phi}_2$ gives

$$\begin{aligned} \frac{T_{\tilde{\alpha}}^{\tilde{a}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_2}} &= \frac{C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{\phi}_2\tilde{a}_2}}{C_{111}^{\tilde{a}_2\tilde{\phi}_2\tilde{a}_2} - C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{\phi}_2\tilde{a}_2}} \\ &= \left(-\frac{1}{3\sqrt{15}}\right) / \left(\frac{-2}{3\sqrt{15}} - \frac{-1}{3\sqrt{15}}\right) = 1, \end{aligned}$$

$$\begin{aligned} \text{or: } \frac{T_{\tilde{\alpha}}^{\tilde{a}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_2}} &= \frac{C_{111}^{\tilde{\phi}_2\tilde{a}_2\tilde{a}_2}}{2C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{a}_2\tilde{a}_2}} \\ &= \left(-\frac{2}{3\sqrt{15}}\right) / 2 * \left(-\frac{1}{3\sqrt{15}}\right) = 1, \end{aligned}$$

$\tilde{a}_2^2\tilde{\phi}_3$ gives

$$\begin{aligned} \frac{T_{\tilde{\alpha}}^{\tilde{a}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_3}} &= \frac{C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{\phi}_3\tilde{a}_2}}{C_{111}^{\tilde{a}_2\tilde{\phi}_3\tilde{a}_2} - C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{\phi}_3\tilde{a}_2}} \\ &= \left(-\frac{\sqrt{10}}{3\sqrt{3}}\right) / \left(\frac{5}{3\sqrt{3}} - \frac{1}{3\sqrt{3}}\right) = -\sqrt{\frac{5}{8}}, \\ \text{or: } \frac{T_{\tilde{\alpha}}^{\tilde{a}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_3}} &= \frac{C_{111}^{\tilde{\phi}_3\tilde{a}_2\tilde{a}_2}}{2C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_3\tilde{a}_2\tilde{a}_2}} \\ &= \left(\frac{5}{3\sqrt{3}}\right) / 2 * \left(-\frac{\sqrt{10}}{3\sqrt{3}}\right) = -\sqrt{\frac{5}{8}}, \end{aligned}$$

$\tilde{a}_2\tilde{\phi}_2^2$ gives

$$\begin{aligned} \frac{T_{\tilde{\alpha}}^{\tilde{a}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_2}} &= \frac{2C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{\phi}_2\tilde{\phi}_2}}{C_{111}^{\tilde{a}_2\tilde{\phi}_2\tilde{\phi}_2}} \\ &= 2 * \left(-\frac{4}{15\sqrt{15}}\right) / \left(-\frac{8}{15\sqrt{15}}\right) = 1, \\ \text{or: } \frac{T_{\tilde{\alpha}}^{\tilde{a}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_2}} &= \frac{C_{111}^{\tilde{\phi}_2\tilde{\phi}_2\tilde{a}_2} - C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{\phi}_2\tilde{a}_2}}{C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{\phi}_2\tilde{\phi}_2}} \\ &= \left(-\frac{8}{15\sqrt{15}} + \frac{4}{15\sqrt{15}}\right) / \left(-\frac{4}{15\sqrt{15}}\right) = 1, \end{aligned}$$

$\tilde{a}_2\tilde{\phi}_3^2$ gives

$$\begin{aligned} \frac{T_{\tilde{\alpha}}^{\tilde{a}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_3}} &= \frac{2C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{\phi}_3\tilde{\phi}_3}}{C_{111}^{\tilde{a}_2\tilde{\phi}_3\tilde{\phi}_3}} \\ &= 2 * \left(-\frac{4\sqrt{2}}{15\sqrt{3}}\right) / \left(\frac{32}{15\sqrt{15}}\right) = -\sqrt{\frac{5}{8}}, \\ \text{or: } \frac{T_{\tilde{\alpha}}^{\tilde{a}_2}}{T_{\tilde{\alpha}}^{\tilde{\phi}_3}} &= \frac{C_{111}^{\tilde{\phi}_3\tilde{\phi}_3\tilde{a}_2} - C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_3\tilde{\phi}_3\tilde{a}_2}}{C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_3\tilde{\phi}_3\tilde{\phi}_3}} \\ &= \left(\frac{32}{15\sqrt{15}} - \frac{22}{15\sqrt{15}}\right) / \left(-\frac{4\sqrt{2}}{15\sqrt{3}}\right) = -\sqrt{\frac{5}{8}}, \end{aligned}$$

$\tilde{S}\tilde{a}_2^2$ gives

$$\begin{aligned} \frac{T_{\tilde{\alpha}}^{\tilde{S}}}{T_{\tilde{\alpha}}^{\tilde{a}_2}} &= \frac{2C_{\tilde{\alpha}\alpha 1}^{\tilde{S}\tilde{a}_2\tilde{a}_2}}{C_{111}^{\tilde{S}\tilde{a}_2\tilde{a}_2}} \\ &= 2 * \left(\frac{1}{2\sqrt{15}}\right) / \left(\frac{1}{\sqrt{15}}\right) = 1, \end{aligned}$$

$$\text{or: } \frac{T_{\tilde{\alpha}}^{\tilde{S}}}{T_{\tilde{\alpha}}^{\tilde{a}_2}} = \frac{C_{111}^{\tilde{a}_2\tilde{a}_2\tilde{S}} - C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{a}_2\tilde{S}}}{C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{S}\tilde{a}_2}} \\ = \left(\frac{1}{\sqrt{15}} - \frac{1}{2\sqrt{15}} \right) / \left(\frac{1}{2\sqrt{15}} \right) = 1,$$

$\tilde{S}\tilde{\phi}_2^2$ gives

$$\frac{T_{\tilde{\alpha}}^{\tilde{S}}}{T_{\tilde{\alpha}}^{\tilde{\phi}_2}} = \frac{2C_{\tilde{\alpha}\alpha 1}^{\tilde{S}\tilde{\phi}_2\tilde{\phi}_2}}{C_{111}^{\tilde{S}\tilde{\phi}_2\tilde{\phi}_2}} \\ = 2 * \left(\frac{1}{12\sqrt{15}} \right) / \left(\frac{1}{6\sqrt{15}} \right) = 1,$$

$$\text{or: } \frac{T_{\tilde{\alpha}}^{\tilde{S}}}{T_{\tilde{\alpha}}^{\tilde{\phi}_2}} = \frac{C_{111}^{\tilde{\phi}_2\tilde{\phi}_2\tilde{S}} - C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{\phi}_2\tilde{S}}}{C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{S}\tilde{\phi}_2}} \\ = \left(\frac{1}{6\sqrt{15}} - \frac{1}{12\sqrt{15}} \right) / \left(\frac{1}{12\sqrt{15}} \right) = 1,$$

$\tilde{S}\tilde{\phi}_3^2$ gives

$$\frac{T_{\tilde{\alpha}}^{\tilde{S}}}{T_{\tilde{\alpha}}^{\tilde{\phi}_3}} = \frac{2C_{\tilde{\alpha}\alpha 1}^{\tilde{S}\tilde{\phi}_3\tilde{\phi}_3}}{C_{111}^{\tilde{S}\tilde{\phi}_3\tilde{\phi}_3}} \\ = 2 * \left(-\frac{1}{3\sqrt{6}} \right) / \left(\frac{4}{3\sqrt{15}} \right) = -\sqrt{\frac{5}{8}},$$

$$\text{or: } \frac{T_{\tilde{\alpha}}^{\tilde{S}}}{T_{\tilde{\alpha}}^{\tilde{\phi}_3}} = \frac{C_{111}^{\tilde{\phi}_3\tilde{\phi}_3\tilde{S}} - C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_3\tilde{\phi}_3\tilde{S}}}{C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_3\tilde{S}\tilde{\phi}_3}} \\ = \left(\frac{4}{3\sqrt{15}} - \frac{11}{12\sqrt{15}} \right) / \left(-\frac{1}{3\sqrt{6}} \right) = -\sqrt{\frac{5}{8}}.$$

(4) According to (7), $\tilde{a}_2\tilde{\phi}_2\tilde{\phi}_3$ gives

$$0 = \begin{pmatrix} -C_{111}^{\tilde{a}_2\tilde{\phi}_2\tilde{\phi}_3} & C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{\phi}_2\tilde{\phi}_3} & C_{\tilde{\alpha}\alpha 1}^{\tilde{a}_2\tilde{\phi}_3\tilde{\phi}_2} \\ C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{a}_2\tilde{\phi}_3} & -C_{111}^{\tilde{\phi}_2\tilde{a}_2\tilde{\phi}_3} & C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{\phi}_3\tilde{a}_2} \\ C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_3\tilde{a}_2\tilde{\phi}_2} & C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_3\tilde{\phi}_2\tilde{a}_2} & -C_{111}^{\tilde{\phi}_3\tilde{a}_2\tilde{\phi}_2} \end{pmatrix} \begin{pmatrix} T_{\tilde{\alpha}}^{\tilde{a}_2} \\ T_{\tilde{\alpha}}^{\tilde{\phi}_2} \\ T_{\tilde{\alpha}}^{\tilde{\phi}_3} \end{pmatrix} \\ = \begin{pmatrix} \frac{2}{3\sqrt{3}} & -\frac{2}{15\sqrt{3}} & \frac{2\sqrt{2}}{3\sqrt{15}} \\ -\frac{2}{15\sqrt{3}} & \frac{2}{3\sqrt{3}} & \frac{2\sqrt{2}}{3\sqrt{15}} \\ \frac{2\sqrt{2}}{3\sqrt{15}} & \frac{2\sqrt{2}}{3\sqrt{15}} & \frac{2}{3\sqrt{3}} \end{pmatrix} \begin{pmatrix} T_{\tilde{\alpha}}^{\tilde{a}_2} \\ T_{\tilde{\alpha}}^{\tilde{\phi}_2} \\ T_{\tilde{\alpha}}^{\tilde{\phi}_3} \end{pmatrix},$$

so that

$$T_{\tilde{\alpha}}^{\tilde{a}_2}:T_{\tilde{\alpha}}^{\tilde{\phi}_2}:T_{\tilde{\alpha}}^{\tilde{\phi}_3} = 1:1:\left(-\sqrt{\frac{8}{5}}\right),$$

and $\tilde{S}\tilde{\phi}_2\tilde{\phi}_3$ gives

$$0 = \begin{pmatrix} -C_{111}^{\tilde{S}\tilde{\phi}_2\tilde{\phi}_3} & C_{\tilde{\alpha}\alpha 1}^{\tilde{S}\tilde{\phi}_2\tilde{\phi}_3} & C_{\tilde{\alpha}\alpha 1}^{\tilde{S}\tilde{\phi}_3\tilde{\phi}_2} \\ C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{S}\tilde{\phi}_3} & -C_{111}^{\tilde{\phi}_2\tilde{S}\tilde{\phi}_3} & C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_2\tilde{\phi}_3\tilde{S}} \\ C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_3\tilde{S}\tilde{\phi}_2} & C_{\tilde{\alpha}\alpha 1}^{\tilde{\phi}_3\tilde{\phi}_2\tilde{S}} & -C_{111}^{\tilde{\phi}_3\tilde{S}\tilde{\phi}_2} \end{pmatrix} \begin{pmatrix} T_{\tilde{\alpha}}^{\tilde{S}} \\ T_{\tilde{\alpha}}^{\tilde{\phi}_2} \\ T_{\tilde{\alpha}}^{\tilde{\phi}_3} \end{pmatrix} \\ = \begin{pmatrix} -\frac{5}{6\sqrt{3}} & \frac{1}{6\sqrt{3}} & -\frac{\sqrt{5}}{3\sqrt{6}} \\ \frac{1}{6\sqrt{3}} & -\frac{5}{6\sqrt{3}} & -\frac{\sqrt{5}}{3\sqrt{6}} \\ -\frac{\sqrt{5}}{3\sqrt{6}} & -\frac{\sqrt{5}}{3\sqrt{6}} & -\frac{5}{6\sqrt{3}} \end{pmatrix} \begin{pmatrix} T_{\tilde{\alpha}}^{\tilde{S}} \\ T_{\tilde{\alpha}}^{\tilde{\phi}_2} \\ T_{\tilde{\alpha}}^{\tilde{\phi}_3} \end{pmatrix},$$

so that

$$T_{\tilde{\alpha}}^{\tilde{S}}:T_{\tilde{\alpha}}^{\tilde{\phi}_2}:T_{\tilde{\alpha}}^{\tilde{\phi}_3} = 1:1:\left(-\sqrt{\frac{8}{5}}\right).$$

Altogether, the Goldstino is consistently

$$\vec{G}_{(3,2,\frac{1}{6})} = \tilde{a}_2\hat{A}_{(24,0)}^{(3,2,\frac{1}{6})} + \tilde{S}\hat{E}_{(24,0)}^{(3,2,\frac{1}{6})} + \frac{2\sqrt{15}}{5}\delta\hat{\Delta}_{(\bar{50},-2)}^{(3,2,\frac{1}{6})} \\ + \tilde{\phi}_2\hat{\Phi}_{(24,0)}^{(3,2,\frac{1}{6})} - \sqrt{\frac{8}{5}}\tilde{\phi}_3\hat{\Phi}_{(75,0)}^{(3,2,\frac{1}{6})}. \quad (29)$$

7 Summary

We have studied the Goldstone modes of SSB in the SUSY SO(10) model with general renormalizable couplings. VEVs and CGCs determine the contents of these Goldstone modes while the parameters of the model are irrelevant. Identities among the CGCs are examined to be in accord with the general conclusions in [38].

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Appendix: Mass matrix for $[(3, 2, \frac{1}{6}) + \text{c.c.}]$

For the SSB of G_{422} and $SU(5)$, the mass matrices for the Goldstone representations can be found in [29,36], respectively. Here we give the mass matrix for $[(3, 2, \frac{1}{6}) + \text{c.c.}]$

using G_{51} as the maximal subgroup of $SO(10)$. This matrix is transformed from [29] through (25)–(28). $[(3, 2, \frac{1}{6}) + c.c.]$

$$\mathbf{c}: \hat{A}_{(24,0)}^{(3,2,\frac{1}{6})}, \hat{E}_{(24,0)}^{(3,2,\frac{1}{6})}, \hat{D}_{(10,-6)}^{(3,2,\frac{1}{6})}, \hat{\Delta}_{(50,-2)}^{(3,2,\frac{1}{6})},$$

$$\hat{\Delta}_{(45,-2)}^{(3,2,\frac{1}{6})}, \hat{\Phi}_{(24,0)}^{(3,2,\frac{1}{6})}, \hat{\Phi}_{(75,0)}^{(3,2,\frac{1}{6})}$$

$$\mathbf{r}: \hat{A}_{(24,0)}^{(\bar{3},2,-\frac{1}{6})}, \hat{E}_{(24,0)}^{(\bar{3},2,-\frac{1}{6})}, \hat{D}_{(10,6)}^{(\bar{3},2,-\frac{1}{6})}, \hat{\Delta}_{(50,2)}^{(\bar{3},2,-\frac{1}{6})},$$

$$\hat{\Delta}_{(45,2)}^{(\bar{3},2,-\frac{1}{6})}, \hat{\Phi}_{(24,0)}^{(\bar{3},2,-\frac{1}{6})}, \hat{\Phi}_{(75,0)}^{(\bar{3},2,-\frac{1}{6})}$$

$$m_{66} = m_1 + \frac{4}{5}\sqrt{\frac{2}{5}}\tilde{a}_1\lambda_7 - \frac{4\tilde{a}_2\lambda_7}{15\sqrt{15}} + \frac{\delta\lambda_{10}}{12\sqrt{15}} + \frac{\lambda_1\tilde{\phi}_1}{2\sqrt{15}} - \frac{7\lambda_1\tilde{\phi}_2}{18\sqrt{15}} + \frac{\lambda_1\tilde{\phi}_3}{18\sqrt{3}},$$

$$m_{67} = \frac{2}{3}\sqrt{\frac{2}{15}}\tilde{a}_2\lambda_7 - \frac{1}{3}\sqrt{\frac{5}{6}}\tilde{S}\lambda_{10} - \frac{1}{9}\sqrt{\frac{5}{6}}\lambda_1\tilde{\phi}_2 + \frac{1}{9}\sqrt{\frac{2}{3}}\lambda_1\tilde{\phi}_3,$$

$$m_{77} = m_1 - \frac{2}{5}\sqrt{\frac{2}{5}}\tilde{a}_1\lambda_7 + \frac{22\tilde{a}_2\lambda_7}{15\sqrt{15}} + \frac{11\delta\lambda_{10}}{12\sqrt{15}} - \frac{\lambda_1\tilde{\phi}_1}{\sqrt{15}} - \frac{11\lambda_1\tilde{\phi}_2}{18\sqrt{15}} + \frac{4\lambda_1\tilde{\phi}_3}{9\sqrt{3}}.$$

$$\begin{pmatrix} m_{11} & \sqrt{\frac{2}{5}}\tilde{a}_1\lambda_9 + \frac{\tilde{a}_2\lambda_9}{2\sqrt{15}} & -\frac{i\tilde{\delta}\lambda_{19}}{\sqrt{10}} & -\frac{\tilde{\delta}\lambda_6}{5} & 0 & m_{16} & m_{17} \\ \sqrt{\frac{2}{5}}\tilde{a}_1\lambda_9 + \frac{\tilde{a}_2\lambda_9}{2\sqrt{15}} & m_5 + \frac{1}{2}\sqrt{\frac{3}{5}}\tilde{S}\lambda_8 & 0 & 0 & -\frac{2\delta\lambda_{12}}{5} & m_{26} & m_{27} \\ \frac{i\delta\lambda_{18}}{\sqrt{10}} & 0 & m_{33} & m_{34} & m_{35} & -\frac{\delta\lambda_{20}}{6\sqrt{10}} & -\frac{\delta\lambda_{20}}{3\sqrt{5}} \\ -\frac{\delta\lambda_6}{5} & 0 & m_{43} & m_{44} & \frac{\tilde{S}\lambda_{12}}{\sqrt{15}} & -\frac{\delta\lambda_2}{30} & -\frac{\sqrt{2}}{15}\delta\lambda_2 \\ 0 & -\frac{2\delta\lambda_{11}}{5} & m_{53} & \frac{\tilde{S}\lambda_{11}}{\sqrt{15}} & m_{55} & 0 & 0 \\ m_{16} & m_{26} & -\frac{\delta\lambda_{21}}{6\sqrt{10}} & -\frac{\delta\lambda_2}{30} & 0 & m_{66} & m_{67} \\ m_{17} & m_{27} & -\frac{\delta\lambda_{21}}{3\sqrt{5}} & -\frac{\sqrt{2}}{15}\delta\lambda_2 & 0 & m_{67} & m_{77} \end{pmatrix},$$

where:

$$\begin{aligned} m_{11} &= m_4 + \frac{\delta\lambda_9}{2\sqrt{15}} - \frac{\lambda_5\tilde{\phi}_1}{\sqrt{15}} - \frac{\lambda_5\tilde{\phi}_2}{3\sqrt{15}} + \frac{\lambda_5\tilde{\phi}_3}{3\sqrt{3}}, \\ m_{16} &= \sqrt{\frac{2}{5}}\tilde{a}_1\lambda_5 - \frac{\tilde{a}_2\lambda_5}{3\sqrt{15}} + \frac{2}{5}\sqrt{\frac{3}{5}}\lambda_7\tilde{\phi}_1 - \frac{4\lambda_7\tilde{\phi}_2}{15\sqrt{15}} - \frac{2\lambda_7\tilde{\phi}_3}{15\sqrt{3}}, \\ m_{17} &= -\frac{1}{3}\sqrt{\frac{10}{3}}\tilde{a}_2\lambda_5 + \frac{2}{3}\sqrt{\frac{2}{15}}\lambda_7\tilde{\phi}_2 - \frac{4}{15}\sqrt{\frac{2}{3}}\lambda_7\tilde{\phi}_3, \\ m_{26} &= \frac{1}{2}\sqrt{\frac{3}{5}}\lambda_{10}\tilde{\phi}_1 + \frac{\lambda_{10}\tilde{\phi}_2}{12\sqrt{15}} + \frac{\lambda_{10}\tilde{\phi}_3}{6\sqrt{3}}, \\ m_{27} &= -\frac{1}{3}\sqrt{\frac{5}{6}}\lambda_{10}\tilde{\phi}_2 - \frac{\lambda_{10}\tilde{\phi}_3}{3\sqrt{6}}, \\ m_{33} &= m_6 - \frac{\delta\lambda_{14}}{3\sqrt{15}} - \frac{\lambda_{15}\tilde{\phi}_1}{3\sqrt{15}} - \frac{2\lambda_{15}\tilde{\phi}_2}{3\sqrt{15}} + \frac{\lambda_{15}\tilde{\phi}_3}{3\sqrt{3}}, \\ m_{34} &= -\frac{i\tilde{a}_2\lambda_{18}}{2\sqrt{6}} + \frac{\lambda_{20}\tilde{\phi}_2}{12\sqrt{6}} - \frac{1}{3}\sqrt{\frac{2}{15}}\lambda_{20}\tilde{\phi}_3, \\ m_{35} &= -\frac{1}{5}i\tilde{a}_1\lambda_{19} - \frac{i\tilde{a}_2\lambda_{19}}{10\sqrt{6}} + \frac{\lambda_{21}\tilde{\phi}_1}{10\sqrt{6}} - \frac{13\lambda_{21}\tilde{\phi}_2}{60\sqrt{6}} - \frac{\lambda_{21}\tilde{\phi}_3}{6\sqrt{30}}, \\ m_{43} &= \frac{i\tilde{a}_2\lambda_{19}}{2\sqrt{6}} + \frac{\lambda_{21}\tilde{\phi}_2}{12\sqrt{6}} - \frac{1}{3}\sqrt{\frac{2}{15}}\lambda_{21}\tilde{\phi}_3, \\ m_{44} &= m_2 + \frac{\tilde{a}_1\lambda_6}{5\sqrt{10}} - \frac{7\tilde{a}_2\lambda_6}{10\sqrt{15}} - \frac{\lambda_2\tilde{\phi}_1}{10\sqrt{15}} - \frac{7\lambda_2\tilde{\phi}_2}{60\sqrt{15}} + \frac{\lambda_2\tilde{\phi}_3}{15\sqrt{3}}, \\ m_{53} &= \frac{1}{5}i\tilde{a}_1\lambda_{18} + \frac{i\tilde{a}_2\lambda_{18}}{10\sqrt{6}} + \frac{\lambda_{20}\tilde{\phi}_1}{10\sqrt{6}} - \frac{13\lambda_{20}\tilde{\phi}_2}{60\sqrt{6}} - \frac{\lambda_{20}\tilde{\phi}_3}{6\sqrt{30}}, \\ m_{55} &= m_2 - \frac{\tilde{a}_1\lambda_6}{5\sqrt{10}} + \frac{7\tilde{a}_2\lambda_6}{10\sqrt{15}} - \frac{\lambda_2\tilde{\phi}_2}{12\sqrt{15}} + \frac{\lambda_2\tilde{\phi}_3}{30\sqrt{3}}, \end{aligned}$$

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