General expressions for extra-dimensional tree amplitudes and all-plus 1-loop integrands in $\mathcal{Q}$ -cut representation

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Abstract In this paper, we give the general expressions for a special series of tree amplitudes of the Yang–Mills theory. This series of amplitudes have two adjacent massless spin-1 particles with extra-dimensional momenta and any number of positive helicity gluons. With special helicity choices, we use the spinor helicity formalism to express these n -point amplitudes in compact forms, and find a clever way to use the BCFW recursion relations to prove the results. Then these amplitudes are used to form the complete 1-loop all-plus integrand with any number of gluons, expressed in the $\mathcal{Q}$ -cut representation.

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1 Introduction

Recently, the $\mathcal{Q}$ -cut construction [1,2] has been developed to compute complete loop integrands of massless field theory. When using a $\mathcal{Q}$ -cut representation to calculate loop integrands, a special series of on-shell tree amplitudes in the general case of D dimensions are required. These tree amplitudes have two adjacent legs with extra-dimensional momenta and other legs with four-dimensional momenta.

For all-plus 1-loop gluon amplitudes in Yang–Mills theory, the helicity of four-dimensional particles (gluons) of the required tree amplitudes are all plus, which is just similar to the MHV amplitudes. Our motivation for this paper is to see if we can generalize these tree amplitudes to include any number of positive helicity gluons and find compact expressions for them. With these tree amplitudes we can calculate general expressions for the integrands of n -point 1-loop all-plus amplitudes in $\mathcal{Q}$ -cut representation.

However, it is difficult to do this work by directly calculating a tremendous number of Feynman diagrams involving n gluons. Fortunately, in calculations of multi-particle amplitudes great progress has been made in the last ten years, starting from the twistor string description of $\mathcal{N} = 4$ Yang–Mills proposed by Witten [3]. New ideas have led to the development of new powerful formalisms. The MHV vertex expansion (CSW)[4] and the BCFW recursion relations [5,6] are two most important ones. Recently, the CHY formula [7,8] was developed, which can be used to calculate scattering amplitudes in arbitrary dimension.

In this paper, to deal with our extra-dimensional cases, we focus on the BCFW recursion relations, which is an on-shell recursive method developed by Britto, Cachazo, Feng, and Witten, where an higher-point tree amplitude can be given by the sum of products of a propagator and two on-shell lower-point amplitudes with shifted, complex momenta. Later, through the work of Badger et al. [9], the BCFW recur-

sion relations have been generalized to include massive particles with spin.

The BCFW recursion relations for massive particles are also applicable in our cases because each helicity state of a massless D -dimensional spin-1 particle is equivalent to a massive particle state. A massless spin-1 particle with D -dimensional momentum has $D - 2$ polarization degrees of freedom instead of 2 for a four-dimensional massless spin-1 particle. So we need to choose more helicity states and this series of extra-dimensional amplitudes contains more kinds of amplitudes. After choosing totally $D - 2$ physical polarization vectors labeled by helicity $+$, $-$ and S_a ($a \in \{4, 5, \dots, D-1\}$), we can follow the similar treatments using the BCFW recursion relations in amplitudes with massive legs and apply the generalized spinor helicity formalism for massive amplitudes to express these tree amplitudes involving 2 adjacent extra-dimensional legs. Our results for this series of different amplitudes are concise and astonishingly have a common structure.

In our cases, the two D -dimensional spin-1 particles are equivalent to two massive particles with equal mass and four-dimensional momenta. We follow the generalization of the spinor helicity formalism for massive particles to express our results in helicity amplitudes. By introducing a null reference vector q_i , a massive momentum p_i can be decomposed along two lightlike directions. In this paper we set all $q_i = q$, so $p_i = p_i^\perp - \frac{m_i^2}{2q \cdot p_i} q$. The amplitudes are then q -dependent. For an introduction to the approaches to generalizing the massless spinor helicity formalism, see [10]. In [11–13], applications of BCFW recursion relations with massive particles can be found. Compact expressions for several towers of tree-level amplitudes with a complex scalar–antiscalar pair or a massive W -boson pair and any numbers of positive helicity gluons are given in [14–16]. Besides, in [15], superamplitudes on the Coulomb branch of $N = 4$ SYM are calculated from massive amplitudes, which implies the massive version of our extra-dimensional amplitudes can be used to study superamplitudes on the Coulomb branch.

To compare with treatments involving massive particles, we review a recent work, a four-dimensional formulation (FDF) of the extra-dimensional regularization of 1-loop scattering amplitudes [17].

This paper is organized as follows. In Sect. 2, we set up conventions, introduce polarization vectors with helicity choices $+$, $-S_a$, compare with the formulation in [17] and give compact expressions for these tree amplitudes. In Sect. 3, a proof by mathematical induction of the results using the BCFW recursion relations is given. Finally, in Sect. 4, we use the expressions to give general expressions for the compact integrands of all-plus 1-loop amplitudes, written in Q -cut representation.

2 Conventions and main results

In this section, we will first introduce our conventions and helicity choices, compared with four-dimensional treatment as massive particles. Then, to set our starting point, we give expressions for related 3-point tree amplitudes and 4-point tree amplitudes. Finally, we summarize our main results for n -point tree amplitudes and analyze their structures.

We only need to calculate some of amplitudes, since color-ordered amplitudes have the cyclic property, $\mathcal{A}_n(1, 2, \dots, n) = \mathcal{A}_n(2, \dots, n, 1)$, and the reflection property, $\mathcal{A}_n(1, 2, \dots, n) = (-1)^n \mathcal{A}_n(n, \dots, 2, 1)$.

2.1 Helicity choices and polarization vectors

The convention of D -dimensional metric is $\eta^{\mu\nu} = \text{diag}(+, -, -, \dots, -)$ throughout the paper. Let us denote an extra-dimensional momentum

$$\widehat{\ell} = (\ell, \vec{\mu}), \quad \vec{\mu} = (\mu_1, \dots, \mu_d), \quad (2.1)$$

where ℓ is the four-dimensional component and $\vec{\mu}$ is a vector in the extra d -dimension ($d = D - 4$). For the Euclidean d -dimensional space, the extra-dimensional basis e_a for $a = 1, \dots, d$ can be chosen as $\vec{\mu}_{e_a} = (\delta_{1a}, \dots, \delta_{ia}, \dots, \delta_{da})$. The massless condition of $\widehat{\ell}$ is then $\ell^2 - \vec{\mu}^2 = 0$.

By introducing a null auxiliary momentum q in four dimensions and $q \cdot \ell \neq 0$, we can define the null momentum as

$$\ell^\perp = \ell - \frac{\vec{\mu}^2}{\langle q | \ell | q \rangle} q, \quad (\ell^\perp)^2 = 0. \quad (2.2)$$

The transverse condition $\epsilon \cdot \widehat{\ell} = 0$ tells that the transverse space is $(D - 2)$ -dimensional. By imposing the extra condition $\epsilon_i \cdot q = 0$, the transverse polarization vectors are fixed. The polarization vectors in [2] are listed here with $D - 2$ helicity choices $+$, $-$ and S_a for spin-1 particles,

$$\begin{aligned} \epsilon_v^+(\widehat{\ell}, q) &= \left(\frac{\langle q | \gamma_v | \ell^\perp \rangle}{\sqrt{2} \langle q | \ell^\perp \rangle}, \vec{0}_d \right), \\ \epsilon_v^-(\widehat{\ell}, q) &= \left(\frac{\langle \ell^\perp | \gamma_v | q \rangle}{\sqrt{2} [\ell^\perp | q]}, \vec{0}_d \right), \\ \epsilon^{S_a}(\widehat{\ell}, q) &= \left(\frac{2\mu_a}{\langle q | \ell | q \rangle} q, e_a \right), \\ a &= 1, \dots, d, \end{aligned} \quad (2.3)$$

where e_a is the basis vector of the extra d dimensions, $\vec{0}_d$ denotes the d -dimensional vanishing vector, and μ_a is the a th component of $\vec{\mu}$ in (2.1). The rest of the two polarization vectors are longitudinal and time-like, which has no physical effect, and are defined as

$$\epsilon^L(\widehat{\ell}, q) = \widehat{\ell}, \quad \epsilon^T(\widehat{\ell}, q) = (q, \vec{0}_d). \quad (2.4)$$

These polarization vectors possess the following properties:

$$\begin{aligned}\epsilon^\pm \cdot q &= \epsilon^\pm \cdot \ell = \epsilon^\pm \cdot \epsilon^\pm = 0, \\ \epsilon^+ \cdot \epsilon^- &= -1, \\ \epsilon^L \cdot \epsilon^T &= \langle q | \ell | q \rangle, \\ \epsilon^{S_a} \cdot \epsilon^{L,T} &= 0, \\ \epsilon^{S_a} \epsilon^{S_b} &= -\delta^{ab}.\end{aligned}\quad (2.5)$$

The metric can be decomposed as

$$\begin{aligned}\eta_{\mu\nu} &= \frac{\epsilon^L \epsilon^T + \epsilon^T \epsilon^L}{\langle q | \ell | q \rangle} \\ &\quad - \epsilon^+ \epsilon^- - \epsilon^- \epsilon^+ - \sum_{a=1}^d \epsilon^{S_a} \epsilon^{S_a}.\end{aligned}\quad (2.6)$$

The convention above can be trivially generalized to $(4 - 2\epsilon)$ dimensions, where we should replace $d \rightarrow -2\epsilon$.

To avoid notation confusion for the symbol $\hat{\ell}$, an important convention is that we use $\hat{\ell}$ in two distinguished situations: to represent extra-dimensional momenta $\hat{\ell}_i$ and to represent BCFW shifted legs. Since we denote the leg in the scattering amplitude with momenta $\hat{\ell}_i$ by ℓ_i during the BCFW recursion and never do BCFW shift to it, these two situations are not hard to distinguish.

In our cases of amplitudes, we have two spin-1 extra-dimensional particles denoted by their four-dimensional momenta ℓ_2 and ℓ_1 . Since there are $D - 2$ helicity states $+$, $-$ and S_a for each spin-1 particles with extra-dimensional momenta and all gluons have positive helicity, there are nine kinds of amplitudes distinguished by the helicity of the two extra-dimensional particles. We denote these tree amplitudes as $\{\ell_2^+, \ell_1^+\}$, $\{\ell_2^+, \ell_1^{S_a}\}$, $\{\ell_2^+, \ell_1^-\}$, $\{\ell_2^-, \ell_1^+\}$, $\{\ell_2^-, \ell_1^{S_a}\}$, $\{\ell_2^-, \ell_1^-\}$, $\{\ell_2^{S_a}, \ell_1^+\}$, $\{\ell_2^{S_a}, \ell_1^-\}$, $\{\ell_2^{S_a}, \ell_1^{S_b}\}$.

One four-dimensional formulation In this part, we review the four-dimensional formulation of D -dimensional particles in [18], as an example to compare with the D -dimensional formulation above.

In [18], D -dimensional momentum $\bar{\ell}$ of mass m is decomposed thus:

$$\bar{\ell} = \ell + \tilde{\ell}, \quad \bar{\ell}^2 = \ell^2 - \mu^2 = m^2, \quad (2.7)$$

and its four-dimensional component ℓ is expressed as

$$\ell = \ell^b + \hat{q}_\ell, \quad \hat{q}_\ell \equiv \frac{m^2 + \mu^2}{2\ell \cdot q_\ell} q_\ell, \quad (2.8)$$

in terms of the two massless momenta ℓ^b and q_ℓ , which is similar to (2.2). For massive spin- $\frac{1}{2}$ particles, one introduced tachyonic spinors for the degrees of freedom of the extra dimension, while for massless spin-1 particles, the degree of

freedom is decomposed as a four-dimensional massive vector bosons' part in terms of $\{\epsilon_\pm, \epsilon_0\}$ and a scalar part represented by the factor $\hat{G}^{AB}$. The metric is decomposed as

$$\begin{aligned}\sum_{i=1}^{D-2} \epsilon_{i(d)}^\mu(\bar{\ell}, \bar{\eta}) \epsilon_{i(d)}^{*\nu}(\bar{\ell}, \bar{\eta}) &= \left(-\eta^{\mu\nu} + \frac{\ell^\mu \ell^\nu}{\mu^2} \right) \\ &\quad - \left(\tilde{\eta}^{\mu\nu} + \frac{\tilde{\ell}^\mu \tilde{\ell}^\nu}{\mu^2} \right),\end{aligned}\quad (2.9)$$

where

$$-\eta^{\mu\nu} + \frac{\ell^\mu \ell^\nu}{\mu^2} = \sum_{\lambda=\pm,0} \epsilon_\lambda^\mu(\ell) \epsilon_\lambda^{*\nu}(\ell), \quad (2.10)$$

in another kind of choices for three polarization vectors of a vector boson of mass μ ,

$$\begin{aligned}\epsilon_+^\mu(\ell) &= -\frac{[\ell^b | \gamma^\mu | \hat{q}_\ell]}{\sqrt{2}\mu}, \\ \epsilon_-^\mu(\ell) &= -\frac{\langle \ell^b | \gamma^\mu | \hat{q}_\ell \rangle}{\sqrt{2}\mu}, \\ \epsilon_0^\mu(\ell) &= \frac{\ell^b \mu - \hat{q}_\ell^\mu}{\mu},\end{aligned}\quad (2.11)$$

and an extra-dimensional part,

$$\tilde{\eta}^{\mu\nu} + \frac{\tilde{\ell}^\mu \tilde{\ell}^\nu}{\mu^2} \rightarrow \hat{G}^{AB} \equiv G^{AB} - Q^A Q^B. \quad (2.12)$$

This decomposition is typical for treating the $D - 2$ degrees of freedom, which is useful in the general unitarity method for 1-loop amplitudes to take the cut of propagator of the loop as massive vector bosons parts and scalar parts.

Different from (2.11), the $\epsilon^\pm$ in (2.3) are equivalent to massive spin-1 states in [14–16] and we no longer have ϵ^0 because we define the rest of the $D - 4$ physical polarization vectors ϵ^{S_a} to be extra-dimensional, each equivalent to a massive scalar in [14–16] instead of (2.12). So the tree amplitudes below are comparable with the massive results in [14–16].

However, in our paper, the motivation to use those helicity choices $+$, $-$ and S_a is that they are useful to express D -dimensional tree-level helicity amplitudes as inputs for the Q -cut representation and apply the BCFW recursion relations; we directly use the extra-dimensional polarization vectors and we can conveniently characterize the longitude degree of freedom by different helicity states.

2.2 3-Point amplitudes

Since the 3-point amplitudes act as building blocks in the BCFW recursion relations, we should calculate 3-point amplitudes first. Here we use simple Feynman rules to obtain

$$\begin{aligned}\mathcal{A}(1, 2, 3) = & -\sqrt{2}[(p_2 \cdot \epsilon_3)(\epsilon_1 \cdot \epsilon_2) \\ & + (p_3 \cdot \epsilon_1)(\epsilon_2 \cdot \epsilon_3) \\ & + (p_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_1)].\end{aligned}\quad (2.13)$$

To use the BCFW recursion relations in our cases, we only need these 3-point amplitudes: $\mathcal{A}_3(1^+, \ell_2^+, \ell_1^+)$, $\mathcal{A}_3(1^+, \ell_2^+, \ell_1^-)$, $\mathcal{A}_3(1^+, \ell_2^+, \ell_1^{S_a})$, $\mathcal{A}_3(1^+, \ell_2^-, \ell_1^-)$, $\mathcal{A}_3(1^+, \ell_2^-, \ell_1^{S_a})$, $\mathcal{A}_3(1^+, \ell_2^-, \ell_1^{S_b})$. Here we choose 1 to represent the particle in four dimensions and ℓ_1, ℓ_2 to represent the particles in D dimensions. Then we use the transverse polarization vectors given in (2.3) to give specific expressions. When we use (2.13), we must pay attention to $(p \cdot \epsilon)$. If the momentum p has extra-dimensional components and $\epsilon = \epsilon^{S_a}$ ($a = 1, 2, \dots, d$), the product gets a contribution from a d -dimensional momentum component: $\epsilon^{S_a}(\widehat{\ell}_1, q) \cdot p_{\widehat{\ell}_2} = \frac{\mu_{\ell_1, a} \langle q | \ell_2^\perp | q \rangle}{\langle q | \ell_1^\perp | q \rangle} - \mu_{\ell_2, a}$.

The expressions for the 3-point amplitudes are

$$\begin{aligned}\mathcal{A}_3(1^+, \ell_2^+, \ell_1^+) &= \mathcal{A}_3(1^+, \ell_2^+, \ell_1^{S_a}) = 0, \\ \mathcal{A}_3(1^+, \ell_2^+, \ell_1^-) &= \frac{[1 \ell_2^\perp]^3}{[\ell_2^\perp \ell_1^\perp][\ell_1^\perp 1]}, \\ \mathcal{A}_3(1^+, \ell_2^-, \ell_1^-) &= \frac{\langle \ell_2^\perp \ell_1^\perp \rangle^3}{\langle \ell_1^\perp 1 \rangle \langle 1 \ell_2^\perp \rangle}, \\ \mathcal{A}_3(1^+, \ell_2^-, \ell_1^{S_a}) &= \frac{\sqrt{2} \mu_{\ell_1, a} \langle \ell_2^\perp q \rangle \langle \ell_1^\perp \ell_2^\perp \rangle^2}{\langle \ell_1^\perp q \rangle \langle 1 \ell_1^\perp \rangle \langle 1 \ell_2^\perp \rangle}, \\ \mathcal{A}_3(1^+, \ell_2^-, \ell_1^{S_b}) &= -\frac{\sqrt{2} \mu_{\ell_2, a} \langle \ell_1^\perp q \rangle \langle \ell_1^\perp \ell_2^\perp \rangle^2}{\langle \ell_2^\perp q \rangle \langle 1 \ell_1^\perp \rangle \langle 1 \ell_2^\perp \rangle}, \\ \mathcal{A}_3(1^+, \ell_2^-, \ell_1^{S_c}) &= -\delta^{ab} \frac{[1 \ell_2^\perp][1 \ell_1^\perp]}{[\ell_1^\perp \ell_2^\perp]}.\end{aligned}\quad (2.14)$$

Notice that the reflection property of $\mathcal{A}_3(1^+, \ell_2^{S_a}, \ell_1^-) = (-1)^3 \mathcal{A}_3(1^+, \ell_1^-, \ell_2^{S_a})$, which is not obvious, is also obeyed because $\mu_{\ell_1, a} = -\mu_{\ell_2, a}$.

2.3 Four-point amplitudes

In the appendix of [2], the expressions that follow are calculated using Feynman rules,

$$\begin{aligned}\mathcal{A}_4(1^+, 2^+, \ell_2^+, \ell_1^+) &= \mathcal{A}_4(1^+, 2^+, \ell_2^+, \ell_1^{S_a}) = 0, \\ \mathcal{A}_4(1^+, 2^+, \ell_2^+, \ell_1^-) &= \frac{\mu^2 \langle \ell_1^\perp q \rangle^2}{\langle \ell_2^\perp q \rangle^2} \frac{[2 1]}{\langle 1 2 \rangle \langle 1 | \ell_1 | 1 \rangle}, \\ \mathcal{A}_4(1^+, 2^+, \ell_2^-, \ell_1^-) &= -\langle \ell_1^\perp \ell_2^\perp \rangle^2 \frac{[2 1]}{\langle 1 2 \rangle \langle 1 | \ell_1 | 1 \rangle}, \\ \mathcal{A}_4(1^+, 2^+, \ell_2^-, \ell_1^{S_a}) &= \frac{\sqrt{2} \mu_{\ell_1, a} \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_2^\perp q \rangle}{\langle \ell_1^\perp q \rangle} \frac{[2 1]}{\langle 1 2 \rangle \langle 1 | \ell_1 | 1 \rangle}, \\ \mathcal{A}_4(1^+, 2^+, \ell_2^-, \ell_1^{S_b}) &= -\frac{\sqrt{2} \mu_{\ell_2, a} \langle \ell_2^\perp \ell_1^\perp \rangle \langle \ell_1^\perp q \rangle}{\langle \ell_2^\perp q \rangle} \frac{[2 1]}{\langle 1 2 \rangle \langle 1 | \ell_1 | 1 \rangle},\end{aligned}$$

$$\mathcal{A}_4(1^+, 2^+, \ell_2^{S_a}, \ell_1^{S_b}) = -\mu^2 \frac{[2 1]}{\langle 1 2 \rangle \langle 1 | \ell_1 | 1 \rangle} \delta^{ab}. \quad (2.15)$$

We have checked $\mathcal{A}_4$ with the BCFW recursion relations. These expressions inspire us to guess the general expression for $\mathcal{A}_n$ for $n \geq 4$. The notations here are the same as in Sect. 2.4.

2.4 General expressions for the n -point amplitudes

To express n -point tree amplitudes, we denote k positive helicity gluons as $1, 2, \dots, k$, and two adjacent massless particles with extra-dimensional momenta by their four-dimensional momenta ℓ_1, ℓ_2 . (In this paper, we also denote this series of amplitudes by $\{\ell_2^{h_2}, \ell_1^{h_1}\}$ for short.)

Using the BCFW recursion relations, we find $n = k + 2$ tree amplitudes do have general expressions for $n \geq 4$. Here we list our main results:

$$\begin{aligned}\mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^+, \ell_1^+) &= \mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^+, \ell_1^{S_a}) = 0, \\ \mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^+, \ell_1^-) &= -\frac{\mu^2 \langle q \ell_1^\perp \rangle^2}{\langle q \ell_2^\perp \rangle^2} \\ &\quad \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1 2 \rangle \dots \langle k-1 k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)}, \\ \mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^-, \ell_1^-) &= \langle \ell_1^\perp \ell_2^\perp \rangle^2 \\ &\quad \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1 2 \rangle \dots \langle k-1 k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)}, \\ \mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^-, \ell_1^{S_a}) &= -\frac{\sqrt{2} \mu_{\ell_1, a} \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_2^\perp q \rangle}{\langle \ell_1^\perp q \rangle} \\ &\quad \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1 2 \rangle \dots \langle k-1 k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)}, \\ \mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^-, \ell_1^{S_b}) &= \frac{\sqrt{2} \mu_{\ell_2, a} \langle \ell_2^\perp \ell_1^\perp \rangle \langle \ell_1^\perp q \rangle}{\langle \ell_2^\perp q \rangle} \\ &\quad \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1 2 \rangle \dots \langle k-1 k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)}, \\ \mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^-, \ell_1^{S_c}) &= \mu^2 \delta^{ab} \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1 2 \rangle \dots \langle k-1 k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)},\end{aligned}\quad (2.16)$$

where q is a null four-dimensional reference momentum and $q \cdot \ell_i \neq 0$, $\ell_i^\perp = \ell_i - \frac{\tilde{\mu}^2}{\langle q | \ell_i | q \rangle} q$, μ^2 is the inner product of

all the extra-dimensional momentum values of two $\widehat{\ell}_i$, while $\mu_{\ell_i,a}$ is the extra-dimensional momentum component of $\widehat{\ell}_i$ in the direction $\vec{e}_a$ of the extra-dimensional polarization vector $\epsilon^{S_a}(\ell_i)$. We borrow the notations $x_{i,\ell_1} = p_i + p_{i+1} + \dots + p_k + p_{\ell_2}$ and $x_{\ell_1,i} = p_{\ell_1} + p_1 + \dots + p_{i-1}$ from [15] and set $[1|\prod_{i=2}^{k-1}(\mu^2 - x_{i,\ell_1}x_{\ell_1,i+1})|2] = [1\ 2]$ for $n = 4(k = 2)$, so these expressions also hold for $\mathcal{A}_4$. Massive versions of $\{\ell_2^{S_a}, \ell_1^{S_b}\}$ and $\{\ell_2^+, \ell_1^-\}$ have been obtained in [14] and [15] separately but the other two, $\{\ell_2^-, \ell_1^-\}$ and $\{\ell_2^-, \ell_1^{S_a}\}$, are new here.

One observation is that these amplitudes have a common structure: they all have one common part $\frac{[1|\prod_{i=2}^{k-1}(\mu^2 - x_{i,\ell_1}x_{\ell_1,i+1})|k]}{\langle 1\ 2 \rangle \dots \langle k-1\ k \rangle \prod_{i=2}^k(\mu^2 + x_{\ell_1,i}^2)}$ changing with the increasing number of k , while the other part stays the same for any number of k and depends on $\ell_1^{h_1}, \ell_2^{h_2}, q, \mu_{\ell_i,a}, \mu^2$. This structure is astonishingly simple!

2.5 Structure of the general expressions

The general expressions for n -points amplitudes with $n > 4$ are not easy to get from Feynman diagrams. Here, based on our assumption that they have a common structure inspired by 4-point cases (2.15), we develop a clever proof by using the BCFW recursion relations.

Let us study the common structure first. We can see $\mathcal{A}_n$ in (2.16) has two parts, which we denote Part A($\ell_1^{h_1}, \ell_2^{h_2}$) and Part B(k).

Part B(k) is the same for all non-zero cases,

$$\text{Part B}(k) = \frac{[1|\prod_{i=2}^{k-1}(\mu^2 - x_{i,\ell_1}x_{\ell_1,i+1})|k]}{\langle 1\ 2 \rangle \dots \langle k-1\ k \rangle \prod_{i=2}^k(\mu^2 + x_{\ell_1,i}^2)}, \quad (2.17)$$

while Part A is different

$$\begin{aligned} \text{Part A}(\ell_1^+, \ell_2^-) &= -\frac{\mu^2 \langle q\ \ell_1^\perp \rangle^2}{\langle q\ \ell_2^\perp \rangle^2}, \\ \text{Part A}(\ell_1^-, \ell_2^-) &= \langle \ell_1^\perp\ \ell_2^\perp \rangle^2, \\ \text{Part A}(\ell_1^-, \ell_2^{S_a}) &= -\frac{\sqrt{2}\mu_{\ell_1,a} \langle \ell_1^\perp\ \ell_2^\perp \rangle \langle \ell_2^\perp\ q \rangle}{\langle \ell_1^\perp\ q \rangle}, \\ \text{Part A}(\ell_1^{S_a}, \ell_2^-) &= \frac{\sqrt{2}\mu_{\ell_2,a} \langle \ell_2^\perp\ \ell_1^\perp \rangle \langle \ell_1^\perp\ q \rangle}{\langle \ell_2^\perp\ q \rangle}, \\ \text{Part A}(\ell_1^{S_a}, \ell_2^{S_b}) &= \mu^2 \delta_{ab}, \end{aligned} \quad (2.18)$$

and we can see Part A($\ell_2^{h_2}, \ell_1^{h_1}$) is independent of all k gluons.

The pole structure of these $\mathcal{A}_n$ is contained in Part B(k). We see there are not only 2-particle poles $\langle 1\ 2 \rangle \dots \langle k-1\ k \rangle$ in Part B(k) but also multi-particle poles $\prod_{i=2}^k(\mu^2 + x_{\ell_1,i}^2)$.

3 Proof by mathematical induction

In this section, we give our proof of the results (2.16) by mathematical induction. First we choose a special shift so that the BCFW recursion relations for these tree amplitudes can be reduced so as to become very simple. Then we separate our proof for the inductive step into two parts.

As the first step of a mathematical induction, known as the base case, we can see $\mathcal{A}_4$ amplitudes given in (2.15) match the expressions of Eq. (2.16) for $n = 4$.

The second step is called the inductive step. Let us assume the expressions of Eq. (2.16) hold for n -point cases. If the expressions of Eq. (2.16) hold for $(n+1)$ -point cases, we can conclude that (2.16) are general expressions for $n \geq 4$. This hypothesis for the expressions of Eq. (2.16), holding for the $(n+1)$ -point cases, is called the inductive hypothesis. By proving the hypothesis, we can prove the expressions hold for any natural number $n \geq 4$.

3.1 Strategies to use the BCFW recursion relations

Using the BCFW recursion relations [5,6], one can get higher-point tree amplitudes from lower ones. An instruction to use the BCFW recursion relations and common notations can be found in textbooks such as [19].

After shifting the momentum with complex parameter z , tree amplitudes without boundary contributions can be calculated by

$$A_n = \sum_{i,j} \sum_h \frac{A_i^h(z_{ij}) A_j^{-h}(z_{ij})}{P(0)^2}, \quad (3.1)$$

where we need to sum all possible partitions of subdiagrams noted by i, j as well as all possible helicity h channels with solutions $z = z_{ij}$ putting the propagator $P(z)^2 = 0$ to get A_i and A_j on-shell. In the following, we will show how to use the recursion relations explicitly in our proof.

In our cases, the intermediate particles have extra-dimensional momentum, so we need to sum over the intermediate particle's helicity $+$, $-S_a$ rather than $+$, $-$ in the 4- D cases.

To calculate $\mathcal{A}_{n+1}(n = k+2)$ from lower-point amplitudes, we choose the BCFW shift to be, with $[k|k+1]^+$,

$$\begin{aligned} \widehat{k} &= |k\rangle - z|k+1\rangle, \\ \widehat{k} &= |k\rangle, \\ \widehat{k+1} &= |k+1\rangle, \\ \widehat{k+1} &= |k+1\rangle + z|k\rangle. \end{aligned} \quad (3.2)$$

This shift, involving two adjacent gluons, mainly has two advantages. One is that the poles $\langle 1\ 2 \rangle \dots \langle k-1\ k \rangle$ and $\prod_{i=2}^k(\mu^2 + x_{\ell_1,i}^2)$ in the denominator of Part B(k) would not change after the shift (Fig. 1).

Fig. 1 The general situation for $\{\ell_2^{h_1}, \ell_1^{h_2}\}$

The other advantage is that we can reduce (3.1) to

$$\mathcal{A}_{n+1}(1^+, 2^+, \dots, (k+1)^+, \ell_2^{h_2}, \ell_1^{h_1}) = \sum_h \frac{A_L(1^+, 2^+, \dots, \widehat{k}^+, \widehat{mQ}^{-h}, \ell_1^{h_1}) \times A_R(\widehat{k+1}^+, \ell_2^{h_2}, \widehat{Q}^h)}{P(0)^2}, \quad (3.3)$$

where $\widehat{P_{mQ}} = -\widehat{P_Q} = \widehat{P_{k+1}} + \widehat{P_{\ell_2}}$ for momentum conservation. The symbol $\widehat{}$ above the legs $1 \cdots k+1$, Q and mQ in (3.3) stands for the leg being on-shell after the shift and we denote legs which have extra-dimension momenta $\widehat{\ell}_i$ by ℓ_i , since we never do BCFW shift to ℓ_i . We see that one only needs to replace ℓ_2 in A_n with $\widehat{mQ}$ to express A_L .

The reason why we can obtain (3.3) is as follows. When both of ℓ_1 and ℓ_2 are in the same diagram A_L or A_R , the intermedia massless particles, denoted $\widehat{Q}$ and $\widehat{mQ}$, will have no extra-dimensional momentum component. One can find that the other diagram which has no leg with extra-dimensional momentum always vanishes. As a conclusion, a non-vanishing $A_L \times A_R$ term should be a product of a n -point amplitude and a 3-point amplitude as (3.3).

For zero cases in (2.16), using (3.3) we find that they do vanish because the right hand side of (3.3) of $\{\ell_2^+, \ell_1^+\}$, $\{\ell_2^+, \ell_1^{S_a}\}$, $\{\ell_2^-, \ell_1^+\}$ has at least one vanishing lower-point amplitude of A_L and A_R .

Finally, let us focus on the non-zero cases. Our strategy for the proof is to deal with Part A and Part B separately in our recursion from n to $n+1$. We rewrite (3.3) as

$$\begin{aligned} \mathcal{A}_{n+1}(1^+, 2^+, \dots, (k+1)^+, \ell_2^{h_1}, \ell_1^{h_2}) &= \sum_h \frac{A_L(1^+, 2^+, \dots, \widehat{k}^+, \widehat{mQ}^{-h}, \ell_1^{h_2}) \times A_R(\widehat{k+1}^+, \ell_2^{h_1}, \widehat{Q}^h)}{P(0)^2} \\ &= \frac{\text{Part B}(\widehat{k}) \sum_h \text{Part A}(\widehat{mQ}^h, \ell_1^{h_1}) \times A_3(\widehat{k+1}^+, \widehat{Q}^{-h}, \ell_2^{h_2})}{P(0)^2}. \end{aligned} \quad (3.4)$$

With these notations, we separate the proof into two parts. The first part is

$$\begin{aligned} \sum_h \text{Part A}(\widehat{mQ}^h, \ell_1^{h_1}) \times A_3(\widehat{k+1}^+, \widehat{Q}^{-h}, \ell_2^{h_2}) \\ \rightarrow \text{Part A}(\ell_2^h, \ell_1^{h_1}) \frac{\langle k|\ell_2|k+1 \rangle}{\langle k|k+1 \rangle}, \end{aligned} \quad (3.5)$$

while the second part is

$$\begin{aligned} \text{Part B}(\widehat{k}) &= \frac{[1|\prod_{i=2}^{k-1}(\mu^2 - x_{i,\ell_1}x_{\ell_1,i+1})|\widehat{k}]}{\langle 12 \rangle \cdots \langle k-1 k \rangle \prod_{i=2}^k(\mu^2 + x_{\ell_1,i}^2)} \\ &\rightarrow \frac{[1|\prod_{i=2}^k(\mu^2 - x_{i,\ell_1}x_{\ell_1,i+1})|k+1]}{\langle 12 \rangle \cdots \langle k-1 k \rangle \prod_{i=2}^k(\mu^2 + x_{\ell_1,i}^2)} \frac{1}{\langle k|\ell_2|k+1 \rangle} \\ &= \text{Part B}(k+1) \frac{\langle k|k+1 \rangle P(0)^2}{\langle k|\ell_2|k+1 \rangle}. \end{aligned} \quad (3.6)$$

According to the partition in (3.3), where two legs $(k+1)$ and ℓ_2 are always on the right side, we can rewrite $P(0)^2$ thus:

$$\begin{aligned} P(0)^2 &= (p_{\ell_2} + p_{k+1})^2 \\ &= (p_{\ell_1} + p_1 + p_2 + \cdots + p_k)^2 + \mu^2. \end{aligned} \quad (3.7)$$

Define $x_{\ell_1,i} = p_{\ell_1} + p_1 + p_2 + \cdots + p_{i-1}$, then

$$P(0)^2 = \mu^2 + x_{\ell_1,k}^2. \quad (3.8)$$

The on-shell condition for $P(z)^2 = 0$ is $P_Q^2 = (\widehat{P_{k+1}} + P_{\ell_2})^2 = 0$, with the following solution for z :

$$z = z_L(k+1) = -\frac{\langle k+1|\ell_2|k+1 \rangle}{\langle k|\ell_2|k+1 \rangle}. \quad (3.9)$$

3.2 Proof of the inductive hypothesis

Here we prove the inductive hypothesis by showing that (3.5) and (3.6) are correct.

The proof for (3.6) is easy.

After the shift, $\text{Part B}(k)$ changes with $|\widehat{k}]$,

$$\begin{aligned} |\widehat{k}] &= |k] - z_L(k+1)|k+1] \\ &= |k] + \frac{\langle k+1|\ell_2|k+1 \rangle}{\langle k|\ell_2|k+1 \rangle} |k+1] \\ &= \frac{\mu^2 - x_{k,\ell_1}x_{\ell_1,k+1}}{\langle k|\ell_2|k+1 \rangle} |k+1], \end{aligned} \quad (3.10)$$

where we use the identity $|k|\ell_2|k+1] + |k+1|\ell_2|k+1] = |\mu^2|k+1] - |(p_k + p_{k+1} + p_{\ell_2})|\ell_1 + 1 + \cdots + k|k+1]$.

Applying (3.10) to $\text{Part B}(k)$, we see (3.6) holds true, where we can see one advantage of the special BCFW shift we choose.

However, the proof for (3.5) is very difficult.

The inductive steps (3.5) are different for each of the non-zero cases. Regarding the results the $\{\ell_2^+, \ell_1^+\}$, $\{\ell_2^+, \ell_1^{S_a}\}$, $\{\ell_2^{S_a}, \ell_1^+\}$ cases are zero, and the remaining non-zero cases are $\{\ell_2^+, \ell_1^-\}$, $\{\ell_2^-, \ell_1^+\}$, $\{\ell_2^-, \ell_1^{S_a}\}$, $\{\ell_2^{S_a}, \ell_1^-\}$, $\{\ell_2^-, \ell_1^-\}$, $\{\ell_2^{S_a}, \ell_1^{S_b}\}$. We only need to calculate four of them: $\{\ell_2^+, \ell_1^-\}$, $\{\ell_2^-, \ell_1^{S_a}\}$, $\{\ell_2^{S_a}, \ell_1^{S_b}\}$, $\{\ell_2^-, \ell_1^-\}$.

Recall that for (3.5) we are to show

$$\sum_h \text{Part A}(\widehat{mQ}^h, \ell_1^{h_1}) \times A_3(\widehat{k+1}^+, \widehat{Q}^{-h}, \ell_2^{h_2}) = \frac{\text{Part A}(\ell_2^{h_2}, \ell_1^{h_1}) \times \langle k|\ell_2|k+1 \rangle}{\langle k k+1 \rangle}. \quad (3.11)$$

The proof for (3.11) is nontrivial. We conclude the proof taking the following steps:

- First, substitute all the spinor brackets and angles of $Q^\perp$ with associated spinor brackets and angles of $mQ^\perp$ due to momentum conservation. Second, multiply every $[\widehat{mQ}^\perp]$ (or $[\widehat{mQ}^\perp]$) with $\langle \widehat{mQ}^\perp q \rangle$ (or $[\widehat{mQ}^\perp q]$) so that we can replace $\widehat{mQ}^\perp$ by $\widehat{mQ}$ for $\widehat{mQ}^\perp = \widehat{mQ} - \frac{\tilde{\mu}^2}{\langle q|\widehat{mQ}|q \rangle} q$.
- Next, before simplifying the spinor products, if there is $\langle q|\widehat{mQ}|q \rangle$ appearing in the denominator, we should eliminate all the $\langle q|\widehat{mQ}|q \rangle$ in the first place, by technically using the Schouten identity to $\langle \ell_i^\perp|\widehat{mQ}|q \rangle$ with another term including $|q\rangle$. We will specifically show this step for each case later.
- Finally, if there is no $\langle q|\widehat{mQ}|q \rangle$ left, we begin to simplify the spinor products using the following identities:

$$\begin{aligned} \langle q|\widehat{k+1} \rangle &= -\frac{\langle k k+1 \rangle [k+1 \ell_2^\perp] \langle \ell_2^\perp q \rangle}{\langle k|\ell_2|k+1 \rangle}, \\ \langle \widehat{k+1} \ell_2^\perp \rangle &= \frac{\mu^2 \langle k k+1 \rangle [k+1 q]}{[\ell_2^\perp q] \langle k|\ell_2|k+1 \rangle}, \\ \langle q|\widehat{mQ}|\ell_2^\perp \rangle &= \frac{\langle \ell_2^\perp q \rangle \langle k k+1 \rangle [k+1 \ell_2^\perp]^2}{\langle k|\ell_2|k+1 \rangle}, \\ \langle \ell_2^\perp|\widehat{mQ}|q \rangle &= -\frac{\mu^2 \langle k k+1 \rangle [q k+1]^2}{\langle k|\ell_2|k+1 \rangle [q \ell_2^\perp]}, \\ \langle q|\widehat{mQ}|k+1 \rangle &= \langle q|\ell_2^\perp|k+1 \rangle, \end{aligned} \quad (3.12)$$

which is not hard to prove by substituting $\widehat{k+1}$, $\widehat{k}$ and $\widehat{mQ}$ and then using the Schouten identity to combine the separate terms.

In the following, we will prove (3.11) holds true for each of the non-zero cases in detail. We have also verified (3.11) numerically using the S@M package [18] for each case.

3.2.1 $\{\ell_2^+, \ell_1^-\}$ and $\{\ell_2^{S_a}, \ell_1^{S_b}\}$

Both of the two cases $\{\ell_2^+, \ell_1^-\}$ and $\{\ell_2^{S_a}, \ell_1^{S_b}\}$ only have one contributing term and no $\langle q|\widehat{mQ}|q \rangle$ will appear in the denominators. We only need to simplify all the spinor products.

Fig. 2 $\{\ell_2^+, \ell_1^-\}$

Fig. 3 $\{\ell_2^{S_a}, \ell_1^{S_b}\}$

For $\{\ell_2^+, \ell_1^-\}$ (Fig. 2), we have

$$\begin{aligned} \text{Part A}(\widehat{mQ}^+, \ell_1^-) \times A_3(\widehat{k+1}, \widehat{Q}^-, \ell_2^+) &= -\frac{u^2 \langle q \ell_1^\perp \rangle^2}{\langle q \widehat{Q}^\perp \rangle^2} \times \frac{[k+1 \ell_2^\perp]^3}{[k+1 \widehat{mQ}^\perp][\widehat{mQ}^\perp \ell_2^\perp]} \\ &= -\frac{u^2 \langle q \ell_1^\perp \rangle^2 [k+1 \ell_2^\perp]^3}{\langle q|\widehat{mQ}|k+1 \rangle \langle q|\widehat{mQ}|\ell_2^\perp \rangle} \\ &= -\frac{\mu^2 \langle q \ell_1^\perp \rangle^2}{\langle q \ell_2^\perp \rangle^2} \times \frac{\langle k|\ell_2|k+1 \rangle}{\langle k k+1 \rangle}, \end{aligned} \quad (3.13)$$

where for the last equality, we use the identities in (3.12).

The case $\{\ell_2^{S_a}, \ell_1^{S_b}\}$ is similar to $\{\ell_2^+, \ell_1^-\}$, and we can easily get (Fig. 3)

$$\begin{aligned} \sum_{S_c} \text{Part A}(\widehat{mQ}^{S_c}, \ell_1^{S_b}) \times A_3(\widehat{k+1}^+, \widehat{Q}^{S_c}, \ell_2^{S_a}) &= \sum_{S_c} -\mu^2 \delta_{cb} \times (-\delta_{ac} \frac{[k+1 \ell_2^\perp][k+1 \widehat{Q}^\perp]}{[\widehat{Q}^\perp \ell_2^\perp]}) \\ &= -\mu^2 \delta_{ab} \frac{[k+1 \ell_2^\perp][k+1 \widehat{mQ}^\perp|q]}{\langle q|\widehat{mQ}^\perp|\ell_2^\perp \rangle} \\ &= -\mu^2 \delta_{ab} \times \frac{\langle k|\ell_2|k+1 \rangle}{\langle k k+1 \rangle}. \end{aligned} \quad (3.14)$$

Fig. 4 $\{\ell_2^-, \ell_1^{S_a}\}$

3.2.2 $\{\ell_2^-, \ell_1^{S_a}\}$

This case has two contributing terms (Fig. 4)

$$\begin{aligned} & \sum_h \text{Part A}(\widehat{mQ}^h, \ell_1^{h_1}) \times A_R(\widehat{k+1}^+, \ell_2^{h_2}, \widehat{Q}^{-h}) \\ &= \text{Part A}(\widehat{mQ}^-, \ell_1^{S_a}) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^+) \\ &+ \sum_{S_b} \text{Part A}(\widehat{mQ}^{S_b}, \ell_1^{S_a}) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^{S_b}). \end{aligned} \quad (3.15)$$

First we need to separate $\mathcal{A}_3(1^+, \ell_2^-, \ell_1^{S_a})$ to two terms

$$\begin{aligned} \mathcal{A}_3(1^+, \ell_2^-, \ell_1^{S_a}) &= \sqrt{2}\mu_{\ell_1, a} \frac{\langle \ell_2^\perp q \rangle [1 q]}{\langle q 1 \rangle [q \ell_2^\perp]} \\ &+ \sqrt{2}\mu_{\ell_1, a} \frac{\langle \ell_2^\perp q \rangle^2 [1 q]}{\langle q 1 \rangle \langle \ell_1^\perp q \rangle [q \ell_1^\perp]}. \end{aligned} \quad (3.16)$$

Then we write them explicitly

$$\begin{aligned} & \text{Part A}(\widehat{mQ}^-, \ell_1^{S_a}) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^+) \\ &= \frac{-\sqrt{2}\mu_{\ell_1, a} \langle \ell_1^\perp \widehat{mQ}^\perp \rangle \langle \widehat{mQ}^\perp q \rangle [\widehat{mQ}^\perp k+1]^3}{\langle \ell_1^\perp q \rangle [k+1 \ell_2^\perp] [\ell_2^\perp \widehat{mQ}^\perp]} \\ &= -\frac{\sqrt{2}\mu_{\ell_1, a} \langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle \langle q \widehat{mQ}^\perp | k+1 \rangle^3}{\langle \ell_1^\perp q \rangle [k+1 \ell_2^\perp] \langle q \widehat{mQ}^\perp | \ell_2^\perp \rangle \langle q \widehat{mQ}^\perp | q \rangle}, \end{aligned} \quad (3.17)$$

$$\begin{aligned} & \sum_{S_b} \text{Part A}(\widehat{mQ}^{S_b}, \ell_1^{S_a}) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^{S_b}) \\ &= \sum_{S_b} \left(\mu^2 \delta_{ab} \sqrt{2}\mu_{\ell_1, b} \frac{\langle \ell_2^\perp q \rangle [k+1 q]}{\langle q \widehat{k+1} \rangle [q \ell_2^\perp]} \right. \\ &\quad \left. + \mu^2 \delta_{ab} \sqrt{2}\mu_{\ell_1, b} \frac{\langle \ell_2^\perp q \rangle^2 [k+1 q]}{\langle q \widehat{k+1} \rangle \langle \widehat{mQ}^\perp q \rangle [q \widehat{mQ}^\perp]} \right) \\ &= \mu^2 \sqrt{2}\mu_{\ell_1, a} \frac{\langle \ell_2^\perp q \rangle [k+1 q]}{\langle q \widehat{k+1} \rangle [q \ell_2^\perp]} \\ &\quad - \mu^2 \sqrt{2}\mu_{\ell_1, a} \frac{\langle \ell_2^\perp q \rangle^2 [k+1 q]}{\langle q \widehat{k+1} \rangle \langle q \widehat{mQ} | q \rangle}. \end{aligned} \quad (3.18)$$

Instead of simplifying all the spinor products directly, the first thing to do is to apply the Schouten identity to $\langle \ell_1^\perp | \widehat{mQ} | q \rangle \langle q | \widehat{mQ} | k+1 \rangle$ to eliminate the unphysical pole $\langle q | \widehat{mQ} | q \rangle$

$$\begin{aligned} \langle \ell_1^\perp | \widehat{mQ} | q \rangle \langle q | \widehat{mQ} | k+1 \rangle &= \langle \ell_1^\perp | \widehat{mQ} | k+1 \rangle \langle q | \widehat{mQ} | q \rangle \\ &+ \mu^2 \langle \ell_1^\perp q \rangle [k+1 q], \end{aligned} \quad (3.19)$$

where $\langle \ell_1^\perp | \widehat{mQ} | k+1 \rangle$ can be written as

$$\begin{aligned} \langle \ell_1^\perp | \widehat{mQ} | k+1 \rangle &= \langle \ell_1^\perp | \ell_2 | k+1 \rangle \\ &= \langle \ell_1^\perp | \ell_2^\perp + \frac{\mu^2}{\langle q | \ell_2 | q \rangle} q | k+1 \rangle \\ &= \langle \ell_1^\perp | \ell_2^\perp \rangle [\ell_2^\perp k+1] \\ &\quad + \frac{\mu^2 \langle \ell_1^\perp q \rangle [q k+1]}{\langle q \ell_2^\perp \rangle [\ell_2^\perp q]}. \end{aligned} \quad (3.20)$$

Surprisingly the result can be merged and simplified

$$\begin{aligned} & \sum_h \text{Part A}(\widehat{mQ}^h, \ell_1^{S_a}) \times \mathcal{A}_3(\widehat{k+1}, \widehat{Q}^{-h}, \ell_2^-) \\ &= \text{Part A}(\widehat{mQ}^-, \ell_1^{S_a}) \times \mathcal{A}_3(\widehat{k+1}, \widehat{Q}^+, \ell_2^-) \\ &\quad + \text{Part A}(\widehat{mQ}^{S_a}, \ell_1^{S_a}) \times \mathcal{A}_3(\widehat{k+1}, \widehat{Q}^{S_a}, \ell_2^-) \\ &= \frac{\sqrt{2}\mu_{\ell_1, a} \langle k | \ell_2 | k+1 \rangle \langle \ell_1^\perp | \widehat{mQ}^\perp | q \rangle \langle q | \widehat{mQ}^\perp | k+1 \rangle}{\langle \ell_1^\perp q \rangle [k+1 \ell_2^\perp] \langle k k+1 \rangle \langle q | \widehat{mQ}^\perp | q \rangle} \\ &\quad - \frac{\mu^2 \sqrt{2}\mu_{\ell_1, a} [k+1 q] \langle k | \ell_2 | k+1 \rangle}{\langle k k+1 \rangle [k+1 \ell_2^\perp] [\ell_2^\perp q]} \\ &\quad + \frac{\mu^2 \sqrt{2}\mu_{\ell_1, a} \langle \ell_2^\perp q \rangle [k+1 q] \langle k | \ell_2 | k+1 \rangle}{\langle k k+1 \rangle [k+1 \ell_2^\perp]} \\ &= \frac{\sqrt{2}\mu_{\ell_1, a} \langle \ell_2^\perp \ell_1^\perp \rangle \langle \ell_1^\perp q \rangle \langle k | \ell_2 | k+1 \rangle}{\langle \ell_2^\perp q \rangle \langle k k+1 \rangle}. \end{aligned} \quad (3.21)$$

Fig. 5 $\{\ell_2^-, \ell_1^-\}$ 3.2.3 $\{\ell_2^-, \ell_1^-\}$

The $\{\ell_2^-, \ell_1^-\}$ case has three contributing terms (Fig. 5):

$$\begin{aligned} & \sum_h \text{Part A}(\widehat{mQ}^h, \ell_1^{h1}) \times A_R(\widehat{k+1}^+, \ell_2^{h2}, m\widehat{Q}^{-h}) \\ &= \text{Part A}(\widehat{mQ}^-, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^+) \\ &+ \text{Part A}(\widehat{mQ}^+, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^-) \\ &+ \sum_{Sc} \text{Part A}(\widehat{mQ}^{Sc}, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^{Sc}). \end{aligned} \quad (3.22)$$

We write them explicitly

$$\begin{aligned} & \text{Part A}(\widehat{mQ}^-, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^+) \\ &= \frac{\langle \widehat{mQ}^\perp \ell_1^\perp \rangle^2 \langle \widehat{mQ}^\perp k+1 \rangle}{[k+1 \ell_2^\perp] [\ell_2^\perp \widehat{mQ}^\perp]} \\ &= \frac{-\langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle^2 \langle q \widehat{mQ}^\perp | k+1 \rangle^3}{[k+1 \ell_2^\perp] \langle q \widehat{mQ}^\perp | q \rangle^2 \langle q \widehat{mQ}^\perp | \ell_2^\perp \rangle}, \\ & \text{Part A}(\widehat{mQ}^+, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^-) \\ &= \frac{-\mu^2 \langle \ell_1^\perp q \rangle^2 \langle \ell_2^\perp \widehat{mQ}^\perp \rangle^3}{\langle k+1 \widehat{mQ}^\perp \rangle \langle k+1 \ell_2^\perp \rangle \langle q \widehat{mQ}^\perp \rangle^2} \\ &= \frac{-\mu^2 \langle \ell_1^\perp q \rangle^2 \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle^3}{\langle k+1 \widehat{mQ}^\perp \rangle \langle q \widehat{mQ}^\perp | q \rangle^2 \langle k+1 \widehat{mQ}^\perp | q \rangle}, \\ & \sum_{Sc} \text{Part A}(\widehat{mQ}^{Sc}, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^{Sc}) \\ &= \frac{2\mu^2 \langle \widehat{mQ}^\perp \ell_1^\perp \rangle \langle \ell_1^\perp q \rangle \langle \ell_2^\perp q \rangle \langle \ell_2^\perp \widehat{mQ}^\perp \rangle^2}{\langle k+1 \ell_2^\perp \rangle \langle k+1 \widehat{mQ}^\perp \rangle \langle \widehat{mQ}^\perp q \rangle^2} \\ &= \frac{2\mu^2 \langle \ell_1^\perp q \rangle \langle \ell_2^\perp q \rangle \langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle^2}{\langle k+1 \ell_2^\perp \rangle \langle q \widehat{mQ}^\perp | q \rangle^2 \langle k+1 \widehat{mQ}^\perp | q \rangle}. \end{aligned} \quad (3.23)$$

So

$$\sum_h \text{Part A}(\widehat{mQ}^h, \ell_1^-) \times A_R(\widehat{k+1}^+, \ell_2^-, m\widehat{Q}^{-h})$$

$$\begin{aligned} &= \frac{-\langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle^2 \langle q \widehat{mQ}^\perp | k+1 \rangle^3}{[k+1 \ell_2^\perp] \langle q \widehat{mQ}^\perp | q \rangle^2 \langle q \widehat{mQ}^\perp | \ell_2^\perp \rangle} \\ &+ \frac{-\mu^2 \langle \ell_1^\perp q \rangle^2 \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle^3}{\langle k+1 \widehat{mQ}^\perp \rangle \langle q \widehat{mQ}^\perp | q \rangle^2 \langle k+1 \widehat{mQ}^\perp | q \rangle} \\ &+ \frac{\mu^2 \langle \ell_1^\perp q \rangle \langle \ell_2^\perp q \rangle \langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle^2}{\langle k+1 \ell_2^\perp \rangle \langle q \widehat{mQ}^\perp | q \rangle^2 \langle k+1 \widehat{mQ}^\perp | q \rangle} \\ &+ \frac{\mu^2 \langle \ell_1^\perp q \rangle \langle \ell_2^\perp q \rangle \langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle^2}{\langle k+1 \ell_2^\perp \rangle \langle q \widehat{mQ}^\perp | q \rangle^2 \langle k+1 \widehat{mQ}^\perp | q \rangle} \\ &= \frac{-\mu^2 \langle \ell_2^\perp q \rangle \langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle^2 \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle}{\langle k+1 \ell_2^\perp \rangle \langle q \widehat{mQ}^\perp | q \rangle^2 \langle k+1 \widehat{mQ}^\perp | q \rangle} \\ &+ \frac{\mu^2 \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_2^\perp q \rangle \langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle}{\langle k+1 \ell_2^\perp \rangle \langle q \widehat{mQ}^\perp | q \rangle \langle k+1 \widehat{mQ}^\perp | q \rangle} \\ &+ \frac{\mu^2 \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_1^\perp q \rangle \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle^2}{\langle k+1 \ell_2^\perp \rangle \langle q \widehat{mQ}^\perp | q \rangle \langle k+1 \widehat{mQ}^\perp | q \rangle} \\ &+ \frac{\langle \ell_2^\perp q \rangle^3 \langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle^2 [\ell_2^\perp k+1]^2}{\langle q \widehat{mQ}^\perp | q \rangle^2 \langle q \widehat{mQ}^\perp | \ell_2^\perp \rangle} \\ &= \frac{1}{\langle q \widehat{mQ}^\perp | q \rangle^2} \left(\frac{-\mu^2 \langle \ell_2^\perp q \rangle \langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle^2 \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle}{\langle k+1 \ell_2^\perp \rangle \langle k+1 \widehat{mQ}^\perp | q \rangle} \right. \\ &\quad \left. - \frac{\langle \ell_2^\perp q \rangle^3 \langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle^2 [\ell_2^\perp k+1]^2}{\langle q \widehat{mQ}^\perp | \ell_2^\perp \rangle} \right) \\ &+ \frac{1}{\langle q \widehat{mQ}^\perp | q \rangle} \left(\frac{\mu^2 \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_2^\perp q \rangle \langle \ell_1^\perp \widehat{mQ}^\perp | q \rangle \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle}{\langle k+1 \ell_2^\perp \rangle \langle k+1 \widehat{mQ}^\perp | q \rangle} \right. \\ &\quad \left. + \frac{\mu^2 \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_1^\perp q \rangle \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle^2}{\langle k+1 \ell_2^\perp \rangle \langle k+1 \widehat{mQ}^\perp | q \rangle} \right) \\ &= \frac{-\mu^2 \langle \ell_1^\perp \ell_2^\perp \rangle^2 \langle \ell_2^\perp \widehat{mQ}^\perp | q \rangle}{\langle k+1 \ell_2^\perp \rangle \langle k+1 \widehat{mQ}^\perp | q \rangle} \\ &= \frac{\langle \ell_1^\perp \ell_2^\perp \rangle^2 \langle k \ell_2 | k+1 \rangle}{\langle k k+1 \rangle}. \end{aligned} \quad (3.24)$$

For the first equality we substitute (3.23) into (3.22). To get the second equality, we technically use the Schouten identity as (3.25) to factorize the last two terms,

$$\begin{aligned}
 & \langle \ell_1^\perp q \rangle \langle \ell_2^\perp | \widehat{mQ}^\perp | q \rangle \\
 &= \langle \ell_1^\perp \ell_2^\perp \rangle \langle q | \widehat{mQ}^\perp | q \rangle + \langle \ell_1^\perp | \widehat{mQ}^\perp | q \rangle \langle \ell_2^\perp q \rangle, \\
 & \langle \ell_2^\perp q \rangle \langle \ell_1^\perp | \widehat{mQ}^\perp | q \rangle \\
 &= \langle \ell_2^\perp \ell_1^\perp \rangle \langle q | \widehat{mQ}^\perp | q \rangle + \langle \ell_2^\perp | \widehat{mQ}^\perp | q \rangle \langle \ell_1^\perp q \rangle. \quad (3.25)
 \end{aligned}$$

And for the third equality in (3.24), we combine terms according to the power of $\langle q | \widehat{mQ} | q \rangle$. To get the fourth equality, we apply (3.25) and (3.12) technically to eliminate $\langle q | \widehat{mQ} | q \rangle$ and get the remaining term without $\langle q | \widehat{mQ} | q \rangle$. The final equality is easy to get using (3.12).

4 All-plus 1-loop integrand in $\mathcal{Q}$ -cut representation

After we get the general expressions for these tree amplitudes, we can use them to calculate the compact loop integrand for a color-ordered n -point gluon amplitude $\mathcal{A}_n^{1\text{-loop}}(1^+, 2^+, \dots, n^+)$ by the $\mathcal{Q}$ -cut construction.

Below we follow the calculation for $\mathcal{A}_4^{1\text{-loop}}(1^+, 2^+, 3^+, 4^+)$ in [2]. In the following steps, we find that the product $\mathcal{A}_L \mathcal{A}_R$ are independent of q , thus the loop integrand is also indepen-

According to (4.1), the $\mathcal{Q}$ -cut representation of the n -point all-plus 1-loop integrand is given by

$$\begin{aligned}
 \mathcal{I}^{\mathcal{Q}}(\tilde{\ell}) &= \sum_{k=2}^{n-2} \sum_{h_1, h_2} \mathcal{A}_L(1^+, 2^+, \dots, k^+, \widehat{\ell}_R^{h_1}, \\
 & \quad -\widehat{\ell}_L^{h_2}) \frac{1}{\tilde{\ell}^2} \frac{1}{(-2\tilde{\ell} \cdot p_L + p_L^2)} \mathcal{A}_R(\widehat{\ell}_L^{\bar{h}_2}, \\
 & \quad -\widehat{\ell}_R^{\bar{h}_1}, (k+1)^+, \dots, n^+) \\
 & \quad + \text{Cyclic}\{p_1, p_2, \dots, p_{n-1}, p_n\} \\
 &= \sum_{k=2}^{n-2} (\mathcal{A}_L(1^+, 2^+, \dots, k^+, \widehat{\ell}_R^+, -\widehat{\ell}_L^-) \mathcal{A}_R(\widehat{\ell}_L^-, \\
 & \quad -\widehat{\ell}_R^+, (k+1)^+, \dots, n^+) \\
 & \quad + \mathcal{A}_L(1^+, 2^+, \dots, k^+, \widehat{\ell}_R^-, \\
 & \quad -\widehat{\ell}_L^+) \mathcal{A}_R(\widehat{\ell}_L^+, -\widehat{\ell}_R^-, (k+1)^+, \dots, n^+) \\
 & \quad + \sum_{S_A} \mathcal{A}_L(1^+, 2^+, \dots, k^+, \widehat{\ell}_R^{S_A}, -\widehat{\ell}_L^{S_A}) \mathcal{A}_R(\widehat{\ell}_L^{S_A}, \\
 & \quad -\widehat{\ell}_R^{S_A}, (k+1)^+, \dots, n^+) \frac{1}{\tilde{\ell}^2} \frac{1}{(-2\tilde{\ell} \cdot p_L + p_L^2)} \\
 & \quad + \text{Cyclic}\{p_1, p_2, \dots, p_{n-1}, p_n\}. \quad (4.2)
 \end{aligned}$$

Writing the D -dimensional vector as $\widehat{\ell} = (\ell, \mu, \eta)$, we have

$$\begin{aligned}
 & \mathcal{A}_L(1^+, 2^+, \dots, k^+, \widehat{\ell}_R^+, -\widehat{\ell}_L^-) \mathcal{A}_R(\widehat{\ell}_L^+, -\widehat{\ell}_R^-, (k+1)^+, \dots, n^+) \\
 &= \frac{(\mu^2 + \eta^2)^2}{\langle 1 \ 2 \rangle \dots \langle k-1 \ k \rangle \langle k+1 \ k+2 \rangle \dots \langle n-1 \ n \rangle} \\
 & \quad \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 + \eta^2 - x_{i, \ell_L} x_{\ell_L, i+1}) | k] [k+1 | \prod_{i=k+2}^{n-1} (\mu^2 + \eta^2 - x_{i, \ell_R} x_{\ell_R, i+1}) | n]}{\prod_{i=2}^k (\mu^2 + \eta^2 + x_{\ell_L, i}^2) \prod_{i=k+2}^n (\mu^2 + \eta^2 + x_{\ell_R, i}^2)}. \quad (4.3)
 \end{aligned}$$

dent of q , which serves as a consistency check for our tree amplitudes.

The $\mathcal{Q}$ -cut representation of the loop integrand is

$$\begin{aligned}
 \mathcal{I}_n^{\mathcal{Q}}(\ell) &= \sum_{P_L} \sum_{h_1, h_2} \mathcal{A}_L(\dots, \widehat{\ell}_R^{h_1}, \\
 & \quad -\widehat{\ell}_L^{h_2}) \frac{1}{\tilde{\ell}^2} \frac{1}{(-2\tilde{\ell} \cdot p_L + p_L^2)} \mathcal{A}_R(\widehat{\ell}_L^{\bar{h}_2}, -\widehat{\ell}_R^{\bar{h}_1}, \dots), \quad (4.1)
 \end{aligned}$$

where $\widehat{\ell} = \alpha_L(\ell + \eta)$, $\widehat{\ell}_R \equiv \widehat{\ell}_L - P_L$ with $\alpha_L = P_L^2/(2\ell \cdot P_L) \neq 0$, $\eta^2 = \ell^2$. We review shortly that two deformations have been applied to the loop momentum ℓ : the first one is the dimensional deformation $\ell \rightarrow \ell + \eta$ with η in extra dimensions, and the second one is the scale deformation $\ell \rightarrow \alpha\ell$. The details of the 1-loop $\mathcal{Q}$ -cut construction was clarified in [2], and generalizations to two loops or more was illustrated in [1].

The remaining two diagrams that contribute are

$$\begin{aligned}
 & \mathcal{A}_L(\widehat{\ell}_R^+, -\widehat{\ell}_L^-) \mathcal{A}_R(\widehat{\ell}_L^+, -\widehat{\ell}_R^-) \text{ and } \mathcal{A}_L(\widehat{\ell}_R^{S_A}, -\widehat{\ell}_L^{S_A}) \\
 & \mathcal{A}_R(\widehat{\ell}_L^{S_A}, -\widehat{\ell}_R^{S_A}),
 \end{aligned}$$

which leads to exactly the same results.

Under the massless conditions of $\widehat{\ell}_L, \widehat{\ell}_R$, we can make the replacement $\eta^2 \rightarrow \tilde{\ell}^2$ where $\tilde{\ell} = (\ell, \mu)$, $\tilde{\ell} \rightarrow \alpha_L \tilde{\ell}$ and $\eta \rightarrow \alpha_L \eta$, as well as $\alpha_L = p_L^2/(2\tilde{\ell} \cdot p_L)$ in succession. After changing η , the general integrand in the $\mathcal{Q}$ -cut representation is

$$\begin{aligned}
\mathcal{I}^Q(\tilde{\ell}) = & (2 - 2\epsilon) \sum_{k=2}^{n-2} \frac{1}{(1\ 2) \cdots (k-1\ k)(k+1\ k+2) \cdots (n-1\ n)} \frac{(\mu^2 + \tilde{\ell}^2)^2 (p_L^2 / (2\tilde{\ell} \cdot p_L))^2}{\tilde{\ell}^2 (-2\tilde{\ell} \cdot p_L + p_L^2)} \\
& \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 + \tilde{\ell}^2 - x_{i,\ell_L} x_{\ell_L,i+1}) | k] [k+1 | \prod_{i=k+2}^{n-1} (\mu^2 + \tilde{\ell}^2 - x_{i,\ell_R} x_{\ell_R,i+1}) | n]}{\prod_{i=2}^k (\mu^2 + \tilde{\ell}^2 + x_{\ell_L,i}^2) \prod_{i=k+2}^n (\mu^2 + \tilde{\ell}^2 + x_{\ell_R,i}^2)} \\
& + \text{Cyclic}\{p_1, p_2, \dots, p_{n-1}, p_n\},
\end{aligned} \tag{4.4}$$

where we have summed over the helicity states in $(4 - 2\epsilon)$ dimensions (especially including the S_A components in $\dim[\mu] = (-2\epsilon)$ dimensions).

5 Summary

Our results for the series of color-ordered tree amplitudes are very concise. Each of these amplitudes shares a common structure where one part of the amplitude relies on the helicity difference of the pair of legs with extra-dimensional momenta and the other part containing pole structures is the same for each case. Our strategies to use the BCFW recursion relations are quite efficient to prove the general expressions for these amplitudes and reveal how different parts of the amplitudes evolve during the recursion.

Conversely, we emphasize that our results are also correct for the associated massive amplitudes, which can be used in the unitarity cut method for 1-loop amplitudes and to build superamplitudes on the Coulomb branch. Two of our results, $\{\ell_2^-, \ell_1^-\}$ and $\{\ell_2^-, \ell_1^a\}$, are completely new.

Using these tree amplitudes we successfully form the complete 1-loop all-plus integrand with any numbers of gluons, regardless of traditional integral reduction method. This complete integrand also has a good structure, despite the fact that it consists of cut constructive parts as well as rational parts, showing the power of the $\mathcal{Q}$ -cut construction.

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