Singlet structure function F_1 in double-logarithmic approximation

B. I. Ermolaev^{1,a}, S. I. Troyan²

¹ Ioffe Physico-Technical Institute, 194021 St.Petersburg, Russia

² St.Petersburg Institute of Nuclear Physics, 188300 Gatchina, Russia

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Abstract The conventional ways to calculate the perturbative component of the DIS singlet structure function F_1 involve approaches based on BFKL which account for the single-logarithmic contributions accompanying the Born factor $1/x$. In contrast, we account for the double-logarithmic (DL) contributions unrelated to $1/x$ and because of that they were disregarded as negligibly small. We calculate the singlet F_1 in the double-logarithmic approximation (DLA) and account at the same time for the running α_s effects. We start with a total resummation of both quark and gluon DL contributions and obtain the explicit expression for F_1 in DLA. Then, applying the saddle-point method, we calculate the small- x asymptotics of F_1 , which proves to be of the Regge form with the leading singularity $\omega_0 = 1.066$. Its large value compensates for the lack of the factor $1/x$ in the DLA contributions. Therefore, this Reggeon can be identified as a new Pomeron, which can be quite important for the description of all QCD processes involving the vacuum (Pomeron) exchanges at very high energies. We prove that the expression for the small- x asymptotics of F_1 scales: it depends on a single variable Q^2/x^2 only instead of x and Q^2 separately. Finally, we show that the small- x asymptotics reliably represent F_1 at $x \leq 10^{-6}$.

1 Introduction

A description of the singlet structure function F_1 in the framework of Collinear Factorization usually involves DGLAP [1–4] to calculate the perturbative contributions. In this case F_1 is represented in the form of two convolutions:

$$F_1 = C_q(x/y) \otimes \Delta q(y, Q^2) + C_g(x/y) \otimes \Delta g(y, Q^2), \quad (1)$$

where C_q and C_g are the coefficient functions and Δq and Δg denote the evolved (with respect to Q^2) quark and gluon dis-

tributions, respectively. These distributions are solutions to the DGLAP equations which govern the Q^2 -evolution of the initial quark and gluon distributions $\delta q(x, \mu^2)$ and $\delta g(x, \mu^2)$, evolving them from the scale μ^2 to Q^2 . Both δq and δg are defined at $x \sim 1$ and $Q^2 = \mu^2 \sim 1 \text{ GeV}^2$. The parameter μ is also called the factorization scale. The x -dependence of F_1 is described by the coefficient functions $C_{q,g}$ as well as by the phenomenological factors in $\delta q, \delta g$. In the framework of DGLAP the evolution in the $k_\perp$ -space is separated from the evolution with respect to x . Such a separation takes place at $x \sim 1$ only and breaks at small x as was shown in Refs. [5–7]. It is the theoretical reason not to use DGLAP at small x . A practical reason is that DGLAP, by its design, accounts for the total resummation of $\ln^n Q^2$, while contributions $\sim \ln^n x$ are present in the DGLAP expressions in few first orders in α_s only (through the coefficient functions in NLO, NNLO, etc.).

On the other hand, such contributions are very important at small x , so it would be appropriate to substitute the DGLAP expressions for the DIS structure functions by new ones which include the total resummation of all double-logarithmic (DL) contributions. In the first place there are DL terms $\sim (\alpha_s \ln^2(1/x))^n$, then the terms $\sim (\alpha_s \ln(1/x) \ln Q^2)^n$, etc. Expressions accounting for resummation of DL contributions and for the running α_s effects were obtained for several structure functions with non-vacuum exchanges in the t -channel: the spin structure function g_1 (the singlet and non-singlet components) and the non-singlet component of F_1 (see the overview of [8] and the references therein). Besides, the expressions were obtained for g_1 and non-singlet F_1 combining the DGLAP results and resummation of the DL contributions, which made it possible to apply these expressions at arbitrary x and Q^2 .

However, a similar generalization of DGLAP was not obtained for the singlet F_1 . The point is that by that time F_1 in the small- x region has been intensively investigated in terms of approaches based on BFKL [9–13] and this looked as the only way to study F_1 at small x . Indeed, the lead-

^a e-mail: boris.ermolaev@cern.ch

ing x -dependent contributions to F_1 proved to be the single-logarithmic (SL) terms accompanying the “Born” factor $1/x$:

$$(1/x)[1 + c_1\alpha_s \ln(1/x) + c_2(\alpha_s \ln(1/x))^2 + \dots], \quad (2)$$

while the DL contributions proportional to $1/x$, i.e. the terms

$$(1/x)[1 + c_1^{DL}\alpha_s \ln^2(1/x) + c_2^{DL}(\alpha_s \ln(1/x))^4 + \dots], \quad (3)$$

cancel each other (i.e. $c_k^{DL} = 0$ for $k = 1, 2, \dots$) as was found first in Ref. [14]. As a result, the common strategy for investigating the QCD processes with vacuum exchanges in the t -channel was based on the use of the BFKL results. In particular, SL contributions to the structure functions $F_{1,2}$ was presented in Refs. [15–17]; SL contributions to F_2 in Refs. [18, 19] were calculated with inclusion of the resummed anomalous dimensions in the renormalization group equation, while F_2 in Ref. [20] was calculated with direct unification of DGLAP and BFKL.

The solution to the BFKL equation is expressed through the series of the high-energy asymptotics of the Regge form, with the leading asymptotics commonly addressed as the BFKL Pomeron, so at $x \rightarrow 0$

$$F_1 \sim x^{-(1+\Delta_P)}, \quad (4)$$

where Δ_P is the Pomeron intercept. As $\Delta_P > 0$ for both the LO and the NLO BFKL Pomerons, they are called supercritical. As we are not going to use BFKL or its modifications like [21, 22] in the present paper, we just mention that extensive literature on this issue can be found in Ref. [23].

Instead of using the BFKL results or trying to increase the accuracy of the method of Refs. [16, 17], in the present paper we account for total resummation of the double-logarithmic contributions to F_1 . In the first place we account for the x -dependent contributions

$$1 + c'_1\alpha_s \ln^2(1/x) + c'_2(\alpha_s \ln^2(1/x))^2 + \dots \quad (5)$$

and then for DL terms combining logs of x and Q^2 . These DL contributions do not involve the large factor $1/x$ and for this reason they have been neglected in the BFKL approach. We calculate the singlet structure function F_1 in DLA, summing DL contributions coming from virtual gluon and quark exchanges. As a result, our expressions for coefficient functions and anomalous dimensions contain total resummations of appropriate DL terms. To calculate F_1 we compose and solve Infra-Red Evolution Equations (IREEs) in the same way as we did for calculating the DIS singlet structure function g_1 (see Ref. [8]), investigating the cases of fixed and running α_s . We recall that the IREE method was suggested by Lipatov in Refs. [24–27]. It is based on factorization of DL contributions of the partons with minimal transverse

momenta first noticed by Gribov in Ref. [28] in the context of QED of hadrons. The technology of implementation of this method to DIS is described in detail in Ref. [8]. In contrast to DGLAP and BFKL equations, we compose the two-dimensional evolution equations: They control the evolutions in both x and Q^2 . We obtain the explicit expression for F_1 and then, applying the saddle-point method, we calculate the small- x asymptotics of F_1 automatically complemented by the asymptotic Q^2 -dependence. The asymptotics proves to be of the Regge form. The large value of the intercept compensates for the lack of the factor $1/x$ in the DL contributions and thereby renders the DLA asymptotics of the same order as the BFKL one. This proves that the DL contributions to F_1 at small x are, at least, no less important than the contributions coming from the BFKL Pomeron.

Our paper is outlined as follows: in Sect. 2 we compose and solve IREE for the Compton amplitudes $A_{q,g}$ related to F_1 by the optical theorem. In this section we express $A_{q,g}$ through the amplitudes of the $2 \rightarrow 2$ scattering of partons. Those amplitudes are calculated in Sect. 3. In Sect. 4 we apply the saddle-point method to obtain an explicit expression for the small- x asymptotics of F_1 and prove that this asymptotics depends on the single variable Q^2/x^2 instead of depending separately on Q^2 and x . In Sect. 5 we consider in detail the intercept of the Pomeron in DLA, embracing the cases of fixed and running α_s . We also fix the region where the small- x asymptotics can reliably represent F_1 . Finally, Sect. 6 is for our concluding remarks.

2 IREE for the amplitudes of Compton scattering off partons

Following the DGLAP pattern, we consider F_1 in the framework of Collinear factorization and represent F_1 through the convolutions of the perturbative components T_q and T_g with non-perturbative initial quark and gluon distributions $\phi_{q,g}$, respectively:

$$F_1 = F_1^q \otimes \phi_q + F_1^g \otimes \phi_g. \quad (6)$$

Throughout the paper we will consider the perturbative objects $F_1^{q,g}$ only. It is convenient to consider the Compton amplitudes A_q and A_g related to $T_1^{q,g}$ by the optical theorem:

$$F_1^{q,g}(x, Q^2/\mu^2) = -\frac{1}{2\pi} \Im A_{q,g}(x, Q^2/\mu^2), \quad (7)$$

where we have introduced the factorization scale μ and used the standard notation $x = Q^2/w$, with $w = 2pq$ and $Q^2 = -q^2$. The next step is to represent $A_{q,g}$ in terms of the Mellin transform:

$$\begin{aligned}
A_{q,g}(w/\mu^2, Q^2/\mu^2) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} (w/\mu^2)^\omega \xi^{(+)}(\omega) F_{q,g}(\omega, Q^2/\mu^2) \\
&\approx \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{\omega\rho} F_{q,g}(\omega, y), \quad (8)
\end{aligned}$$

where we have introduced the signature factor $\xi^{(+)}(\omega) = (1 + e^{-i\omega})/2 \approx 1$ and the logarithmic variables ρ, y (using the standard notation $w = 2pq$):

$$\rho = \ln(w/\mu^2), \quad y = \ln(Q^2/\mu^2). \quad (9)$$

In what follows we will call F_q, F_g Mellin amplitudes and will use the same form of the Mellin transform for the other amplitudes as well. For instance, the Mellin transform for the color singlet amplitude A_{gg} of the elastic gluon–gluon scattering in the forward kinematics is

$$\begin{aligned}
A_{gg} &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} (w/\mu^2)^\omega \xi^{(+)}(\omega) f_{gg}(\omega) \\
&\approx \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{\omega\rho} f_{gg}(\omega). \quad (10)
\end{aligned}$$

We have presumed in Eq. (10) that the virtualities of all external gluons are $\sim \mu^2$. Let us notice that the only difference between the Mellin representation for the Compton amplitudes $A_{q,g}$ and similar amplitudes related to the singlet g_1 is in the signature factors only: the signature factor for g_1 is $\xi^{(-)}(\omega) = (-1 + e^{-i\omega})/2$. Otherwise, the technology of composing and solving IREE for A_{gg} and singlet g_1 is the same. Because of that, we present IREE for F_q, F_g (and for auxiliary amplitudes as well) with short comments only. The full-length derivation of all involved IREE can be found in Ref. [8]. Now all is set up to construct IREEs for $F_{q,g}$. In the kinematics where

$$w \gg Q^2 \gg \mu^2, \quad (11)$$

the amplitudes F_q, F_g obey the partial differential equations:

$$\begin{aligned}
[\partial/\partial y + \omega] F_q(\omega, y) &= F_q(\omega, y) h_{qq}(\omega) \\
&\quad + F_g(\omega, y) h_{gq}(\omega), \\
[\partial/\partial y + \omega] F_g(\omega, y) &= F_q(\omega, y) h_{qg}(\omega) \\
&\quad + F_g(\omega, y) h_{gg}(\omega), \quad (12)
\end{aligned}$$

where we have used the following convenient notation:

$$h_{rr'} = \frac{1}{8\pi^2} f_{rr'}, \quad (13)$$

with $r, r' = q, g$ and $f_{rr'}$ being the parton–parton amplitudes. We will calculate $h_{rr'}$ in the next section. Actually, the equations in (12) manifest a strong resemblance with the

DGLAP equations. Indeed, the first factor in brackets in the l.h.s. of (12) exists in DGLAP too. The second term vanishes when the Mellin factor $(s/\mu^2)^\omega$ is replaced by the factor $x^{-\omega}$, which is used in the DGLAP equations. When the parton amplitudes $f_{rr'}$ are in the Born approximation, Eq. (12) coincides with the DGLAP equations. A general solution to Eq. (12) is

$$\begin{aligned}
F_q(\omega, y) &= e^{-\omega y} [C_{(+)} e^{\Omega_{(+)} y} + C_{(-)} e^{\Omega_{(-)} y}], \\
F_g(\omega, y) &= e^{-\omega y} \left[C_{(+)} \frac{h_{gg} - h_{qq} + \sqrt{R}}{2h_{qg}} e^{\Omega_{(+)} y} \right. \\
&\quad \left. + C_{(-)} \frac{h_{gg} - h_{qq} - \sqrt{R}}{2h_{qg}} e^{\Omega_{(-)} y} \right], \quad (14)
\end{aligned}$$

where $C_{(\pm)}(\omega)$ are arbitrary factors whereas

$$\Omega_{(\pm)} = \frac{1}{2} [h_{gg} + h_{qq} \pm \sqrt{R}] \quad (15)$$

and

$$\begin{aligned}
R &= (h_{gg} + h_{qq})^2 - 4(h_{qg}h_{gg} - h_{qg}h_{gq}) \\
&= (h_{gg} - h_{qq})^2 + 4h_{qg}h_{gq}. \quad (16)
\end{aligned}$$

We specify the factors $C_{(\pm)}(\omega)$ by the matching with the Compton amplitudes f_q, f_g calculated in the kinematics $Q^2 \approx \mu^2$, i.e. at $y = 0$. The matching condition is

$$F_q(\omega, y)|_{y=0} = f_q(\omega), \quad F_g(\omega, y)|_{y=0} = f_g(\omega), \quad (17)$$

which leads to the following expressions:

$$\begin{aligned}
C_{(+)} &= \frac{h_{qg}f_g(\omega) - (h_{gg} - h_{qq} - \sqrt{R})f_q(\omega)}{2\sqrt{R}}, \\
C_{(-)} &= \frac{-h_{qg}f_g(\omega) + (h_{gg} - h_{qq} + \sqrt{R})f_q(\omega)}{2\sqrt{R}}. \quad (18)
\end{aligned}$$

Now let us express f_q, f_g through the parton–parton amplitudes $h_{rr'}$. To this end, we construct IREE for them. As f_q, f_g do not depend on Q^2 , the IREE for them are algebraic:

$$\begin{aligned}
\omega f_q(\omega) &= a_{\gamma q} + f_q(\omega) h_{qq}(\omega) + f_g(\omega) h_{gq}(\omega), \\
\omega f_g(\omega) &= f_q(\omega) h_{qg}(\omega) + f_g(\omega) h_{gg}(\omega), \quad (19)
\end{aligned}$$

where $a_{\gamma q} = e^2$, with e^2 being the total electric charge of the involved quarks, so that $a_{\gamma q}/\omega$ is the Born value of amplitude $f_q(\omega)$. There is no similar term in the equation for $f_g(\omega)$. The only difference between the r.h.s. of (19) and (12) is the factor $a_{\gamma q}$ in Eq. (19). The solution to Eq. (19) is

$$\begin{aligned}
f_q(\omega) &= a_{\gamma q} \frac{(\omega - h_{gg})}{G(\omega)}, \\
f_g(\omega) &= a_{\gamma q} \frac{h_{qg}}{G(\omega)}, \quad (20)
\end{aligned}$$

with $G(\omega)$ being the determinant of the system (19):

$$G = (\omega - h_{qq})(\omega - h_{gg}) - h_{gg}h_{qq}. \quad (21)$$

Combining Eqs. (18) and (20), we express $C_{\pm}$ through the parton-parton amplitudes:

$$\begin{aligned} C_{(+)} &= a_{\gamma q} \frac{h_{qg}h_{gq} - (\omega - h_{gg})(h_{gg} - h_{qq} - \sqrt{R})}{2G\sqrt{R}}, \\ C_{(-)} &= a_{\gamma q} \frac{-h_{qg}h_{gq} + (\omega - h_{gg})(h_{gg} - h_{qq} + \sqrt{R})}{2G\sqrt{R}}. \end{aligned} \quad (22)$$

Combining Eqs. (22, 15) and (14), we can easily express $F_{q,g}$ in terms of the parton-parton amplitudes $h_{rr'}$.

3 Parton-parton amplitudes

In this section we obtain explicit expressions for the parton amplitudes $h_{rr'}$. The IREE for $h_{rr'}$ are quite similar to Eq. (19):

$$\begin{aligned} \omega h_{qq} &= b_{qq} + h_{qq}h_{qq} + h_{qg}h_{gq}, \\ \omega h_{qg} &= b_{qg} + h_{qq}h_{qg} + h_{qg}h_{gg}, \\ \omega h_{gq} &= b_{gq} + h_{gq}h_{qq} + h_{gg}h_{gq}, \\ \omega h_{gg} &= b_{gg} + h_{gq}h_{qg} + h_{gg}h_{gg}, \end{aligned} \quad (23)$$

where the terms $b_{rr'}$ include the Born factors $a_{rr'}$ and contributions of non-ladder graphs $V_{rr'}$:

$$b_{rr'} = a_{rr'} + V_{rr'}. \quad (24)$$

The Born factors are (see Ref. [8] for details):

$$\begin{aligned} a_{qq} &= \frac{A(\omega)C_F}{2\pi}, \quad a_{qg} = \frac{A'(\omega)C_F}{\pi}, \quad a_{gq} \\ &= -\frac{A'(\omega)n_f}{2\pi}, \quad a_{gg} = \frac{2NA(\omega)}{\pi}, \end{aligned} \quad (25)$$

where A and A' stand for the running QCD couplings:

$$\begin{aligned} A &= \frac{1}{b} \left[\frac{\eta}{\eta^2 + \pi^2} - \int_0^\infty \frac{dz e^{-\omega z}}{(z + \eta)^2 + \pi^2} \right], \\ A' &= \frac{1}{b} \left[\frac{1}{\eta} - \int_0^\infty \frac{dz e^{-\omega z}}{(z + \eta)^2} \right], \end{aligned} \quad (26)$$

with $\eta = \ln(\mu^2/\Lambda_{QCD}^2)$ and b being the first coefficient of the Gell-Mann-Low function. When the running effects for the QCD coupling are neglected, $A(\omega)$ and $A'(\omega)$ are replaced by α_s . The terms $V_{rr'}$ are represented in a similar albeit more involved way (see Ref. [8] for details):

$$V_{rr'} = \frac{m_{rr'}}{\pi^2} D(\omega), \quad (27)$$

with

$$\begin{aligned} m_{qq} &= \frac{C_F}{2N}, \quad m_{gg} = -2N^2, \quad m_{gq} \\ &= n_f \frac{N}{2}, \quad m_{qg} = -NC_F, \end{aligned} \quad (28)$$

and

$$\begin{aligned} D(\omega) &= \frac{1}{2b^2} \int_0^\infty dz e^{-\omega z} \ln((z + \eta)/\eta) \left[\frac{z + \eta}{(z + \eta)^2 + \pi^2} \right. \\ &\quad \left. - \frac{1}{z + \eta} \right]. \end{aligned} \quad (29)$$

Let us note that $D = 0$ when the running coupling effects are neglected. It corresponds to the total compensation of DL contributions of non-ladder Feynman graphs to scattering amplitudes with the positive signature as was first noticed in Ref. [29]. When α_s is running, such a compensation is only partial. A solution to Eq. (23) is

$$\begin{aligned} h_{qq} &= \frac{1}{2} \left[\omega - Z - \frac{b_{gg} - b_{qq}}{Z} \right], \quad h_{qg} = \frac{b_{qg}}{Z}, \\ h_{gq} &= \frac{1}{2} \left[\omega - Z + \frac{b_{gg} - b_{qq}}{Z} \right], \quad h_{gg} = \frac{b_{gg}}{Z}, \end{aligned} \quad (30)$$

where

$$Z = \frac{1}{\sqrt{2}} \sqrt{Y + W}, \quad (31)$$

with

$$Y = \omega^2 - 2(b_{qq} + b_{gg}) \quad (32)$$

and

$$\begin{aligned} W &= \sqrt{(\omega^2 - 2(b_{qq} + b_{gg}))^2 - 4(b_{qq} - b_{gg})^2 - 16b_{qg}b_{gq}}. \end{aligned} \quad (33)$$

The algebraic equations (23) are non-linear, so they yield four expressions for Z . We selected in Eq. (31) the solution obeying the matching with the Born amplitudes $h_{rr'}^{\text{Born}}$: at large ω

$$h_{rr'} \rightarrow h_{rr'}^{\text{Born}} = a_{rr'}/\omega. \quad (34)$$

Substituting the expressions of Eq. (30) in (20), we obtain explicit expressions for the amplitudes f_q, f_g . Combining them with Eqs. (22), (15) and (14), we obtain explicit expressions for F_q and F_g . Substituting them in Eq. (8), we arrive at the explicit expressions for the Compton amplitudes A_q and A_g . Finally, applying the optical theorem (7) to A_q and A_g , we arrive at the singlet structure function F_1 .

4 Small- x asymptotics of the structure function F_1

The regular way to obtain the small- x asymptotics of A_q and A_g is to write explicit expressions for F_q and F_g in Eq. (8), then push $x \rightarrow 0$ and apply the saddle-point method. However, before doing this, let us consider in detail how to calculate the asymptotics of the gluon-gluon scattering amplitude A_{gg} , presuming virtualities of all external gluons $\sim \mu^2$.

4.1 Asymptotics of F_1

The small- x asymptotics of A_q and A_g can be obtained with applying the saddle-point method to Eq. (8). As $\Omega_{(+)} > \Omega_{(-)}$, we neglect the terms $C_{(-)}$ in (14) and represent Eq. (8) in the following form:

$$\begin{aligned} A_q &\approx \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} e^{\omega\xi} \tilde{F}_q(\omega) e^{\Omega_{(+)}y} = \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} e^{\Psi_q}, \\ A_g &\approx \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} e^{\omega\xi} \tilde{F}_g(\omega) e^{\Omega_{(+)}y} = \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} e^{\Psi_g}, \end{aligned} \quad (35)$$

with $\xi = \ln(1/x)$ and

$$\tilde{F}_q = C_{(+)}, \quad \tilde{F}_g = C_{(+)} \frac{(h_{gg} - h_{qq} + \sqrt{R})}{2h_{qg}} \quad (36)$$

and

$$\Psi_q = \omega\xi + \ln \tilde{F}_q, \quad \Psi_g = \omega\xi + \ln \tilde{F}_g. \quad (37)$$

The stationary point at $x \rightarrow 0$ of Ψ_q is given by the rightmost root ω_0 of the following equation:

$$d\Psi_q/d\omega = \xi + \frac{\tilde{F}'_q(\omega_0)}{\tilde{F}_q(\omega_0)} = 0. \quad (38)$$

When $\xi \rightarrow \infty$, it must be equated by some negative singular contribution in the second term of Eq. (38). Using the explicit formulas for $\tilde{F}_{q,g}$, one can conclude that such a contribution comes from the factor $1/W$. So, the stationary point ω_0 is the rightmost root of the equation

$$\begin{aligned} (\omega^2 - 2b_{qq} - 2b_{gg})^2 - 4(b_{qq} - b_{gg})^2 \\ - 16b_{qg}b_{gq} = 0. \end{aligned} \quad (39)$$

We consider in detail solutions to Eq. (39) at fixed and running α_s in the next section. In the vicinity of ω_0 we can represent Eq. (38) as

$$\Psi'_q = \xi + \frac{\partial \tilde{F}_q}{\tilde{F}_q} \frac{dW}{d\omega} = \xi + \frac{\partial \tilde{F}_q}{\tilde{F}_q} \frac{\lambda}{W} = \xi - \varphi \frac{\lambda}{W} = 0, \quad (40)$$

with

$$\varphi_q = -\partial \ln \tilde{F}_q / \partial W \quad (41)$$

and

$$\lambda = 2\omega(\omega^2 - 2(b_{+}) + b_{(-)}), \quad (42)$$

so in vicinity of the singularity ω_0

$$W \approx W_0 = \varphi \frac{\lambda}{\xi}. \quad (43)$$

Expanding Ψ_q in a series, we obtain

$$\Psi_q(\omega) \approx \Psi_q(\omega_0) + (1/2)\Psi''_q(\omega_0)(\omega - \omega_0)^2. \quad (44)$$

In order to calculate Ψ''_q we notice that the most singular contributions come from differentiation of the numerator in Eq. (40), so

$$\Psi''_q \approx -\frac{\partial \tilde{F}_q}{\tilde{F}_q} \frac{\lambda dW^{-1}}{d\omega} = \varphi \frac{\lambda^2}{W^3} = \frac{\xi^3}{\lambda \varphi_q^2} \quad (45)$$

and therefore the asymptotics of A_q at $x \rightarrow 0$ is

$$\begin{aligned} A_q(x, Q^2/\mu^2) &\sim A_q^{as}(x, Q^2/\mu^2) \\ &= C_{(+)}(\omega_0) \varphi_q(\omega_0) \sqrt{\frac{\lambda}{2\pi\xi^3}} \left(\frac{1}{x}\right)^{\omega_0} \left(\frac{Q^2}{\mu^2}\right)^{\Omega_{(+)}(\omega_0)}. \end{aligned} \quad (46)$$

Repeating the reasoning above for A_g and applying to them the optical theorem, we conclude that the small- x asymptotics of F_1 is

$$F_1 \sim \Pi(\omega_0, \xi) \left(\frac{1}{x}\right)^{\omega_0} \left(\frac{Q^2}{\mu^2}\right)^{\Omega_{(+)}(\omega_0)}, \quad (47)$$

where the factor $\Pi(\omega_0, \xi)$ is

$$\begin{aligned} \Pi(\omega_0, \xi) &= C_{(+)} \sqrt{\frac{\lambda}{2\pi\xi^3}} \left[\varphi_q \delta q \right. \\ &\quad \left. + \varphi_g \left(\frac{h_{gg}(\omega_0) - h_{qq}(\omega_0) + \sqrt{R(\omega_0)}}{2h_{qg}(\omega_0)} \right) \delta g \right], \end{aligned} \quad (48)$$

with δq and δg being the initial quark and gluon densities. They do not include singular factors $\sim x^{-a}$, with positive a . The Regge form of the asymptotics is brought about entirely by the perturbative contributions. Let us notice that $\Pi \sim \ln^{-3/2}(1/x)$. Equation (47) exhibits that the total resummation of DL contributions leads to the Regge behavior of F_1 at small x .

4.2 Asymptotic scaling

Substituting the explicit expressions for $h_{rr'}$ of Eq. (30) in Eq. (15) and using Eq. (39), we obtain $\Omega_{(+)}(\omega_0) = \omega_0/2$. This allows us to write the asymptotics of F_1 of Eq. (47) in the following way:

$$\begin{aligned} F_1 &\sim \Pi(\omega_0, \xi) \left(\frac{1}{x}\right)^{\omega_0} \left(\frac{Q^2}{\mu^2}\right)^{\omega_0/2} \\ &= \Pi(\omega_0, \xi) \left(\frac{Q^2}{x^2 \mu^2}\right)^{\omega_0/2}. \end{aligned} \quad (49)$$

Equation (49) manifests that $F_1(x, Q^2)$ at asymptotically high energies depends on the single variable Q^2/x^2 only. We name such a confluence of the x and Q^2 dependence an asymptotic scaling. The same form of asymptotic scaling was obtained earlier for the structure function g_1 and the non-singlet component of F_1 (see Ref. [8] for details). We stress that the asymptotic scaling for F_1 can be checked by an analysis of available experimental data. Moreover, $F_2 = 2xF_1$ at very small x , which proves the asymptotic scaling for F_2 . Finally, let us notice that the leading singularity ω_0 in Eq. (49) does not depend on Q^2 .

5 Anatomy of the leading singularity ω_0

In this section we consider in detail the leading singularity ω_0 , which is the rightmost root of Eq. (39). In order to make the asymptotics of F_1 look similar to Eq. (4), we denote

$$\omega_0 = 1 + \Delta, \quad (50)$$

so that ω_0 could look similar to the BFKL leading singularity; see Eq. (4). Now let us discuss different scenarios for calculating ω_0 . In what follows we will call Δ the DL Pomeron intercept. We recall that in the straightforward Reggeology concept $\Delta = 0$ and the Pomeron with $\Delta > 0$ is called the supercritical Pomeron.

5.1 Intercept under approximation of fixed QCD coupling

In the first place let us estimate ω_0 for the case of fixed α_s . In this case DL contributions of non-ladder graphs totally cancel each other, so that $D = 0$ and $b_{rr'} = a_{rr'}$, with $a_{rr'}$ defined in Eq. (25), where $A(\omega)$ and $A'(\omega)$ should be replaced by α_s . Then the solution to Eq. (39) is

$$\begin{aligned} \omega_0^{\text{fix}} &= \left(\frac{\alpha_s^{\text{fix}}}{\pi}\right)^{1/2} \left[4N + C_F \right. \\ &\quad \left. + \sqrt{(4N - C_F)^2 - 8n_f C_F} \right]^{1/2} \approx 2.63\sqrt{\alpha_s}, \end{aligned} \quad (51)$$

Fig. 1 Dependence of ω_0 (upper solid curve) and ω_1 (lower solid curve) on η . The dashed line shows the maximum value of $\omega_0 = 1.066$ reached at $\eta = 5.25$, which corresponds to $\mu/\Lambda = 13.8$

with the standard notations of the color factors $N = 3$, $C_F = (N^2 - 1)/(2N)$ and $n_f = 4$ is the flavor number. According to Ref. [30], in this case $\alpha_s^{\text{fix}} \approx 0.24$, which gives $\omega_0^{\text{fix}} = 1.29$. Using the representation of ω_0 of Eq. (50), we obtain

$$\Delta^{\text{fix}} = \omega_0^{\text{fix}} - 1 = 0.29, \quad (52)$$

which fairly coincides with the well-known LO BFKL intercept Δ_{LO} . However, Δ_{LO} corresponds to accounting for gluon contributions only, while Δ^{fix} accommodates both gluon and quark contributions. When the quark contributions in Eq. (51) are dropped, the purely gluonic intercept Δ_g^{fix} becomes somewhat greater:

$$\Delta_g^{\text{fix}} = 0.35, \quad (53)$$

which again bears a strong resemblance to the LO BFKL intercept. However, we are positive that the approximation of fixed α_s can be used for rough estimating only, so we will not pursue this approximation any longer.

5.2 Intercept for the case of running coupling

Now we account for the running coupling effects in Eq. (39). Because of that, Eq. (39) can be solved only numerically. As the couplings A and A' included in the factors $b_{rr'}$ depend on μ through $\eta = \ln(\mu^2/\Lambda^2)$, the solution, ω_0 , is also μ -dependent. Numerical calculations yield the plot of the η -dependence of ω_0 presented in Fig. 1.

The curve in Fig. 1 has the maximum $\omega_0^{\text{DLA}} = 1.066$ at $\mu/\Lambda = 13.8$. We call

$$\mu_0 = 13.8\Lambda \quad (54)$$

the optimal mass scale and call

$$\Delta = \omega_0^{\text{DLA}} - 1 = 0.066 \quad (55)$$

the intercept of the Pomeron in DLA. It is interesting to notice that Δ is close to the NLO BFKL intercept. In contrast, when the quark contributions are neglected, the purely gluonic intercept Δ_g^{DLA} is much greater:

$$\Delta_g = 0.254. \quad (56)$$

Confronting Eqs. (53)–(52) and (56)–(55) demonstrates that accounting for the quark contributions decreases the intercept. Similarly, confronting Eqs. (52)–(55) exhibits that accounting for the running α_s effects essentially decreases the intercept value. We also would like to stress that despite our values of Δ in Eqs. (53) and (50) being close to the values of the LO BFKL and NLO BFKL intercepts, respectively, this similarity is just a coincidence: our intercepts are obtained from resummation of DL contributions, while the BFKL sums the single-logarithmic terms. Moreover, Eq. (55) corresponds to the case of α_s running in every vertex of all involved Feynman graph, while BFKL operates with fixed α_s and includes the setting of its scale a posteriori.

5.3 Applicability region of the small- x asymptotics

It is obvious that the small- x asymptotic expressions, like Eq. (47), are always much simpler than non-asymptotic expressions. However, it is important to find at which values of x the asymptotics can reliably be used. To answer this question we numerically investigate R_{as} , defined as follows:

$$R_{as}(x, Q^2) = \frac{A_q^{as}(x, Q^2)}{A_q(x, Q^2)}. \quad (57)$$

The x -dependence of R_{as} at fixed Q^2 is shown in Fig. 2 for the case when $Q^2 \approx \mu^2$.

Fig. 2 Approach of A_q to its asymptotics A_q^{as} at fixed $Q^2 = \mu^2$

Fig. 3 Approach of A_q to its asymptotics A_q^{as} at fixed $Q^2 = 100\mu^2$

Figure 2 demonstrates that $R_{as} = 0.9$ at $x \approx 8 \cdot 10^{-5}$, while the curve in Fig. 3, where $Q^2 = 100\mu^2$, grows slower and reaches the value $R_{as} = 0.9$ much later, at $x \approx 3 \cdot 10^{-7}$.

Therefore, the applicability region of the small- x asymptotics essentially depends on the Q^2 value. The plots in Figs. 2 and 3 lead us to conclude that the small- x asymptotics reliably represent F_1 in the wide range of Q^2 when $x < x_{\max}$, with $x_{\max} \approx 10^{-6}$.

6 Summary and outlook

In this paper we have calculated the perturbative contributions F_1^q and F_1^g to the structure function F_1 in the Double-Logarithmic Approximation, by collecting the DL contributions and at the same time accounting for the running α_s effects. We obtained the explicit expressions for $F_1^{q,g}$ and then, applying the saddle-point method, calculated the small- x asymptotics of F_1 , arriving at a new DL contribution to the QCD Pomeron. We demonstrated that despite the lack of the factor $1/x$ in the DL contributions, the impact of their total resummation makes this Pomeron supercritical, albeit the value of the intercept strongly depends on the accuracy of the calculations. The maximal value of the intercept corresponds to the roughest approximation where the quark contributions are neglected and α_s is fixed. Then the value of the intercept decreases when the accuracy of the calculations increases: first, when the quark contributions are accounted for and then, notably, when the running α_s effects are taken into account. Nevertheless, the Pomeron remains supercritical as $\omega_0 = 1.066$. Such a monotonic decrease allows us to suggest that further accounting for sub-leading contributions can decrease the value of the intercept down to zero, so that eventually the intercept will satisfy the Froissart bound. We proved that the x - and Q^2 -dependencies of F_1 converge at small x in dependence on the single variable Q^2/x^2 . We call this convergence asymptotic scaling. We stress that this prediction of asymptotic scaling can be confirmed by an analysis of the available experimental data. As asymptotically

$F_2 \sim 2x F_1$, the asymptotic scaling should also take place for F_2 . Investigating the applicability region for the asymptotics, we found that F_1 can reliably be represented by its asymptotics at $x \leq x_{\max}$, with $x_{\max} \approx 10^{-6}$.

Although we have discussed the structure function F_1 , we would like to notice that the experimental data available in the literature are mostly on the structure functions F_2 and F_L , so it would be interesting to apply our approach to calculating F_2 and F_L as well. Calculating F_2 in DLA can be done in a way quite similar to that we have used for F_1 . As a result, we see that F_2 in DLA can be represented through F_1 :

$$F_2 = 2x F_1, \quad (58)$$

which coincides with the well-known Born relation between F_1 and F_2 . Equation (58) entails that in DLA $F_L = 0$. In order to estimate the deviation of F_L from zero, one should account for sub-leading contributions to both F_1 and F_2 . In the first place, such contributions are the single-logarithmic (SL) ones. In this regard we recall that the SL contributions to F_2 following from emission of gluons with momenta widely separated in rapidity and not ordered in transverse momenta were accounted for in Refs. [15–19], which involved dealing with the BFKL characteristic function. However, there are the SL contributions unrelated to BFKL, i.e., for the case of F_1 , the SL terms unaccompanied by the factor $1/x$, similar to the DL terms in Eq. (5). In contrast to the DL contributions (5), there is not a general technology in the literature for resummations of such non-BFKL SL terms. On the other hand, we were able to modify the IREE method for the spin structure function g_1 (see Ref. [8] and the references therein) to account for the SL contributions which are complementary to the ones calculated in Refs. [15–19], namely, the SL following from emission of the partons with momenta ordered in the $k_\perp$ -space and disordered in the longitudinal space. We plan to adapt this approach to calculating the SL contributions to $F_{1,2}$.

Finally, we stress that in contrast to DGLAP we do not need singular factors $\sim x^{-a}$ in fits for the initial parton distributions for F_1 . Such factors cause a steep rise of the structure functions at small x and lead to the Regge asymptotics of F_1 . However, we have shown in Sect. 4 that the resummation of the DL contributions to F_1 automatically leads to the Regge asymptotics, which makes an inclusion unnecessary of the singular terms into the fits. This result agrees with our earlier results (see Ref. [8]) for the non-singlet structure functions g_1 and F_1 and also with the results of Refs. [18, 19] obtained for the small- x behavior of the structure function F_2 . The latter agreement is especially interesting because the approaches used in Refs. [18, 19] and in the present paper are totally different.

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