Charged reflecting stars supporting charged massive scalar field configurations

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Abstract The recently published no-hair theorems of Hod, Bhattacharjee, and Sarkar have revealed the intriguing fact that horizonless compact reflecting stars *cannot* support spatially regular configurations made of scalar, vector and tensor fields. In the present paper we explicitly prove that the interesting no-hair behavior observed in these studies is not a generic feature of compact reflecting stars. In particular, we shall prove that charged reflecting stars *can* support *charged* massive scalar field configurations in their exterior spacetime regions. To this end, we solve analytically the characteristic Klein–Gordon wave equation for a linearized charged scalar field of mass μ , charge coupling constant q , and spherical harmonic index l in the background of a spherically symmetric compact reflecting star of mass M , electric charge Q , and radius $R_s \gg M, Q$. Interestingly, it is proved that the discrete set $\{R_s(M, Q, \mu, q, l; n)\}_{n=1}^{n=\infty}$ of star radii that can support the charged massive scalar field configurations is determined by the characteristic zeroes of the confluent hypergeometric function. Following this simple observation, we derive a remarkably compact analytical formula for the discrete spectrum of star radii in the intermediate regime $M \ll R_s \ll 1/\mu$. The analytically derived resonance spectrum is confirmed by direct numerical computations.

1 Introduction

Classical black-hole spacetimes are characterized by one-way membranes (event horizons) that irreversibly absorb matter and radiation fields. Event horizons are therefore characterized by purely ingoing boundary conditions, a remarkable physical property which naturally suggests, as nicely summarized by Wheeler’s no-hair conjecture [1–3], that asymptotically flat black holes cannot support spatially reg-

ular static field configurations in their exterior spacetime regions.

The no-hair conjecture for classical black-hole spacetimes has attracted much attention from physicists and mathematicians over the years. In particular, the elegant no-hair theorems presented in [4–14] have explicitly proved that asymptotically flat black holes *cannot* support physically acceptable (spatially regular) static configurations made of scalar, spinor, or vector fields¹ [15–44].

The characteristic absorbing property of the black-hole horizon has played a key role in the classical no-hair theorems [4–14]. It is therefore quite remarkable that a recently published theorem has extended the no-scalar hair property to the physically opposite regime of horizonless compact stars with *reflecting* (rather than *absorbing*) boundary conditions [45–47]. These reflecting stars describe gravitating physical objects whose compact surfaces are characterized by reflecting Dirichlet ($\Psi = 0$ at the compact surface) or Newmann ($d\Psi/dr = 0$ at the compact surface) boundary conditions [see Eq. (8) below]. Moreover, in a very interesting paper, Bhattacharjee and Sarkar [48] have recently extended the regime of validity of the no-hair theorem for horizonless spacetimes and proved that compact reflecting stars cannot support spatially regular configurations made of vector (spin-1) and tensor (spin-2) fields.

In the present paper we shall explicitly prove that the intriguing no-hair behavior observed in [45–48] is *not* a generic property of horizonless reflecting stars.² In partic-

¹ It is worth mentioning that, as explicitly shown in [15–44], spinning black holes can support spatially regular external configurations made of *stationary* massive scalar fields.

² It is well known that the compact surfaces (horizons) of black holes are characterized by purely ingoing (*absorbing*) boundary conditions. As explicitly revealed by the interesting no-hair theorem of Mayo and Bekenstein [8], asymptotically flat black holes with absorbing surfaces cannot support spatially regular static matter configurations made of charged scalar fields. The main goal of the present analysis

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ular, we shall use analytical techniques in order to show that spherically symmetric asymptotically flat charged stars with reflecting surfaces *can* support spatially regular static matter configurations made of linearized *charged* massive scalar fields.

Interestingly, below we shall prove that, for a spherically symmetric compact reflecting star of mass M and electric charge Q , there exists a *discrete* set $\{R_s(M, Q, \mu, q, l; n)\}_{n=1}^{n=\infty}$ of star radii that can support an external spatially regular charged massive scalar field of proper mass μ , charge coupling constant q , and spherical harmonic index l .^{3,4} In particular, we shall explicitly show that the physical properties of the composed charged-reflecting-star-linearized-charged-massive-scalar-field configurations can be studied *analytically* in the intermediate radii regime $M \ll R_s \ll 1/\mu$.

2 Description of the system

We shall analyze the physical and mathematical properties of an asymptotically flat spacetime which is composed of a central spherically symmetric reflecting star of radius R_s , mass M , and electric charge Q , which is linearly coupled to a static scalar field Ψ of proper mass μ and charge coupling constant q . The external spacetime of the charged star is described by the curved spherically symmetric line element [49]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \text{for } r \geq R_s, \quad (1)$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \quad (2)$$

Substituting the mathematical decomposition⁵

$$\Psi(r, \theta, \phi) = \sum_{lm} e^{im\phi} S_{lm}(\theta) R_{lm}(r) \quad (3)$$

Footnote 2 continued

is to explore the opposite regime of *reflecting* (rather than absorbing) compact objects. In particular, below we shall explicitly prove that, as opposed to black holes with absorbing boundary conditions, spatially compact stars with reflecting boundary conditions can support spatially regular charged matter configurations.

³ We shall use natural units in which $G = c = \hbar = k_B = 1$.

⁴ Note that the physical parameters μ and q , which characterize the charged massive scalar field [see Eq. (4) below], stand respectively for $\mu/\hbar$ and $q/\hbar$. These field parameters therefore have the dimensions of $(\text{length})^{-1}$.

⁵ Here the integer parameters l and m (with $-l \leq m \leq l$) are respectively the spherical harmonic index and the azimuthal harmonic index of the linearized charged massive scalar field mode.

for the static charged massive scalar field into the characteristic Klein–Gordon wave equation [50–57]

$$[(\nabla^\nu - iqA^\nu)(\nabla_\nu - iqA_\nu) - \mu^2]\Psi = 0 \quad (4)$$

(here $A_\nu = -\delta_\nu^0 Q/r$ is the electromagnetic potential of the spherically symmetric charged star), and using the metric components of the curved line element (1), one finds the ordinary differential equation [50–59]⁶

$$\frac{d}{dr} \left[r^2 f(r) \frac{dR_{lm}}{dr} \right] + \left[\frac{(qQ)^2}{f(r)} - (\mu r)^2 - l(l+1) \right] R_{lm} = 0 \quad (5)$$

for the radial part of the scalar eigenfunction. A simple inspection of the radial equation (5) for the charged massive scalar field in the curved spacetime of the spherically symmetric charged star reveals the fact that it is invariant under the symmetry transformation $qQ \rightarrow -qQ$. We shall henceforth assume without loss of generality the dimensionless relation

$$qQ > 0. \quad (6)$$

The spatially regular charged massive scalar field configurations that we shall analyze in the present paper are characterized by normalizable eigenfunctions that decay exponentially at radial infinity:

$$\Psi(r \rightarrow \infty) \sim r^{-1-\alpha} e^{-\mu r}, \quad (7)$$

where $\alpha \equiv M\mu$ is the dimensionless star-field mass parameter [see Eq. (15) below]. In addition, we shall assume that the central compact star is characterized by a reflecting surface of radius R_s . In particular, below we shall consider two types,

$$\begin{cases} \Psi(r = R_s) = 0 & \text{Dirichlet B. C.;} \\ d\Psi(r = R_s)/dr = 0 & \text{Neumann B. C.,} \end{cases} \quad (8)$$

of inner reflecting boundary conditions for the scalar field in the curved spacetime of the star.

The radial scalar equation (5), supplemented by the boundary conditions (7) and (8), determines two *discrete* spectra of radii, $\{R_s^{\text{Dirichlet}}(M, Q, \mu, q, l; n)\}_{n=1}^{n=\infty}$ and $\{R_s^{\text{Neumann}}(M, Q, \mu, q, l; n)\}_{n=1}^{n=\infty}$, which, for a given set $\{M, Q, \mu, q, l\}$ of the star-field physical parameters, characterize the charged reflecting stars that can support the spatially regular static

⁶ Here the integer $l(l+1)$ is the dimensionless eigenvalue of the spherical harmonic eigenfunction $S_{lm}(\theta)$ which characterizes the angular behavior of the spatially regular charged massive scalar field mode [58,59].

configurations of the charged massive scalar fields. Interestingly, in the next section we shall explicitly prove that, for charged reflecting stars in the large-radii regime

$$R_s \gg M, Q, \quad (9)$$

the ordinary differential equation (5) for the charged massive scalar fields in the background of the spherically symmetric charged star is amenable to an *analytical* treatment.

3 The resonance conditions of the composed charged-reflecting-star-charged-massive-scalar-field configurations

In the present section we shall use analytical techniques, which are valid in the large-radius regime (9), in order to derive the resonance conditions for the discrete sets of radii, $\{R_s^{\text{Dirichlet}}(M, Q, \mu, q, l; n)\}_{n=1}^{n=\infty}$ and $\{R_s^{\text{Neumann}}(M, Q, \mu, q, l; n)\}_{n=1}^{n=\infty}$, which characterize the composed charged-reflecting-star-linearized-charged-massive-scalar-field configurations.

It is convenient to define the new radial eigenfunction

$$\psi_{lm} = r f^{1/2}(r) R_{lm}. \quad (10)$$

Substituting (10) into (5) and neglecting terms of order $O(M/r^3, Q^2/r^4)$ [see (9)], one obtains the Schrödinger-like ordinary differential equation⁷

$$\frac{d^2\psi}{dr^2} + \left[-\mu^2 - \frac{2M\mu^2}{r} + \frac{(qQ)^2 - (4M^2 - Q^2)\mu^2 - l(l+1)}{r^2} \right] \psi = 0. \quad (11)$$

Next, defining the dimensionless radial coordinate

$$x = 2\mu r, \quad (12)$$

one can bring (11) into the familiar form

$$\frac{d^2\psi}{dx^2} + \left[-\frac{1}{4} - \frac{M\mu}{x} + \frac{(qQ)^2 - (4M^2 - Q^2)\mu^2 - l(l+1)}{x^2} \right] \psi = 0 \quad (13)$$

of the Whittaker radial differential equation (see Eq. 13.1.31 of [58]).

The general mathematical solution of the radial differential equation (13) is given by (see Eqs. 13.1.32 and 13.1.33 of [58]):

$$\psi(x) = e^{-\frac{1}{2}x} x^{\frac{1}{2}+i\beta} \left[A \cdot U\left(\frac{1}{2} + i\beta + \alpha, 1 + 2i\beta, x\right) + B \cdot M\left(\frac{1}{2} + i\beta + \alpha, 1 + 2i\beta, x\right) \right], \quad (14)$$

⁷ For brevity, we shall henceforth omit the angular harmonic indices $\{l, m\}$ of the static linearized charged massive scalar field modes.

where $\{A, B\}$ are normalization constants. Here $M(a, b, z)$ and $U(a, b, z)$ are respectively the confluent hypergeometric functions of the first and second kinds [58], and

$$\alpha \equiv M\mu; \quad \beta^2 \equiv (qQ)^2 - (4M^2 - Q^2)\mu^2 - \left(l + \frac{1}{2}\right)^2 \quad (15)$$

are dimensionless physical parameters which characterize the composed charged-star-charged-field system. We shall henceforth assume that⁸

$$\beta \in \mathbb{R}. \quad (16)$$

Using Eqs. 13.1.4 and 13.1.8 of [58], one finds that the spatial behavior of the radial scalar eigenfunction (14) at radial infinity is given by

$$\psi(x \rightarrow \infty) = A \cdot x^{-\alpha} e^{-\frac{1}{2}x} + B \cdot \frac{\Gamma(1 + 2i\beta)}{\Gamma(\frac{1}{2} + i\beta + \alpha)} x^{\alpha} e^{\frac{1}{2}x}. \quad (17)$$

Taking cognizance of the asymptotic boundary condition (7), which characterizes the physically acceptable (normalizable) eigenfunctions of the bound-state massive scalar fields in the curved spacetime of the central charged star, one concludes that the normalization constant of the unphysical (exponentially exploding) term in the asymptotic expression (17) should vanish:

$$B = 0. \quad (18)$$

We therefore find that the spatially regular bound-state configurations of the static charged massive scalar fields in the curved spacetime of the spherically symmetric charged massive star are characterized by the remarkably compact radial eigenfunction

$$\psi(x) = A \cdot e^{-\frac{1}{2}x} x^{\frac{1}{2}+i\beta} U\left(\frac{1}{2} + i\beta + \alpha, 1 + 2i\beta, x\right) \quad \text{for } R_s \gg M, Q. \quad (19)$$

Taking cognizance of the inner (Dirichlet/Neumann) boundary conditions (8), which characterize the behavior of the scalar fields at the reflecting compact surface of the spherically symmetric star, one obtains the two characteristic resonance equations

$$U\left(\frac{1}{2} + i\beta + \alpha, 1 + 2i\beta, 2\mu R_s\right) = 0 \quad \text{for Dirichlet B. C.} \quad (20)$$

⁸ We shall assume, without loss of generality, that $\beta > 0$.

and

$$\frac{d}{dr} \left[e^{-\mu r} r^{-\frac{1}{2} + i\beta} f^{-1/2}(r) U \left(\frac{1}{2} + i\beta + \alpha, 1 + 2i\beta, 2\mu r \right) \right]_{r=R_s} = 0 \quad \text{for Neumann B. C.} \quad (21)$$

for the composed charged-reflecting-star-charged-massive-scalar-field configurations⁹ [60,61].

Interestingly, as we shall explicitly show in the next section, the analytically derived resonance conditions (20) and (21) determine the characteristic sets of discrete star radii, $\{R_s^{\text{Dirichlet}}(M, Q, \mu, q, l; n)\}_{n=1}^{n=\infty}$ and $\{R_s^{\text{Neumann}}(M, Q, \mu, q, l; n)\}_{n=1}^{n=\infty}$, which can support the spatially regular bound-state configurations of the linearized static charged massive scalar fields.

4 The characteristic resonance spectra of the composed charged-reflecting-star-charged-massive-scalar-field configurations

The analytically derived resonance conditions (20) and (21), which determine the unique families of charged reflecting stars that can support the spatially regular static configurations of the linearized charged massive scalar fields, can easily be solved numerically. Interestingly, we find that, for a given set $\{M, Q, \mu, q, l\}$ of the physical parameters that characterize the composed charged-reflecting-star-charged-massive-scalar-field system and for a given reflecting inner boundary condition [see Eq. (8)], there exists a *discrete* set of star radii,

$$\cdots R_s(n=3) < R_s(n=2) < R_s(n=1) \\ \equiv R_{\text{max}}(M, Q, \mu, q, l), \quad (22)$$

which characterize the compact spherically symmetric charged reflecting stars that can support the bound-state configurations of the spatially regular charged massive scalar fields.

From the resonance equations (20) and (21) one learns that the dimensionless radii $\{\mu R_s(n)\}$ of the central supporting stars depend on the dimensionless star-field physical parameter α and β [see Eq. (15)]. In Table 1 we present, for various values of the dimensionless star-field mass parameter $\alpha \equiv M\mu$, the *largest* possible dimensionless radii $\{\mu R_{\text{max}}(\alpha)\}$ of the central charged reflecting stars that can support the spatially regular configurations of the static charged massive scalar fields. The data presented in Table 1 reveal the fact

Table 1 Composed charged-reflecting-star-charged-massive-scalar-field configurations. We present, for various values of the dimensionless physical parameter $\alpha \equiv M\mu$, the largest possible dimensionless radii $\mu R_{\text{max}}^{\text{Dirichlet}}(\alpha)$ of the spherically symmetric charged reflecting stars that can support the static spatially regular configurations of the charged massive scalar fields. One finds that the maximally allowed radii $\mu R_{\text{max}}^{\text{Dirichlet}}(\alpha)$ of the central supporting stars are a monotonically decreasing function of the star-field dimensionless mass parameter $\alpha \equiv M\mu$. The data presented is for the case $\beta = 5$ [see Eq. (15)]. Note that the assumption $R_s \gg M$ [see (9)] is valid for $\alpha \lesssim 0.25$

$\alpha \equiv M\mu$	0.05	0.1	0.15	0.20	0.25	0.30
$\mu R_{\text{max}}^{\text{Dirichlet}}$	2.3952	2.3662	2.3377	2.3096	2.2819	2.2546

Table 2 Composed charged-reflecting-star-charged-massive-scalar-field configurations. We present, for various values of the dimensionless physical parameter β [see Eq. (15)], the largest possible dimensionless radii $\mu R_{\text{max}}^{\text{Dirichlet}}(\beta)$ of the spherically symmetric charged reflecting stars that can support the spatially regular configurations of the static charged massive scalar fields. One finds that the maximally allowed radii $\mu R_{\text{max}}^{\text{Dirichlet}}(\beta)$ of the central supporting stars are a monotonically increasing function of the dimensionless star-field physical parameter β . The data presented is for the case $\alpha = 0.1$ [$M\mu = 0.1$, see Eq. (15)], which implies that the assumption $R_s \gg M$ is valid for $\beta \gtrsim 3$

β	1	3	5	7	9	11
$\mu R_{\text{max}}^{\text{Dirichlet}}$	0.0552	0.9803	2.3662	3.9205	5.5631	7.2611

that, for a fixed value of the dimensionless physical parameter β , the dimensionless star radii $\{\mu R_{\text{max}}(\alpha)\}$ are a monotonically decreasing function of the dimensionless star-field mass parameter $\alpha \equiv M\mu$. We find qualitatively similar results for the case of reflecting Neumann boundary conditions with the characteristic property $R_{\text{max}}^{\text{Neumann}}(\alpha) > R_{\text{max}}^{\text{Dirichlet}}(\alpha)$.

In Table 2 we present, for various values of the dimensionless physical parameter β [see Eq. (15)], the *largest* possible dimensionless radii $\{\mu R_{\text{max}}(\beta)\}$ of the spherically symmetric charged stars with reflecting Dirichlet boundary conditions that can support the spatially regular charged massive scalar field configurations. From the data presented in Table 2 one finds that, for a fixed value of the physical parameter α (which corresponds to a fixed value of the dimensionless star-field mass parameter $M\mu$), the dimensionless star radii $\{\mu R_{\text{max}}(\beta)\}$ are a monotonically increasing function of β . Again, one finds qualitatively similar results for the case of reflecting Neumann boundary conditions with the characteristic property $R_{\text{max}}^{\text{Neumann}}(\beta) > R_{\text{max}}^{\text{Dirichlet}}(\beta)$.

5 Analytical treatment of the composed star-field system in the intermediate radii regime $M \ll R_s \ll 1/\mu$

Interestingly, as we shall now prove explicitly, the resonance equations (20) and (21), which respectively determine the discrete sets of radii $\{R_s^{\text{Dirichlet}}(M, Q, \mu, q, l; n)\}_{n=1}^{n=\infty}$ and

⁹ It is worth noting that, in the dimensionless limit $\max\{M, Q\} \cdot \mu \rightarrow 0$ of a flat Minkowski spacetime with $\alpha = 0$ [see Eq. (15)], one can use the characteristic mathematical relation $U(1/2 + \nu, 1 + 2\nu, 2z) = \pi^{-1/2} e^z (2z)^{-\nu} K_\nu(z)$ (see Eq. 13.6.21 of [58]) in order to reduce the analytically derived resonance condition (20) to the flat-space resonance equation $K_{i\beta}(\mu R_s) = 0$ derived in [60,61] for the static case, where $K_\nu(z)$ is the modified Bessel function of the second kind [58].

$\{R_s^{\text{Neumann}}(M, Q, \mu, q, l; n)\}_{n=1}^{n=\infty}$ of the charged reflecting stars that can support the spatially regular bound-state configurations of the charged massive scalar fields, can be solved *analytically* in the intermediate radii regime

$$M \ll R_s \ll 1/\mu. \quad (23)$$

Using Eqs. 13.1.3 and 13.5.5 of [58], one can express the (Dirichlet) resonance condition (20) of the composed charged-star-charged-field configurations in the form

$$x^{2i\beta} = \frac{\Gamma(1+2i\beta)\Gamma(\frac{1}{2}-i\beta+\alpha)}{\Gamma(1-2i\beta)\Gamma(\frac{1}{2}+i\beta+\alpha)} \quad \text{for } M \ll R_s \ll 1/\mu. \quad (24)$$

Likewise, using Eqs. 13.1.3, 13.4.21, and 13.5.5 of [58], one can express the (Neumann) resonance condition (21) of the composed charged-star-charged-field configurations in the form

$$x^{2i\beta} = \frac{\Gamma(2+2i\beta)\Gamma(\frac{1}{2}-i\beta+\alpha)}{\Gamma(2-2i\beta)\Gamma(\frac{1}{2}+i\beta+\alpha)} \quad \text{for } M \ll R_s \ll 1/\mu. \quad (25)$$

From the resonance equations (24) and (25), which are valid in the intermediate radii regime (23), one obtains the remarkably compact dimensionless resonance spectra^{10, 11, 12}

$$\mu R_s^{\text{Dirichlet}}(n) = \frac{e^{-\pi n/\beta}}{2} \left[\frac{\Gamma(1+2i\beta)\Gamma(\frac{1}{2}-i\beta+\alpha)}{\Gamma(1-2i\beta)\Gamma(\frac{1}{2}+i\beta+\alpha)} \right]^{1/2i\beta}; \quad n \in \mathbb{Z} \quad (26)$$

and

$$\mu R_s^{\text{Neumann}}(n) = \frac{e^{-\pi n/\beta}}{2} \left[\frac{\Gamma(2+2i\beta)\Gamma(\frac{1}{2}-i\beta+\alpha)}{\Gamma(2-2i\beta)\Gamma(\frac{1}{2}+i\beta+\alpha)} \right]^{1/2i\beta}; \quad n \in \mathbb{Z} \quad (27)$$

for the *discrete* radii of the spherically symmetric charged reflecting stars that, for a given set of the star-field physical parameters $\{M, Q, \mu, q, l\}$, can support the spatially regular bound-state configurations of the static charged massive scalar fields.

¹⁰ Here we have used the relation $1 = e^{-2i\pi n}$, where the integer n is the resonance parameter which characterizes the composed charged-reflecting-star-linearized-charged-massive-scalar-field configurations.

¹¹ Since each inequality in the intermediate radii regime (23) roughly corresponds to an order-of-magnitude difference between two physical parameters (that is, $M/R_s \lesssim 10^{-1}$ and $\mu R_s \lesssim 10^{-1}$), the analytically derived resonance spectra (26) and (27) for the discrete star radii are expected to be valid in the small mass regime $M\mu \lesssim 10^{-2}$.

¹² It is worth noting that, using Eq. 6.1.23 of [58], one finds the characteristic dimensionless relations $\Gamma(1+2i\beta)/\Gamma(1-2i\beta) = e^{i\phi_1}$, $\Gamma(2+2i\beta)/\Gamma(2-2i\beta) = e^{i\phi_2}$, and $\Gamma(1/2-i\beta+\alpha)/\Gamma(1/2+i\beta+\alpha) = e^{i\phi_3}$ for $\beta \in \mathbb{R}$ [see Eq. (16)], where $\{\phi_1, \phi_2, \phi_3\} \in \mathbb{R}$. These relations imply that $\{\mu R_s^{\text{Dirichlet}}(n), \mu R_s^{\text{Neumann}}(n)\} \in \mathbb{R}$ [see Eqs. (26) and (27)].

6 Numerical confirmation

It is of physical interest to confirm the validity of the analytically derived discrete resonance spectra (26) and (27) which characterize the spherically symmetric charged reflecting stars that can support the spatially regular external configurations of the static charged massive scalar fields.

In Table 3 we present the dimensionless radii $\{\mu R_s^{\text{analytical}}(n)\}$ of the spherically symmetric charged stars with reflecting Dirichlet boundary conditions as obtained from the analytically derived resonance spectrum (26) in the intermediate radii regime $M \ll R_s \ll 1/\mu$ [see (23)]. We also present the corresponding dimensionless radii $\{\mu R_s^{\text{numerical}}(n)\}$ of the spherically symmetric supporting stars as obtained numerically from the Dirichlet resonance condition (20). From the data presented in Table 3 one finds a remarkably good agreement¹³ between the exact star radii [as computed numerically from the characteristic resonance condition (20)] and the corresponding approximated radii of the spherically symmetric compact stars [as calculated directly from the analytically derived discrete resonance spectrum (26)].

7 Summary and discussion

It is well known that asymptotically flat black holes, which are characterized by compact event horizons with purely ingoing (*absorbing*) boundary conditions, cannot support spatially regular static configurations made of scalar, spinor, or vector fields [1–14].

Intriguingly, it has recently been proved that horizonless compact stars with reflecting (that is, *repulsive* rather than *attractive*) boundary conditions cannot support spatially regular configurations made of neutral scalar (spin-0) fields [45–47]. Moreover, in a very interesting work, Bhattacharjee and Sarkar [48] have later extended the no-hair theorem of [45] and proved that horizonless compact stars with reflective boundary conditions cannot support spatially regular configurations made of vector (spin-1) and tensor (spin-2) fields.

In the present paper we have proved that the interesting no-hair behavior revealed in [45–48] for compact reflecting stars is *not* a generic feature of these horizonless objects. In particular, we have explicitly shown that spherically symmetric asymptotically flat *charged* stars with compact reflecting sur-

¹³ It is interesting to point out that the data presented in Table 3 reveal the fact that the agreement between the approximated (analytically derived) resonance formula (26) for the discrete spectrum of star radii and the corresponding numerically computed supporting star radii [as obtained numerically from the characteristic resonance condition (20)] is quite good already in the regime $\mu R_s = O(1)$. This observation is quite surprising since the analytically derived resonance spectra (26) and (27) are formally valid in the small mass-radii regime $\mu R_s \ll 1$ [see (23)].

Table 3 Composed charged-reflecting-star-charged-massive-scalar-field configurations with $\alpha \equiv M\mu = 0.01$ (see footnote 11) and $\beta = 10$. We display the discrete resonance spectrum $\{\mu R_s^{\text{analytical}}(n)\}$ of the dimensionless radii which characterize the spherically symmetric charged stars with reflecting Dirichlet boundary conditions as obtained from the analytically derived resonance formula (26) in the intermediate radii regime $M \ll R_s \ll 1/\mu$. We also display the correspond-

Formula	$\mu R(n = -5)$	$\mu R(n = -4)$	$\mu R(n = -3)$	$\mu R(n = -2)$	$\mu R(n = -1)$	$\mu R(n = 0)$	$\mu R(n = 1)$
Analytical [Eq. (26)]	2.2597	1.6505	1.2055	0.8805	0.6431	0.4697	0.3431
Numerical [Eq. (20)]	2.2899	1.6621	1.2101	0.8823	0.6438	0.4700	0.3432

faces (that is, with Dirichlet/Neumann reflecting boundary conditions) can support spatially regular bound-state configurations made of linearized charged massive scalar fields.

Interestingly, we have proved that, for a given set $\{M, Q, \mu, q, l\}$ of the star-field physical parameters, there exist two discrete spectra of radii, $\{R_s^{\text{Dirichlet}}(M, Q, \mu, q, l; n)\}_{n=1}^{\infty}$ and $\{R_s^{\text{Neumann}}(M, Q, \mu, q, l; n)\}_{n=1}^{\infty}$, which characterize the charged compact reflecting stars that can support the external bound-state charged massive scalar field configurations. In particular, it has been explicitly shown that the physical properties of the composed charged-reflecting-star-linearized-charged-massive-scalar-field configurations can be studied *analytically* in the regime $M \ll R_s \ll 1/\mu$ [see Eq. (23)]. In this intermediate radii regime we have used analytical techniques in order to derive the remarkably compact formula [see Eqs. (26) and (27)]

$$\mu R_s = \mathcal{R} \times e^{-\pi n/\beta}; \quad n \in \mathbb{Z} \quad (28)$$

with

$$\mathcal{R}(M, Q, \mu, q, l) = \frac{1}{2} \left[\frac{\Gamma(1 + 2i\beta)\Gamma(\frac{1}{2} - i\beta + \alpha)}{\Gamma(1 - 2i\beta)\Gamma(\frac{1}{2} + i\beta + \alpha)} \right]^{1/2i\beta};$$

$$\nabla = \begin{cases} 1 & \text{Dirichlet B. C.} \\ \frac{1+2i\beta}{1-2i\beta} & \text{Neumann B. C.} \end{cases} \quad (29)$$

for the discrete spectra of radii which characterize the spherically symmetric charged reflecting stars that can support the spatially regular charged massive scalar field configurations.

We have further shown that the analytically derived resonance formula (28) for the characteristic discrete spectra of compact star radii that can support the spatially regular charged massive scalar field configurations agrees remarkably well (see footnote 13) with direct numerical computations of the corresponding discrete star radii (see Table 3).

Finally, it is important to emphasize again that in the present study we have treated the spatially regular bound-state scalar configurations at the linear (perturbative) level. As we explicitly demonstrated in this paper, the main advantage of this small amplitude (linearized) treatment stems

from the fact that the physical properties of the composed charged-reflecting-star-charged-massive-scalar-field system can be studied *analytically* in the linear regime. We believe, however, that it would be physically interesting to use more sophisticated numerical techniques in order to explore the physical properties of these composed star-field configurations in the fully non-linear regime.

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