Stable exponential cosmological solutions with 3- and l -dimensional factor spaces in the Einstein–Gauss–Bonnet model with a Λ -term

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Abstract A D -dimensional gravitational model with a Gauss–Bonnet term and the cosmological term Λ is studied. We assume the metrics to be diagonal cosmological ones. For certain fine-tuned Λ , we find a class of solutions with exponential time dependence of two scale factors, governed by two Hubble-like parameters $H > 0$ and h , corresponding to factor spaces of dimensions 3 and $l > 2$, respectively and $D = 1 + 3 + l$. The fine-tuned $\Lambda = \Lambda(x, l, \alpha)$ depends upon the ratio $h/H = x$, l and the ratio $\alpha = \alpha_2/\alpha_1$ of two constants (α_2 and α_1) of the model. For fixed Λ , α and $l > 2$ the equation $\Lambda(x, l, \alpha) = \Lambda$ is equivalent to a polynomial equation of either fourth or third order and may be solved in radicals (the example $l = 3$ is presented). For certain restrictions on x we prove the stability of the solutions in a class of cosmological solutions with diagonal metrics. A subclass of solutions with small enough variation of the effective gravitational constant G is considered. It is shown that all solutions from this subclass are stable.

1 Introduction

In this paper we study a D -dimensional gravitational model with Gauss–Bonnet term and cosmological term Λ , i.e. we deal with the so-called Einstein–Gauss–Bonnet model (in short, EGB-, or more precisely EGB Λ -model). The so-called Gauss–Bonnet term appeared in string theory as a correction to the string effective action [1–5].

At the moment there is a certain interest to Einstein–Gauss–Bonnet (EGB) gravitational model and its modifica-

tions, see [6–30] and Refs. therein. They are intensively studied in cosmology, e.g. for possible explanation of accelerating expansion of the Universe which follow from supernovae (type Ia) observational data [31–33].

Here we consider the cosmological solutions with diagonal metrics. They are governed by $n = 3 + l > 5$ scale factors which depend upon the synchronous time variable. We deal with solutions which have exponential dependence of scale factors. We present a class of such solutions with two scale factors, which correspond to factor spaces of dimensions 3 and $l > 2$, and are described by two Hubble-like parameters $H > 0$ and h , respectively. Here the total dimension is $D = 1 + 3 + l$. Any of these solutions is presented in parametrized form: the cosmological constant Λ is fine-tuned, it depends upon the ratio $h/H = x$, l and a ratio two coupling constants. Any solution describes an exponential expansion of $3d$ factor space with Hubble parameter $H > 0$ [34].

Here we study the stability of the solutions in a class of cosmological solutions with diagonal metrics and single out a subclass of stable solutions. Our analysis is based on earlier results of Refs. [25, 26] (see also the approach of Ref. [23]).

We also consider a subclass of solutions which correspond to a small enough variation of the effective gravitational constant G in the Jordan frame [35, 36] (see also [37–39] and Refs. therein). We show that all these solutions are stable.

2 The setup

The action of the model has the following form

$$S = \int_M d^D z \sqrt{|g|} \{ \alpha_1 (R[g] - 2\Lambda) + \alpha_2 \mathcal{L}_2[g] \}, \quad (2.1)$$

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where $g = g_{MN} dz^M \otimes dz^N$ is a smooth metric defined on a smooth manifold M , $\dim M = D$, $|g| = |\det(g_{MN})|$, Λ is the cosmological term, $R[g]$ is scalar curvature and

$$\mathcal{L}_2[g] = R_{MNPQ} R^{MNPQ} - 4R_{MN} R^{MN} + R^2$$

is the standard Gauss–Bonnet term. Here α_1, α_2 are nonzero constants.

Our choice of the manifold is as follows

$$M = \mathbb{R} \times M_1 \times \cdots \times M_n. \quad (2.2)$$

We deal with the metric

$$g = -dt \otimes dt + \sum_{i=1}^n B_i e^{2v^i t} dy^i \otimes dy^i, \quad (2.3)$$

where $B_i > 0$ are constants, $i = 1, \dots, n$, and $M_1, \dots, M_n$ are one-dimensional manifolds (e.g. $\mathbb{R}$ or S^1) and $n > 3$, $D = n + 1$.

Equations of motion for the action (2.1) give us the set of polynomial equations [25]

$$G_{ij} v^i v^j + 2\Lambda - \alpha G_{ijkl} v^i v^j v^k v^l = 0, \quad (2.4)$$

$$\left[2G_{ij} v^j - \frac{4}{3} \alpha G_{ijkl} v^j v^k v^l \right] \sum_{i=1}^n v^i - \frac{2}{3} G_{ij} v^i v^j + \frac{8}{3} \Lambda = 0, \quad (2.5)$$

$i = 1, \dots, n$, where $\alpha = \alpha_2/\alpha_1$. Here we use the notations from Refs. [17, 18].

$$G_{ij} = \delta_{ij} - 1, \quad G_{ijkl} = G_{ij} G_{ik} G_{il} G_{jk} G_{jl} G_{kl} \quad (2.6)$$

which are, respectively, the components of two metrics (2-metric and 4-metric) on $\mathbb{R}^n$. For $n > 3$ we get a set of forth-order polynomial equations.

In what follows we deal with anisotropic solutions. The isotropic solutions with $v^1 = \dots = v^n = H, \alpha < 0$ (and $n > 3$) were considered in Refs. [17, 18] and [20] for $\Lambda = 0$ and $\Lambda \neq 0$, respectively. As it was shown in Refs. [17, 18] there are no more than three different numbers among $v^1, \dots, v^n$ when $\Lambda = 0$. This is valid also in the case $\Lambda \neq 0$, when the additional restriction $\sum_{i=1}^n v^i \neq 0$ is imposed [26].

3 Solutions with two Hubble-like parameters

In this section we deal with solutions to the set of equations (2.4), (2.5) of the following form:

$$v = \left(\underbrace{H, H, H}_{\text{"our" space}}, \underbrace{\overbrace{h, \dots, h}^l}_{\text{internal space}} \right). \quad (3.1)$$

where H is the Hubble-like parameter corresponding to the 3-dimensional factor space and h is the Hubble-like parameter corresponding to the l -dimensional factor space, $l > 2$.

We set

$$H > 0 \quad (3.2)$$

for a description of an accelerated expansion of the 3-dimensional subspace (which may describe our Universe). The evolution of the l -dimensional internal factor space is described by the Hubble-like parameter h .

It is widely known that the $4d$ effective gravitational constant $G = G_{eff}$ in the Brans–Dicke–Jordan (or simply Jordan) frame [35] (see also [36]) is proportional to the inverse volume scale factor of the internal space, see [37, 39] and references therein.

It follows from Ref. [26] (for a more general scheme see [21]) that if we consider the ansatz (3.1) with two Hubble-like parameters H and h obeying two restrictions imposed

$$3H + lh \neq 0, \quad H \neq h, \quad (3.3)$$

we may reduce relations (2.4) and (2.5) to the following set of equations

$$\begin{aligned} E = 3H^2 + lh^2 - (3H + lh)^2 + 2\Lambda \\ - \alpha [24lH^3h + 36l(l-1)H^2h^2 \\ + 12l(l-1)(l-2)Hh^3 \\ + l(l-1)(l-2)(l-3)h^4] = 0, \end{aligned} \quad (3.4)$$

$$Q = 2H^2 + 4(l-1)Hh + (l-1)(l-2)h^2 = -\frac{1}{2\alpha}. \quad (3.5)$$

Using Eq. (3.5) we get for $l > 2$

$$H = (-2\alpha\mathcal{P})^{-1/2}, \quad (3.6)$$

where

$$\mathcal{P} = \mathcal{P}(x, l) \equiv 2 + 4(l-1)x + (l-1)(l-2)x^2, \quad (3.7)$$

$$x \equiv h/H, \quad (3.8)$$

and

$$\alpha\mathcal{P} < 0. \quad (3.9)$$

Due to restrictions (3.3) we have for x from (3.8)

$$x \neq x_d = x_d(l) \equiv -3/l, \quad x \neq x_a \equiv 1. \quad (3.10)$$

The relation (3.5) is valid if

$$\mathcal{P}(x, l) \neq 0. \quad (3.11)$$

For $\mathcal{P}(x, l) = 0$ the Eq. (3.5) is not satisfied.

Substituting relation (3.6) into (3.4) we obtain

$$\Lambda\alpha = \lambda = \lambda(x, l) \equiv \frac{1}{4}(\mathcal{P}(x, l))^{-1}\mathcal{M}(x, l) + \frac{1}{8}(\mathcal{P}(x, l))^{-2}\mathcal{R}(x, l), \quad (3.12)$$

$$\mathcal{M}(x, l) \equiv 3 + lx^2 - (3 + lx)^2, \quad (3.13)$$

$$\mathcal{R}(x, l) \equiv 24lx + 36l(l-1)x^2 + 12l(l-1)(l-2)x^3 + l(l-1)(l-2)(l-3)x^4. \quad (3.14)$$

From (3.11) we get

$$x \neq x_{\pm} = x_{\pm}(l) \equiv \frac{-2(l-1) \pm \sqrt{\Delta(l)}}{(l-1)(l-2)}, \quad (3.15)$$

$$\Delta(l) \equiv 2(l-1)l, \quad (3.16)$$

where $x_{\pm}(l)$ are roots of the quadratic equation $\mathcal{P}(x, l) = 0$. They obey the identities

$$x_+(l)x_-(l) = \frac{2}{(l-1)(l-2)}, \quad (3.17)$$

$$x_+(l) + x_-(l) = -\frac{4}{l-2}, \quad (3.18)$$

which imply the following inequalities

$$x_-(l) < x_+(l) < 0. \quad (3.19)$$

It follows from (3.9) and (3.12) that

$$\Lambda = \alpha^{-1}\lambda(x, l), \quad (3.20)$$

where

$$x_-(l) < x < x_+(l) \text{ for } \alpha > 0 \quad (3.21)$$

and

$$x < x_-(l), \text{ or } x > x_+(l) \text{ for } \alpha < 0. \quad (3.22)$$

For $\alpha < 0$ we obtain

$$\lim_{x \rightarrow \pm\infty} \lambda(x, l) = \lambda_{\infty}(l) \equiv -\frac{l(l+1)}{8(l-1)(l-2)} < 0 \quad (3.23)$$

and hence

$$\lim_{x \rightarrow \pm\infty} \Lambda = \Lambda_{\infty} \equiv -\frac{l(l+1)}{8\alpha(l-1)(l-2)} > 0, \quad (3.24)$$

$l > 2$. For $x = 0$ we get

$$\Lambda = \Lambda_0 = \alpha^{-1}\lambda(0, l) = -\frac{3}{4\alpha} > 0, \quad (3.25)$$

which does not depend upon l . In this case the Hubble-like parameters read

$$H = H_0 = (-4\alpha)^{-1/2}, \quad h = 0 \quad (3.26)$$

and our ansatz (2.2), (2.3) gives us the product of (a part of) 4-dimensional de-Sitter space and l -dimensional Euclidean space.

Let us consider the behaviour of the function $\lambda(x, l)$ in the vicinity of the points $x_-(l)$ and $x_+(l)$. Here the following proposition is valid.

Proposition 1 For $l > 2$

$$\lambda(x, l) \sim B_{\pm}(l)(x - x_{\pm}(l))^{-2}, \quad (3.27)$$

as $x \rightarrow x_{\pm} = x_{\pm}(l)$, where $B_{\pm}(l) < 0$ and hence

$$\lim_{x \rightarrow x_{\pm}} \lambda(x, l) = -\infty. \quad (3.28)$$

In the proof of the Proposition 1 the following lemma is used.

Lemma For all $l > 2$

$$\mathcal{R}_{\pm}(l) \equiv \mathcal{R}(x_{\pm}(l), l) < 0. \quad (3.29)$$

The Lemma is proved in the Appendix A.

The proof of Proposition 1 By using the relation $\mathcal{P}(x, l) = (l-1)(l-2)(x-x_+)(x-x_-)$ and Lemma we are led to relation (3.27) with

$$B_{\pm}(l) = \frac{\mathcal{R}_{\pm}(l)}{8(l-1)^2(l-2)^2(x_+ - x_-)^2} = \frac{\mathcal{R}_{\pm}(l)}{64(l-1)l} < 0 \quad (3.30)$$

for $l > 2$. Relation (3.28) just follows from (3.27) and (3.30). The Proposition 1 is proved. $\square$

Now we study the behaviour of the function $\lambda(x, l)$ for fixed l and $x \neq x_{\pm}(l)$. First, we find the extremum points which obey $\frac{\partial}{\partial x}\lambda(x, l) = 0$. The calculations give us

$$\frac{\partial}{\partial x}\lambda(x, l) = -f(x, l)(\mathcal{P}(x, l))^{-3}, \quad (3.31)$$

$$f(x, l) = 2(l-1)(x-1)(lx+3) \times [(l-2)x+2][(l-1)x+1], \quad (3.32)$$

$x \neq x_{\pm}(l)$. By using these relations we find the following extremum points

$$x_a \equiv 1, \quad (3.33)$$

$$x_b = x_b(l) \equiv -\frac{2}{l-2} < 0, \quad (3.34)$$

$$x_c = x_c(l) \equiv -\frac{1}{l-1} < 0, \quad (3.35)$$

$$x_d = x_d(l) \equiv -\frac{3}{l} < 0. \quad (3.36)$$

We also obtain

$$x_b(l) < x_c(l) \quad (3.37)$$

since

$$x_c(l) - x_b(l) = \frac{l}{(l-1)(l-2)} > 0 \quad (3.38)$$

for all $l > 2$.

The points x_b, x_c, x_d from (3.34), (3.35), (3.36) belong to the interval (x_-, x_+) , i.e.

$$x_i(l) \in (x_-(l), x_+(l)), \quad (3.39)$$

$i = b, c, d$ for $l > 2$. This follows from relations $\mathcal{P}_i(l) = \mathcal{P}(x_i(l), l) < 0, i = b, c, d$. Indeed,

$$\mathcal{P}_b(l) = -\frac{2l}{l-2} < 0, \quad (3.40)$$

$$\mathcal{P}_c(l) = -\frac{l}{l-1} < 0, \quad (3.41)$$

$$\mathcal{P}_d(l) = -\frac{w(l)}{l^2} < 0, \quad (3.42)$$

for $l > 2$, where

$$w(l) = l^2 + 15l - 18 > 0. \quad (3.43)$$

Using relations

$$x_d - x_c = \frac{3-2l}{l(l-1)}, \quad (3.44)$$

$$x_d - x_b = \frac{6-l}{l(l-2)}, \quad (3.45)$$

we obtain

$$(A) \ x_b < x_d < x_c, \quad \text{for } 2 < l < 6, \quad (3.46)$$

$$(B) \ x_d < x_b < x_c, \quad \text{for } l > 6, \quad (3.47)$$

and

$$(B_0) \ x_d = x_b < x_c, \quad \text{for } l = 6. \quad (3.48)$$

Now we calculate $\lambda_i = \lambda(x_i, l), i = a, b, c, d$. We find

$$\lambda_a = -\frac{(l+2)(l+3)}{8l(l+1)} < 0, \quad (3.49)$$

$$\lambda_b = \frac{l^2 - 4l + 6}{4(l-2)l} > 0, \quad (3.50)$$

$$\lambda_c = \frac{3l^2 - 7l + 6}{8(l-1)l} > 0, \quad (3.51)$$

$$\lambda_d = \frac{3l(l+3)}{8w(l)} > 0, \quad (3.52)$$

for $l > 2$ [$w(l)$ is defined in (3.43)].

We also obtain

$$\lambda_b - \lambda_c = \frac{l(3-l)}{8(l-2)(l-1)} \begin{cases} = 0, & \text{if } l = 3, \\ < 0, & \text{if } l > 3, \end{cases} \quad (3.53)$$

and

$$\lambda_d - \lambda_c = \frac{(3-2l)^3}{2l(l-1)w(l)}, \quad (3.54)$$

$$\lambda_d - \lambda_b = \frac{(l-1)(l-6)^3}{8l(l-2)w(l)}, \quad (3.55)$$

for $l > 2$. Using these relations we find

$$\lambda_d - \lambda_c < 0, \quad (3.56)$$

for $l > 2$ and

$$\lambda_d - \lambda_b \begin{cases} > 0, & \text{if } l > 6, \\ = 0, & \text{if } l = 6, \\ < 0, & \text{if } 2 < l < 6. \end{cases} \quad (3.57)$$

Now we analyze the behaviour of the function $\lambda(x, l)$ with respect to x for fixed $l > 2$. We calculate $n(\Lambda, \alpha)$, which is the number of solutions (in variable x) of the relation $\Lambda\alpha = \lambda(x, l)$. In what follows we use relations (3.33), (3.34), (3.35), (3.36), (3.37), (3.39), (3.46), (3.47), (3.48), (3.49), (3.50), (3.51), (3.52), (3.53), (3.54), (3.55), (3.56), (3.57), corresponding to points of extremum x_i and λ_i ($i = a, b, c, d$) and relations (3.28), (3.32).

First, we consider the case $\alpha > 0$ and $x_- < x < x_+$. We keep in mind that the solution $x = x_d$ is excluded.

(A) $2 < l < 6$. We get $x_b < x_d < x_c$ and $\lambda_d < \lambda_c, \lambda_d < \lambda_b$. Points x_b, x_c are points of local maximum and x_d is a point of local minimum.

We split this case on two subcases: (A₀) $l = 3$ and (A₋) $3 < l < 6$.

Fig. 1 The function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$, $l = 3$

Fig. 2 The function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$, $l = 4$

(A₀) $l = 3$. In this subcase $\lambda_d < \lambda_c = \lambda_b$ and hence

$$n(\Lambda, \alpha) = \begin{cases} 0, & \Lambda\alpha > \lambda_b = \lambda_c, \\ 2, & \Lambda\alpha = \lambda_b = \lambda_c, \\ 4, & \lambda_d < \Lambda\alpha < \lambda_b = \lambda_c, \\ 2, & \Lambda\alpha = \lambda_d, \\ 2, & \Lambda\alpha < \lambda_d. \end{cases} \quad (3.58)$$

An example of the function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$ and $l = 3$ is depicted at Fig. 1. At this and other figures we mark by i the point (x_i, λ_i) , where $i = a, b, c, d$.

(A₋) $3 < l < 6$ (or $l = 4, 5$). In this subcase $\lambda_d < \lambda_b < \lambda_c$ and

$$n(\Lambda, \alpha) = \begin{cases} 0, & \Lambda\alpha > \lambda_c, \\ 1, & \Lambda\alpha = \lambda_c, \\ 2, & \lambda_b < \Lambda\alpha < \lambda_c, \\ 3, & \Lambda\alpha = \lambda_b, \\ 4, & \lambda_d < \Lambda\alpha < \lambda_b, \\ 2, & \Lambda\alpha = \lambda_d, \\ 2, & \Lambda\alpha < \lambda_d. \end{cases} \quad (3.59)$$

An example of the function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$ and $l = 4$ is depicted at Fig. 2.

(B) $l > 6$. We have $x_d < x_b < x_c$ and $\lambda_b < \lambda_d < \lambda_c$. Points x_c and x_d are points of local maximum (x_c is a point of maximum on interval (x_-, x_+)) and x_b is a point of local

minimum. We find

$$n(\Lambda, \alpha) = \begin{cases} 0, & \Lambda\alpha > \lambda_c, \\ 1, & \Lambda\alpha = \lambda_c, \\ 2, & \lambda_d < \Lambda\alpha < \lambda_c, \\ 2, & \Lambda\alpha = \lambda_d, \\ 4, & \lambda_b < \Lambda\alpha < \lambda_d, \\ 3, & \Lambda\alpha = \lambda_b, \\ 2, & \Lambda\alpha < \lambda_b. \end{cases} \quad (3.60)$$

An example of the function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$ and $l = 12$ is depicted at Fig. 3.

(B₀) $l = 6$. We have $x_d = x_b < x_c$ and $\lambda_b = \lambda_d < \lambda_c$. The point x_c is a point of maximum on interval (x_-, x_+) and $x_b = x_d$ is a point of inflection. We obtain

$$n(\Lambda, \alpha) = \begin{cases} 0, & \Lambda\alpha > \lambda_c, \\ 1, & \Lambda\alpha = \lambda_c, \\ 2, & \lambda_d < \Lambda\alpha < \lambda_c, \\ 1, & \Lambda\alpha = \lambda_d, \\ 2, & \Lambda\alpha < \lambda_d. \end{cases} \quad (3.61)$$

An example of the function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$ and $l = 6$ is depicted at Fig. 4. Thus, we see that for $\alpha > 0$ and small enough value of Λ there exist at least two solutions x_1, x_2 : $x_- < x_1 < x_2 < x_+ < 0$.

Now we consider the case $\alpha < 0$. Here we remember that the solution $x = x_d$ is excluded. We get $\Lambda(x)|\alpha| = -\lambda(x)$, where $x < x_-$ or $x > x_+$. According to the identities (3.23),

Fig. 3 The function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0, l = 12$

Fig. 4 The function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0, l = 6$

(3.32) and Proposition 1 the function $\Lambda(x)|\alpha| = -\lambda(x)$ is monotonically increasing: in the interval $(-\infty, x_-)$ from $-\lambda_\infty$ to $+\infty$ and in the interval $(x_a = 1, +\infty)$ from $-\lambda_a + 0$ to $-\lambda_\infty$. It is monotonically decreasing in the interval (x_+, x_a) from $+\infty$ to $-\lambda_a + 0$. Here x_a is a point of local minimum of the function $\Lambda(x)|\alpha| = -\lambda(x)$ and $-\lambda_a < -\lambda_\infty$. The last relation (inequality) may be verified explicitly. Indeed, due to relations (3.23), (3.49) we obtain

Fig. 5 The function $\Lambda(x) = \lambda(x)/\alpha$ for $\alpha = \pm 1, l = 3$

Fig. 6 The function $\lambda(x)$ for $\alpha < 0$ and $l = 3$

$$\lambda_a - \lambda_\infty = \frac{4(2l^2 + 2l - 3)}{8(l-1)(l-2)l(l+1)} > 0, \quad (3.62)$$

for $l > 2$. The graphical representation of functions $\Lambda(x) = \Lambda(x, \alpha) = \lambda(x)/\alpha$ for $\alpha = +1, -1$, respectively, and $l = 3$ is given at Fig. 5.

The function $\lambda(x)$ for $\alpha < 0$ and $l = 3$ is presented at Fig. 6.

For the number of solutions for $\alpha < 0$ we obtain

$$n(\Lambda, \alpha) = \begin{cases} 2, & \Lambda|\alpha| > |\lambda_\infty|, \\ 1, & \Lambda|\alpha| = |\lambda_\infty|, \\ 2, & |\lambda_a| < \Lambda|\alpha| < |\lambda_\infty|, \\ 0, & \Lambda|\alpha| \leq |\lambda_a|. \end{cases} \quad (3.63)$$

Here we use $x \neq x_a = 1$. Thus, for $\alpha < 0$ and big enough values of Λ there exist two solutions x_1, x_2 : $x_1 < x_- < 0$ and $x_2 > x_+$.

Master equation The Eq. (3.12) may be written in the following form

$$2\mathcal{P}(x, l)\mathcal{M}(x, l) + \mathcal{R}(x, l) - 8\lambda(\mathcal{P}(x, l))^2 = 0. \quad (3.64)$$

We call this equation as a master equation. It is of fourth order (in x) for $\lambda \neq \lambda_\infty(l)$ and of third order for $\lambda = \lambda_\infty(l)$. For any $l > 2$ the master equation can be solved in radicals.

Example for $l = 3$ As an example we consider the solution for $l = 3$. In this case $x_\pm = -2 \pm \sqrt{3}$, $x_b = -2$, $x_c = -1/2$, $x_d = -1$ and $\lambda_a = -5/16$, $\lambda_\infty = -3/4$, $\lambda_b = \lambda_c = 1/4$, $\lambda_d = 3/16$. The solutions obey $x \neq x_\pm$. The master Eq. (3.64) for $m = l = 3$ reads

$$(4\lambda + 3)x^4 + (32\lambda + 12)x^3 + (72\lambda + 15)x^2 + (32\lambda + 12)x + (4\lambda + 3) = 0. \quad (3.65)$$

For $\lambda \neq -3/4$ we get the following solution

$$x = \frac{-16\lambda + \varepsilon_1\sqrt{3}\sqrt{(-32\lambda - 12)\varepsilon_2\sqrt{-4\lambda + 1} + 64\lambda^2 + 20\lambda + 3} + 3\varepsilon_2\sqrt{-4\lambda + 1} - 6}{8\lambda + 6}, \quad (3.66)$$

where $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$, while for $\lambda = -3/4$ we have

$$x = \frac{-13 \pm \sqrt{105}}{8}, \quad \text{or } x = 0. \quad (3.67)$$

4 Stability analysis

Here we outline main relations from [25, 26], devoted to stability analysis of exponential solutions (2.3) with non-static volume factor [proportional to $\exp(\sum_{i=1}^n v^i t)$], which obey

$$K = K(v) = \sum_{i=1}^n v^i \neq 0. \quad (4.1)$$

For a general cosmological diagonal metric

$$g = -dt \otimes dt + \sum_{i=1}^n e^{2\beta^i(t)} dy^i \otimes dy^i, \quad (4.2)$$

the equations of motion for the action (2.1) gives us the set equations [25]

$$E = G_{ij}h^i h^j + 2\Lambda - \alpha G_{ijkl}h^i h^j h^k h^l = 0, \quad (4.3)$$

$$Y_i = \frac{dL_i}{dt} + \left(\sum_{j=1}^n h^j \right) L_i - \frac{2}{3} (G_{sj}h^s h^j - 4\Lambda) = 0, \quad (4.4)$$

where $h^i = \dot{\beta}^i$,

$$L_i = L_i(h) = 2G_{ij}h^j - \frac{4}{3}\alpha G_{ijkl}h^j h^k h^l, \quad (4.5)$$

$i = 1, \dots, n$.

We set the following restriction

$$\det(L_{ij}(v)) \neq 0 \quad (4.6)$$

on the matrix

$$L = (L_{ij}(v)) = \left(2G_{ij} - 4\alpha G_{ijks}v^k v^s \right). \quad (4.7)$$

It was proved in [26] that a fixed point solution $(h^i(t)) = (v^i)$ ($i = 1, \dots, n; n > 3$) to Eqs. (4.3), (4.4) obeying restrictions (4.1), (4.6) is stable under perturbations

$$h^i(t) = v^i + \delta h^i(t), \quad (4.8)$$

$i = 1, \dots, n$, (as $t \rightarrow +\infty$) if

$$K(v) = \sum_{k=1}^n v^k > 0 \quad (4.9)$$

and it is unstable (as $t \rightarrow +\infty$) if

$$K(v) = \sum_{k=1}^n v^k < 0. \quad (4.10)$$

The set of equations for perturbations is presented in Appendix B.

For our ansatz with $K(v) = 3H + lh$ and $H > 0$ the restriction (4.9) is equivalent to the inequality

$$x > -\frac{3}{l} = x_d, \quad (4.11)$$

while the restriction (4.10) is equivalent to another inequality

$$x < -\frac{3}{l} = x_d. \quad (4.12)$$

It follows from Ref. [26] that for the vector v from (3.1), obeying relations (3.3), the matrix L has a block-diagonal form

$$(L_{ij}) = \text{diag}(L_{\mu\nu}, L_{\alpha\beta}), \quad (4.13)$$

where

$$L_{\mu\nu} = G_{\mu\nu}(2 + 4\alpha S_{HH}), \quad (4.14)$$

$$L_{\alpha\beta} = G_{\alpha\beta}(2 + 4\alpha S_{hh}) \quad (4.15)$$

and

$$S_{HH} = 2lHh + l(l-1)h^2, \quad (4.16)$$

$$S_{hh} = 6H^2 + 6(l-2)Hh + (l-2)(l-3)h^2. \quad (4.17)$$

The matrix (4.13) is invertible (for $l > 1$) if and only if [26]

$$S_{HH} \neq -\frac{1}{2\alpha}, \quad (4.18)$$

$$S_{hh} \neq -\frac{1}{2\alpha}. \quad (4.19)$$

Now, let us prove that inequalities (4.18), (4.19) are satisfied if

$$x \neq -\frac{1}{l-1} = x_c. \quad (4.20)$$

and

$$x \neq -\frac{2}{l-2} = x_b \quad (4.21)$$

for $l > 2$.

Let us suppose that (4.18) is not satisfied, i.e. $S_{HH} = -\frac{1}{2\alpha}$. Then using (3.5) we get

$$S_{HH} - Q = -2(H-h)(H+(l-1)h) = 0, \quad (4.22)$$

which implies due to $H-h \neq 0$:

$$H + (l-1)h = 0. \quad (4.23)$$

which is in contradiction with our restriction (4.20). This contradiction proves the inequality (4.18).

Now we suppose that (4.19) is not valid, i.e. $S_{hh} = -\frac{1}{2\alpha}$. Then using (3.5) we get

$$S_{hh} - Q = -2(h-H)((l-2)h+2H) = 0, \quad (4.24)$$

which implies due to $H-h \neq 0$:

$$(l-2)h+2H=0. \quad (4.25)$$

which contradicts the restriction (4.21). The contradiction proves the inequality (4.19).

Thus, we proved that relations (4.18) and (4.19) are valid. Hence the restriction (4.6) is satisfied for our solutions.

Thus we are led to the following proposition.

Proposition 2 *The cosmological solutions under consideration obeying $x = h/H \neq x_i$, $i = a, b, c, d$, where $x_a = 1$, $x_b = -\frac{2}{l-2}$, $x_c = -\frac{1}{l-1}$, $x_d = -\frac{3}{l}$, are stable if (i) $x > x_d$ and unstable if (ii) $x < x_d$.*

We note that for the anisotropic solutions under consideration the points $x = x_a$ and $x = x_d$ are excluded. Nevertheless, the solutions are defined for special extremal cases, when $x = x_b$ or $x = x_c$, if $x \neq x_d$. The stability analysis of these special solutions can not be deduced just from the equations for perturbations (see relations (B.3), (B.4) from the Appendix) in the linear approximation. They need a special consideration [30].

We note that for the example $l = 3$ from Sect. 3 we get stable cosmological solutions for $x > -1$ and $x \neq -1/2$, $x \neq 1$.

Let us denote by $n_+(\Lambda, \alpha)$ the number of non-special stable solutions which are given by Proposition 2 [see item (i)]. By using the results (e.g. figures) from the previous section we find for $\alpha > 0$:

(A) $2 < l < 6$

$$n_+(\Lambda, \alpha) = \begin{cases} 0, & \Lambda\alpha \geq \lambda_c, \\ 2, & \lambda_d < \Lambda\alpha < \lambda_c, \\ 1, & \Lambda\alpha \leq \lambda_d; \end{cases} \quad (4.26)$$

(B) $l > 6$

$$n_+(\Lambda, \alpha) = \begin{cases} 0, & \Lambda\alpha \geq \lambda_c, \\ 2, & \lambda_d \leq \Lambda\alpha < \lambda_c, \\ 3, & \lambda_b < \Lambda\alpha < \lambda_d, \\ 1, & \Lambda\alpha \leq \lambda_b; \end{cases} \quad (4.27)$$

(B₀) $l = 6$

$$n_+(\Lambda, \alpha) = \begin{cases} 0, & \Lambda\alpha \geq \lambda_c, \\ 2, & \lambda_d < \Lambda\alpha < \lambda_c, \\ 1, & \Lambda\alpha \leq \lambda_d. \end{cases} \quad (4.28)$$

Thus, for $\alpha > 0$ and small enough value of Λ there exists at least one stable solution with $x \in (x_-, x_+)$.

For $\alpha < 0$ we obtain

$$n_+(\Lambda, \alpha) = \begin{cases} 1, & \Lambda|\alpha| \geq |\lambda_\infty|, \\ 2, & |\lambda_a| < \Lambda|\alpha| < |\lambda_\infty|, \\ 0, & \Lambda|\alpha| \leq |\lambda_a|. \end{cases} \quad (4.29)$$

Here we use $x \neq x_a = 1$. Thus, for $\alpha < 0$ and big enough value of Λ there exists at least one stable solution with x obeying $x > x_+$. The solution with $x < x_-$ is unstable.

5 Solutions with small enough variation of G

The solutions under consideration may be analysed on a variation of the effective gravitational constant G , which is proportional (in the Jordan frame) to the inverse volume scale factor of the anisotropic internal space, i.e.

$$G = \text{const} \exp(-lht), \quad (5.1)$$

see [22, 37–39] and references therein.

From (5.1) we get the following relation for a dimensionless parameter of temporal variation of G :

$$\delta \equiv \frac{\dot{G}}{GH} = -lx, \quad x = h/H. \quad (5.2)$$

We remind that $H > 0$ is the Hubble parameter.

Due to experimental (or observational) data, the variation of the gravitational constant is allowed at the level of 10^{-13} per year and less. In Ref. [22] the following constraint on the value of the dimensionless variation of the effective gravitational constant was used:

$$-0,65 \cdot 10^{-3} < \frac{\dot{G}}{GH} < 1,12 \cdot 10^{-3}. \quad (5.3)$$

It comes from the most stringent limitation on G -dot obtained in Ref. [40] (by the set of ephemerides)

$$\dot{G}/G = (0.16 \pm 0.6) \cdot 10^{-13} \text{ year}^{-1} \quad (5.4)$$

allowed at 95% confidence (2σ) level and the present value of the Hubble parameter [34]

$$H_0 = (67.80 \pm 1.54) \text{ km/s Mpc}^{-1} \\ = (6.929 \pm 0,157) \cdot 10^{-11} \text{ year}^{-1}, \quad (5.5)$$

with 95% confidence level.

For a given value of δ we get from (5.2)

$$x = x_0(\delta, l) \equiv -\frac{\delta}{l}. \quad (5.6)$$

Our solutions are defined if

$$x_0(\delta, l) \neq x_\pm(l), \quad (5.7)$$

or, equivalently, if

$$\mathcal{P}(x_0(\delta, l), l) \neq 0. \quad (5.8)$$

The calculation of quadratic polynomial (3.7) for $x = x_0(\delta, l)$ gives us

$$\mathcal{P}(x_0(\delta, l), l) = 2 - 4 \frac{(l-1)(l-3)}{l^2} \delta + \frac{(l-1)(l-2)}{l^2} \delta^2. \quad (5.9)$$

The inequality (5.8) is satisfied due to the bounds (5.3) for any $l > 2$. Hence, relation (5.7) is valid.

Now we analyse the stability of the solutions with small enough variation of G . The main condition for the stability $x_0(\delta, l) > x_d$ is satisfied since

$$x_0(\delta, l) - x_d = \frac{3-\delta}{l} > 0 \quad (5.10)$$

due to our bounds (5.3).

Other three conditions (see Proposition 2) $x_0(\delta, l) \neq x_a$, $x_0(\delta, l) \neq x_b$ and $x_0(\delta, l) \neq x_c$ give us

$$\delta \neq \delta_a = -l, \quad \delta \neq \delta_b = \frac{2l}{l-2}, \quad \delta \neq \delta_c = \frac{l}{l-1}, \quad (5.11)$$

which are satisfied due to bounds (5.3) and inequalities: $\delta_a \leq -3$, $\delta_b > 2$ and $\delta_c > 1$ for $l > 2$.

Thus, we have shown that all (well-defined) solutions under consideration obeying the bounds (5.3) (coming from the physical bounds on variation of G) are stable. We note that the proof of this fact is also valid for less restrictive bounds for δ than (5.3).

6 Conclusions

Here we have considered the Einstein–Gauss–Bonnet (EGB) model in dimension $D = 1 + 3 + l$, $l > 2$, with the Λ -term and two non-zero constants α_1 and α_2 . By using the ansatz with diagonal cosmological metrics, we have found, for certain fine-tuned $\Lambda = \Lambda(x, l, \alpha)$, where $\alpha = \alpha_2/\alpha_1$, a class of solutions with exponential time dependence of two scale factors, governed by two Hubble-like parameters $H > 0$ and $h = xH$, corresponding to submanifolds of dimensions 3 and $l > 2$, respectively. The parameter $x = h/H$ obey the restrictions $x \neq x_a = 1$, $x \neq x_d = -3/l$ and $\mathcal{P}(x, l) = 2 + 4(l-1)x + (l-1)(l-2)x^2 \neq 0$. Moreover, it should be imposed: $\mathcal{P}(x, l) < 0$ for $\alpha > 0$ and $\mathcal{P}(x, l) > 0$ for $\alpha < 0$.

For fixed Λ , α and $l > 2$ the equation $\Lambda(x, l, \alpha) = \Lambda$ is equivalent to a polynomial equation of either fourth or third order and hence may be solved in radicals.

Any of solutions describes an exponential expansion of $3d$ subspace (our space) with the Hubble parameter $H > 0$ and either contraction or expansion (with Hubble-like parameter h), or stabilization ($h = 0$) of l -dimensional internal subspace.

By using results of Ref. [26] we have proved that the cosmological solution (under consideration) is stable as $t \rightarrow +\infty$, if it obey the following restrictions: $x > x_d = -3/l$, $x \neq x_b = -\frac{2}{l-2}$ and $x \neq x_c = -\frac{1}{l-1}$. Here the points x_a, x_b, x_c, x_d are points of extremum of the function $\lambda(x, l) = \alpha \Lambda(x, l, \alpha)$ for any $l > 2$.

We have also shown that all (well-defined) solutions with small enough variation of the effective gravitational constant G (in the Jordan frame), obeying physical bounds, are stable.

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Appendix A: The proof of the Lemma

Here we prove the Lemma from Sect. 2. The calculations (by using Mathematica) gives us

$$\mathcal{R}_{\pm}(l) = \mathcal{R}(x_{\pm}(l), l) = \frac{A(l) \pm B(l)\sqrt{\Delta(l)}}{C(l)}, \quad (\text{A.1})$$

where $A(l) = -4lA_*(l)$, $A_*(l) = l^3 + 5l^2 + 8l - 12$, $B(l) = 32l^2$, $\Delta(l) = 2(l-1)l$, $C(l) = (l-2)^3(l-1)$.

To prove $\mathcal{R}_-(l) < 0$ it is sufficient to verify that $A_*(l) > 0$ (for $l > 2$). We have: $A_*(l) = l^3 + 5l^2 + 8l - 12 \geq 84$ (as $l \geq 3$). Thus, the relation $\mathcal{R}_-(l) < 0$ (for $l > 2$) is proved.

Now we prove $\mathcal{R}_+(l) < 0$ ($l > 2$). We should prove the inequality $A(l) + B(l)\sqrt{\Delta(l)} < 0$, or, equivalently,

$$(l^3 + 5l^2 + 8l - 12)^2 - 128l^3(l-1) = (l-2)^3(l+1)(l^2 + 15l - 18) > 0, \quad (\text{A.2})$$

for $l > 2$. But this is valid since $w(l) = l^2 + 15l - 18 > 0$ for $l > 2$.

This completes the proof of the Lemma.

Appendix B: Equations for perturbations δh^i

Here we outline for a completeness the set of equations for perturbations of Hubble-like parameters δh^i (in the linear approximation) from Refs. [25, 26]:

$$C_i(v)\delta h^i = 0, \quad (\text{B.3})$$

$$L_{ij}(v)\delta \dot{h}^j = B_{ij}(v)\delta h^j, \quad (\text{B.4})$$

where

$$C_i(v) = 2v_i - 4\alpha G_{ijks}v^jv^kv^s, \quad (\text{B.5})$$

$$L_{ij}(v) = 2G_{ij} - 4\alpha G_{ijks}v^kv^s, \quad (\text{B.6})$$

$$B_{ij}(v) = -\left(\sum_{k=1}^n v^k\right)L_{ij}(v) - L_i(v) + \frac{4}{3}v_j, \quad (\text{B.7})$$

$v_i = G_{ij}v^j$, $L_i(v) = 2v_i - \frac{4}{3}\alpha G_{ijks}v^jv^kv^s$ and $i, j, k, s = 1, \dots, n$.

It was proved in Ref. [26] that the set of linear equations (B.3), (B.4) has the following solution

$$\delta h^i = A^i \exp(-K(v)t), \quad (\text{B.8})$$

$$\sum_{i=1}^n C_i(v)A^i = 0, \quad (\text{B.9})$$

(A^i are constants) $i = 1, \dots, n$, when restrictions (4.1), (4.6) are imposed.

References

1. B. Zwiebach, Curvature squared terms and string theories. Phys. Lett. B **156**, 315 (1985)
2. E.S. Fradkin, A.A. Tseytlin, Effective field theory from quantized strings. Phys. Lett. B **158**, 316–322 (1985)
3. E.S. Fradkin, A.A. Tseytlin, Effective action approach to superstring theory. Phys. Lett. B **160**, 69–76 (1985)
4. D. Gross, E. Witten, Superstrings modifications of Einstein's equations. Nucl. Phys. B **277**, 1 (1986)
5. R.R. Metsaev, A.A. Tseytlin, Two loop beta function for the generalized bosonic sigma model. Phys. Lett. B **191**, 354 (1987)
6. H. Ishihara, Cosmological solutions of the extended Einstein gravity with the Gauss–Bonnet term. Phys. Lett. B **179**, 217 (1986)
7. N. Deruelle, On the approach to the cosmological singularity in quadratic theories of gravity: the Kasner regimes. Nucl. Phys. B **327**, 253–266 (1989)
8. S. Nojiri, S.D. Odintsov, Introduction to modified gravity and gravitational alternative for Dark Energy. Int. J. Geom. Methods Mod. Phys. **4**, 115–146 (2007). [arXiv:hep-th/0601213](https://arxiv.org/abs/hep-th/0601213)
9. G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov, S. Zerbini, One-loop effective action for non-local modified Gauss–Bonnet gravity in de Sitter space. Eur. Phys. J. C **64**(3), 483–494 (2009). [arXiv:0905.0543](https://arxiv.org/abs/0905.0543)
10. E. Elizalde, A.N. Makarenko, V.V. Obukhov, K.E. Osetrin, A.E. Filippov, Stationary vs. singular points in an accelerating FRW

- cosmology derived from six-dimensional Einstein–Gauss–Bonnet gravity. *Phys. Lett. B* **644**, 1–6 (2007). [arXiv:hep-th/0611213](#)
11. K. Bamba, Z.-K. Guo, N. Ohta, Accelerating cosmologies in the Einstein–Gauss–Bonnet theory with dilaton. *Prog. Theor. Phys.* **118**, 879–892 (2007). [arXiv:0707.4334](#)
 12. A. Toporensky, P. Tretyakov, Power-law anisotropic cosmological solution in 5+1 dimensional Gauss–Bonnet gravity. *Grav. Cosmol.* **13**, 207–210 (2007). [arXiv:0705.1346](#)
 13. S.A. Pavluchenko, A.V. Toporensky, A note on differences between $(4 + 1)$ - and $(5 + 1)$ -dimensional anisotropic cosmology in the presence of the Gauss–Bonnet term. *Mod. Phys. Lett. A* **24**, 513–521 (2009)
 14. I.V. Kirnos, A.N. Makarenko, Accelerating cosmologies in Lovelock gravity with dilaton. *Open Astron. J.* **3**, 37–48 (2010). [arXiv:0903.0083](#)
 15. S.A. Pavluchenko, On the general features of Bianchi-I cosmological models in Lovelock gravity. *Phys. Rev. D* **80**, 107501 (2009). [arXiv:0906.0141](#)
 16. I.V. Kirnos, A.N. Makarenko, S.A. Pavluchenko, A.V. Toporensky, The nature of singularity in multidimensional anisotropic Gauss–Bonnet cosmology with a perfect fluid. *Gen. Relativ. Gravit.* **42**, 2633–2641 (2010). [arXiv:0906.0140](#)
 17. V.D. Ivashchuk, On anisotropic Gauss–Bonnet cosmologies in $(n + 1)$ dimensions, governed by an n -dimensional Finslerian 4-metric. *Grav. Cosmol.* **16**(2), 118–125 (2010). [arXiv:0909.5462](#)
 18. V.D. Ivashchuk, On cosmological-type solutions in multidimensional model with Gauss–Bonnet term. *Int. J. Geom. Methods Mod. Phys.* **7**(5), 797–819 (2010). [arXiv:0910.3426](#)
 19. K-i Maeda, N. Ohta, Cosmic acceleration with a negative cosmological constant in higher dimensions. *JHEP* **1406**, 095 (2014). [arXiv:1404.0561](#)
 20. D. Chirkov, S. Pavluchenko, A. Toporensky, Exact exponential solutions in Einstein–Gauss–Bonnet flat anisotropic cosmology. *Mod. Phys. Lett. A* **29**, 1450093 (11 pages) (2014). [arXiv:1401.2962](#)
 21. D. Chirkov, S.A. Pavluchenko, A. Toporensky, Non-constant volume exponential solutions in higher-dimensional Lovelock cosmologies. *Gen. Relativ. Gravit.* **47**, 137 (33 pages) (2015). [arXiv:1501.04360](#)
 22. V.D. Ivashchuk, A.A. Kobtsev, On exponential cosmological type solutions in the model with Gauss–Bonnet term and variation of gravitational constant. *Eur. Phys. J. C* **75**, 177 (12 pages) (2015). Erratum *ibid.* **76**, 584 (2016). [arXiv:1503.00860](#)
 23. S.A. Pavluchenko, Stability analysis of exponential solutions in Lovelock cosmologies. *Phys. Rev. D* **92**, 104017 (2015). [arXiv:1507.01871](#)
 24. S.A. Pavluchenko, Cosmological dynamics of spatially flat Einstein–Gauss–Bonnet models in various dimensions: low-dimensional Λ -term case. *Phys. Rev. D* **94**, 084019 (2016). [arXiv:1607.07347](#)
 25. K.K. Ernazarov, V.D. Ivashchuk, A.A. Kobtsev, On exponential solutions in the Einstein–Gauss–Bonnet cosmology, stability and variation of G . *Grav. Cosmol.* **22**(3), 245–250 (2016)
 26. V.D. Ivashchuk, On stability of exponential cosmological solutions with non-static volume factor in the Einstein–Gauss–Bonnet model. *Eur. Phys. J. C* **76**, 431 (2016). [arXiv:1607.01244v2](#)
 27. V.D. Ivashchuk, On stable exponential solutions in Einstein–Gauss–Bonnet cosmology with zero variation of G . *Grav. Cosmol.* **22**(4), 329–332 (2016). Erratum *ibid.* **23**(4), 401 (2017). [arXiv:1612.07178](#)
 28. K.K. Ernazarov, V.D. Ivashchuk, Stable exponential cosmological solutions with zero variation of G in the Einstein–Gauss–Bonnet model with a Λ -term. *Eur. Phys. J. C* **77**, 89 (6 pages) (2017). [arXiv:1612.08451](#)
 29. K.K. Ernazarov, V.D. Ivashchuk, Stable exponential cosmological solutions with zero variation of G and three different Hubble-like parameters in the Einstein–Gauss–Bonnet model with a Λ -term. *Eur. Phys. J. C* **77**, 402 (7 pages) (2017). [arXiv:1705.05456](#)
 30. D.M. Chirkov, A.V. Toporensky, On stable exponential cosmological solutions in the EGB model with a cosmological constant in dimensions $D = 5, 6, 7, 8$. *Grav. Cosmol.* **23**(4), 359–366 (2017). [arXiv:1706.08889](#)
 31. A.G. Riess et al., Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. J.* **116**, 1009–1038 (1998)
 32. S. Perlmutter et al., Measurements of Omega and Lambda from 42 High-Redshift Supernovae. *Astrophys. J.* **517**, 565–586 (1999)
 33. M. Kowalski, D. Rubin et al., Improved cosmological constraints from new, old and combined supernova datasets. *Astrophys. J.* **686**(2), 749–778 (2008). [arXiv:0804.4142](#)
 34. P.A.R. Ade et al. [Planck Collaboration], Planck 2013 results. I. Overview of products and scientific results. *Astron. Astrophys.* **571**, A1 (2014). [arXiv:1303.5076](#)
 35. M. Rainer, A. Zhuk, Einstein and Brans–Dicke frames in multi-dimensional cosmology. *Gen. Relativ. Gravit.* **32**, 79–104 (2000). [arXiv:gr-qc/9808073](#)
 36. V.D. Ivashchuk, V.N. Melnikov, Multidimensional gravity with Einstein internal spaces. *Grav. Cosmol.* **2**(3), 211–220 (1996). [arXiv:hep-th/9612054](#)
 37. K.A. Bronnikov, V.D. Ivashchuk, V.N. Melnikov, Time variation of gravitational constant in multidimensional cosmology. *Nuovo Cimento B* **102**, 209–215 (1998)
 38. V.N. Melnikov, Models of G time variations in diverse dimensions. *Front. Phys. China* **4**, 75–93 (2009)
 39. V.D. Ivashchuk, V.N. Melnikov, On time variations of gravitational and Yang–Mills constants in a cosmological model of superstring origin. *Grav. Cosmol.* **20**(1), 26–29 (2014). [arXiv:1401.5491](#)
 40. E.V. Pitjeva, Updated IAA RAS planetary ephemerides-EPM2011 and their use in scientific research. *Astron. Vestnik* **47**(5), 419–435 (2013). [arXiv:1308.6416](#)