Stochastic quantum inflation for a canonical scalar field with linear self-interaction potential

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Received: 20 February 2017 / Accepted: 26 September 2017 / Published online: 13 October 2017
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Abstract We apply Starobinsky's formalism of stochastic inflation to the case of a massless minimally coupled scalar field with linear self-interaction potential. We solve the corresponding Fokker–Planck equation exactly, and we obtain analytical expressions for the stochastic expectation values.

1 Introduction

The inflationary universe [1–4] has become one of the central paradigms in modern cosmology. This is due to the fact that many long-standing problems of the Big Bang model, such as the horizon, flatness, homogeneity and monopole problems, find a natural explanation in the framework of the inflationary universe. However, the essential feature of inflation is that it generates a mechanism to explain the Large-Scale Structure (LSS) of the universe and provides a causal interpretation of the origin of the anisotropies observed in the Cosmic Microwave Background (CMB) radiation [5], since primordial density perturbations may be produced from quantum fluctuations during the inflationary era [6].

Despite its success, a theory of inflation is still missing as we still do not know what the inflaton is, and the inflaton potential cannot be derived from a fundamental theory in a unique way. All we have is a large collection of inflationary models (see e.g. [7, 8], and for a classification of inflationary models see e.g. [9]). In single-field inflationary models with a canonical scalar field in Einstein's General Relativity (GR) there are basically four classes of models, namely large field models, small field models, hybrid models and linear inflation that lies between large and small field models. After the latest Planck results [10] many inflationary models based on monomial potentials have been ruled out or are disfavored by data. There are of course the Starobinsky model [1] and the Higgs inflation model [11, 12], which are in agreement with the data, but they are extensions of GR. The linear inflaton

potential is still in agreement with observations, but difficult to realize in particle physics models. However, recently it was shown that the Coleman–Weinberg potential [13] together with a nonminimal coupling of the inflaton to gravity leads to attractor solutions that interpolate between quadratic inflation, which is ruled out by recent measurements, and linear inflation, which lies within the allowed region [14, 15]. In particular, first in [14] it was shown that the predictions of linear inflation for the observables can be achieved in the context of well-defined quantum field theory, without introducing complicated interactions by hand. Then in [15] the authors extended the previous discussion, presenting a more detailed study of the parameter space, and they also added a discussion of reheating.

During inflation infrared logarithms arise in the expectation values of operators of quantum field theories that contain massless minimally coupled scalar fields or gravitons. For an incomplete list see e.g. [16–20] and the references therein. These terms are powers of the logarithm of the inflationary scale factor $a = \exp(Ht)$, with H being the Hubble constant, and are very exciting because they may signal important quantum effects [21–23]. However, their continued growth implies that when inflation has proceeded for a long time, the large logarithms eventually overcome the small coupling constants. Therefore the effects become non-perturbative and perturbation theory breaks down. A natural approach to obtaining non-perturbative information is the leading logarithm approximation [24], in which one attempts to sum the series comprised of just the leading infrared logarithms at each order. For scalar fields with non-derivative interactions Starobinsky's technique of stochastic inflation [25, 26] recovers the leading infrared logarithms at each order, and the series of these leading effects at all orders can be resummed [27] to give non-perturbative predictions.

Given the observational bound on the tensor-to-scalar ratio $r < 0.07$ [28], the slow-roll parameter $\epsilon = r/16$ is extremely close to zero, $\epsilon < 0.0044$, and therefore the de Sitter space-

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time is an excellent approximation to inflation. The goal of this work is to study the effect of a massless minimally coupled scalar field with a linear self-interaction potential using Starobinsky's formalism. Our work is organized as follows: After this introduction, we briefly summarize stochastic processes in the Langevin and Fokker–Planck approach in Sect. 2. Then we apply this formalism to the case of a canonical scalar field with a linear potential following the Starobinsky technique in Sect. 3. Finally we conclude in Sect. 4.

2 Stochastic processes

The observation that, when suspended in water, small pollen grains are found to be in a very animated and irregular state of motion, was first systematically investigated by the Scottish botanist Brown in 1827, and the observed phenomenon took the name of Brownian motion. First Einstein [29] in 1905 and independently Smoluchowski in 1906 [30], and some time later Langevin [31] in 1908 (see [32] for a translation of the original Langevin paper in English) explained Brownian motion using different but equally successful approaches. On the one hand, Einstein's analysis was based on the diffusion equation

$$\frac{\partial f(t, x)}{\partial t} = D \frac{\partial^2 f(t, x)}{\partial x^2} \quad (1)$$

where D is the diffusion coefficient, with the initial condition $f(x, t = 0) = \delta(x)$ where $\delta(x)$ is Dirac's delta function. The solution of the diffusion equation is given by [33]

$$f(t, x) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) \quad (2)$$

and the mean of the square of displacement is given by $\langle x^2 \rangle = 2Dt$. On the other hand, Langevin started from Newton's equation of motion assuming a Stokes drag force and a random thermal force due to the continuous bombardment from the molecules of the liquid. Although he did not exploit all richness of his model, Langevin obtained in the late-time regime the Einstein result, namely $\langle x^2 \rangle = 2(k_B T / 6\pi a \mu) t$, where a is the radius of the particle, μ is the fluid viscosity, T is the temperature and k_B is the Boltzmann constant, hence bridging Brownian motion, random walk, and diffusion, a view soon quantified experimentally by Perrin [34]. Therefore the diffusion coefficient can be computed in terms of properties of the fluid and the Brownian particles, which is the Einstein–Stokes formula $D = (k_B T) / (6\pi a \mu)$.

In modern times Langevin's approach is still in use. The Langevin equation for the process $x(t)$ (let us call it the position of a moving particle) reads

$$\dot{x} = A(x) + \xi(t) \quad (3)$$

where $A(x)$ is an external applied force, the dot denotes derivative with respect to time, and $\xi(t)$ is assumed to be a Gaussian white noise

$$\langle \xi(t) \rangle = 0, \quad (4)$$

$$\langle \xi(t_1) \xi(t_2) \rangle = 2D \delta(t_1 - t_2). \quad (5)$$

Since the random force $\xi(t)$ is not known, we can only compute mean values of powers of the position, or the moments, $\langle x^n \rangle$, once the density probability function is known. The density probability function $u(t, x)$ satisfies the corresponding Fokker–Planck (FP) equation [35, 36]

$$\frac{\partial u(t, x)}{\partial t} = -\frac{\partial (A(x)u(t, x))}{\partial x} + D \frac{\partial^2 u(t, x)}{\partial x^2} \quad (6)$$

where the first term is due to the external force, while the second term is the diffusion term with a constant diffusion coefficient D . If we ignore the external applied force, the FP equation reduces to the standard diffusion equation. That explains why the Einstein and Langevin approaches were equally successful. Solving the FP equation we then can compute the moments performing the integrals

$$\langle x^n \rangle = \int_{-\infty}^{\infty} dx u(t, x) x^n \quad (7)$$

and they are functions of time. Although seemingly quite different equations, when the diffusion equation and the FP equation at hand have the same number of symmetries, or equivalently when the following condition is satisfied [37]:

$$2DA'(x) + A(x)^2 = c_0 + c_1 x + c_2 x^2, \quad (8)$$

the FP equation can be recast as the diffusion equation and therefore admits an exact analytical solution.

3 Application of Starobinsky's formalism to a massless canonical scalar field with a linear potential

In this section we apply Starobinsky's formalism [25, 26] to a scalar field Ψ described by the Lagrangian

$$\mathcal{L} = -\sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi + V(\Psi) \right) \quad (9)$$

where $V(\Psi) = -M^3 \Psi$ is the self-interaction potential of the scalar field taken to be linear in Ψ , with M being a mass scale. The resulting Klein–Gordon equation reads

$$\ddot{\Psi} + 3H\dot{\Psi} - \frac{\nabla^2 \Psi}{a^2} + V_{,\Psi} = 0 \quad (10)$$

where $V_{,\Psi}$ is the derivative of the scalar potential with respect to the scalar field. We take the onset of inflation to be at $t = 0$, and we work perturbatively around the non-dynamical de Sitter background,

$$ds^2 = -dt^2 + \exp(2Ht) d\mathbf{x} d\mathbf{x}, \quad (11)$$

with the inflationary scale factor being $a(t) = \exp(Ht)$, and H being the Hubble constant.

In the leading logarithm approximation the scalar field behaves like a stochastic variable ϕ satisfying Langevin's equation

$$\dot{\phi}(t, \mathbf{x}) = f(t, \mathbf{x}) - \frac{V_{,\phi}}{3H} \quad (12)$$

where the stochastic source $f(t, \mathbf{x})$ has the properties of the Gaussian white noise

$$\langle f(t_1, \mathbf{x}) f(t_2, \mathbf{x}) \rangle = \frac{H^3}{4\pi^2} \delta(t_1 - t_2) \quad (13)$$

(see, however, Eq. (11) of [26] for the general case where the two space-like points $\mathbf{x}_1, \mathbf{x}_2$ are taken to be different). Langevin's equation above can be rigorously derived from the underlying quantum field theory [25], or it can easily be obtained from the Klein–Gordon equation by applying the stochastic rules of [38], namely

- During inflation the scale factor varies faster than the scalar field, and therefore the Hubble friction term dominates over the spatial derivative and the second time derivative terms.
- We replace the full field Ψ with its stochastic counterpart ϕ .
- We make the substitution $\dot{\phi} \rightarrow \dot{\phi} - f$, where f is the stochastic source.

Since the scalar potential is linear, the corresponding Fokker–Planck equation takes the form

$$\frac{\partial u(t, \phi)}{\partial t} = -\frac{M^3}{3H} \frac{\partial u(t, \phi)}{\partial \phi} + \frac{\Delta}{2} \frac{\partial^2 u(t, \phi)}{\partial^2 \phi} \quad (14)$$

where we have put $\Delta = H^3/(4\pi^2)$. It is trivial to check that the condition (8) is satisfied, and therefore for this model we can find exact analytical solution of the FP equation. According to the previous discussion, the FK equation can be recast in the diffusion equation $\omega_\tau = \omega_{yy}$ and the solution is given by

$$u(t, \phi) = h(t, \phi) \omega(\tau(t, \phi), y(t, \phi)) \quad (15)$$

where $h(t, \phi)$, $y(t, \phi)$, $\tau(t, \phi)$ are given by [37]

$$h(t, \phi) = \exp\left(\frac{M^3 \phi}{3H\Delta} - \frac{M^6 t}{18\Delta H^2}\right), \quad (16)$$

$$y = \phi \quad (17)$$

$$\tau = \Delta t/2. \quad (18)$$

Therefore, the final expression for the solution is given by

$$u(t, \phi) = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{(\phi - \frac{M^3 t}{3H})^2}{2\Delta t}\right). \quad (19)$$

Therefore it is now straightforward to compute the first stochastic expectation values $\langle \phi \rangle$, $\langle \phi^2 \rangle$, $\langle \phi^3 \rangle$ and $\langle \phi^4 \rangle$, which are found to be

$$\langle \phi \rangle = \frac{M^3 t}{3H}, \quad (20)$$

$$\langle \phi^2 \rangle = \left(\frac{M^3 t}{3H}\right)^2 + \Delta t, \quad (21)$$

$$\langle \phi^3 \rangle = \left(\frac{M^3 t}{3H}\right)^3 + \frac{M^3 t^2}{H} \Delta, \quad (22)$$

$$\langle \phi^4 \rangle = \left(\frac{M^3 t}{3H}\right)^4 + 6\Delta \left(\frac{M^3 t}{3H}\right)^2 t^3 + 3(\Delta t)^2, \quad (23)$$

where we have made use of the Gaussian integrals

$$\int_{-\infty}^{\infty} dx \exp(-\alpha x^2) = \frac{\sqrt{\pi}}{\sqrt{\alpha}}, \quad (24)$$

$$\int_{-\infty}^{\infty} dx x^2 \exp(-\alpha x^2) = \frac{\sqrt{\pi}}{2\alpha^{3/2}}, \quad (25)$$

$$\int_{-\infty}^{\infty} dx x^4 \exp(-\alpha x^2) = \frac{3\sqrt{\pi}}{4\alpha^{5/2}}. \quad (26)$$

In the formulas obtained above, it is easy to check that when $M = 0$ we recover the known result for the pure diffusion case. Note that there are two contributions with different time dependence. In particular, the M dependent terms are due to the classical linear potential, while the M independent terms are due to quantum effects, and they are just the ones of massless scalar fields. Equivalently, we switch from the cosmological time t to the inflationary scale factor a by using $t = \ln(a)/H$ and we find the following expressions:

$$\langle \phi \rangle = \frac{M^3 \ln(a)}{3H^2}, \quad (27)$$

$$\langle \phi^2 \rangle = \frac{H^2 \ln(a)}{4\pi^2} \left(1 + \frac{4\pi^2 M^6}{9H^6} \ln(a)\right), \quad (28)$$

$$\langle \phi^3 \rangle = \frac{M^3 \ln^2(a)}{4\pi^2} \left(1 + \frac{4\pi^2 M^6 \ln(a)}{27H^6}\right), \quad (29)$$

$$\langle \phi^4 \rangle = \frac{3H^4 \ln^2(a)}{16\pi^4} \left(1 + \frac{8\pi^2 M^6}{9H^6} \ln(a) + \frac{16\pi^4 M^{12}}{243H^{12}} \ln^2(a)\right). \quad (30)$$

The factors $\ln(a)$ associated with the M independent quantum effects are the IR logarithms mentioned in the Introduction. The model is characterized by two mass scales, H , M , and we can view the ratio M/H as a dimensionless coupling constant. As already mentioned, even if M/H is small, when inflation has proceeded for a long time the large logarithms eventually overcome the small coupling constant.

Powers of Ht due to quantum effects of massless gravitons slow inflation [39]. In this model, however, it is the classical effect of the scalar field rolling down the linear potential that slows inflation. As has been shown in [40, 41], massless canonical scalar fields (with vanishing potential) contribute negligibly to the energy density in the de Sitter case and lead to negligible backreaction. On the contrary, the classical homogeneous field rolling down the linear potential discussed here will eventually develop an energy density comparable to the cosmological constant.

Since the probability density function is Gaussian, it is characterized by only two parameters, and therefore all moments can be given in terms of the first two. However, in the following we shall compute the generic stochastic expectation value ϕ^{2n} for even powers and ϕ^{2n+1} for odd powers in closed form. To this end we change variable $z = \phi - (M^3 t)/(3H)$, use the binomial expansion

$$(a+b)^m = \sum_{k=0}^m \frac{m!}{k!(m-k)!} a^{m-k} b^k, \quad (31)$$

and make use of the integral

$$\int_{-\infty}^{\infty} dx x^{2m} \exp(-\alpha x^2) = \frac{(2m-1)!! \sqrt{\pi}}{2^m \alpha^m \sqrt{\alpha}}. \quad (32)$$

We obtain the final result

$$\frac{\langle \phi^{2n} \rangle}{\left(\frac{M^3 t}{3H}\right)^{2n}} = 1 + \sum_{k=1}^n \frac{(2n)!}{(2k)!(2n-2k)!} \frac{(2k-1)!!}{(2\alpha\bar{\phi}^2)^k} \quad (33)$$

for even powers, and similarly for odd powers we obtain the formula

$$\frac{\langle \phi^{2n+1} \rangle}{\left(\frac{M^3 t}{3H}\right)^{2n+1}} = 1 + \sum_{k=1}^n \frac{(2n+1)!}{(2k)!(2n-2k+1)!} \frac{(2k-1)!!}{(2\alpha\bar{\phi}^2)^k} \quad (34)$$

where we have defined $\alpha^{-1} = 2\Delta t$ and $\bar{\phi} = (M^3 t)/(3H)$, and it is straightforward to check that for $n = 1, 2$ one obtains the previous expressions for $\langle \phi^2 \rangle$, $\langle \phi^3 \rangle$ and $\langle \phi^4 \rangle$.

The last two expressions are the main result of this article. As most of the equations in realistic models are solved either numerically or approximately, it is always desirable to have exact analytical solutions. We find it remarkable that the linear scalar potential: i) can be derived in the framework of

well-established quantum field theory, ii) is still in agreement with observations, and iii) the corresponding Fokker–Planck equation can be recast as a diffusion equation and thus be solved exactly.

A final remark is in order here. Throughout this work we imagine that the scalar field is a spectator to inflation, and we have worked perturbatively around a non-dynamical de Sitter background, which is an excellent approximation to inflation. However, the scalar field can no longer be treated as a spectator when its energy density becomes comparable to the cosmological constant. This happens for a number of e-folds $N_* = Ht$ given by

$$N_* = \frac{9H^4 M_p^2}{M^6} \quad (35)$$

where M_p is the reduced Planck mass. If we require that this happens after $N = 60$ we obtain the following bound on M :

$$M \leq \left(\frac{9H^4 M_p^2}{60} \right)^{1/6}. \quad (36)$$

Before finishing we show that it is possible to obtain the same results following another approach by solving the Langevin equation directly. With the initial condition $\phi(0) = 0$, the solution for $\phi(t)$ is given by

$$\phi(t) = \frac{M^3 t}{3H} + \int_0^t ds f(s). \quad (37)$$

From this one can immediately see that $\langle \phi \rangle = (M^3 t)/(3H)$ since $\langle f(t) \rangle = 0$. Then, by squaring the solution and using the property of the stochastic source, namely $\langle f(t_1) f(t_2) \rangle = \Delta \delta(t_1 - t_2)$, one obtains the previous expression for the second moment. In a similar way the third and the fourth moments can also be computed.

4 Conclusions

In the present article we have applied Starobinsky's technique of stochastic inflation to the case of a minimally coupled scalar field with a linear self-interaction potential. This type of inflaton potential, although still in agreement with the latest Planck results, could not be obtained from a fundamental theory of particle physics. However, it recently became relevant, since it has been shown that the linear potential can be obtained in the context of well-defined quantum field theory from a Coleman–Weinberg potential, provided that a non-minimal coupling to gravity is also present. We have obtained analytical expressions for the stochastic expectation values ϕ^{2n} (even powers) and ϕ^{2n+1} (odd powers) in two ways. First

by solving exactly the corresponding Fokker–Planck equation for the density probability function, and then directly by using the Klein–Gordon–Langevin equation and the properties of the stochastic source. The two approaches give us the same results, as expected.

Acknowledgements The author thanks the Fundação para a Ciência e Tecnologia (FCT), Portugal, for the financial support to the Multidisciplinary Center for Astrophysics (CENTRA), Instituto Superior Técnico, Universidade de Lisboa, through the Grant No. UID/FIS/00099/2013. In addition, he thanks the anonymous reviewer for her/his comments and suggestions, which improved the quality of the manuscript, and R. Woodard for useful correspondence.

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