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Letter

Quintessence background for 5D Einstein–Gauss–Bonnet black holes

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Abstract As we know that the Lovelock theory is an extension of the general relativity to the higher-dimensions, in this theory the first- and the second-order terms correspond to general relativity and the Einstein–Gauss–Bonnet gravity, respectively. We obtain a 5D black hole solution in Einstein–Gauss–Bonnet gravity surrounded by the quintessence matter, and we also analyze their thermodynamical properties. Owing to the quintessence corrected black hole, the thermodynamic quantities have also been corrected except for the black hole entropy, and a phase transition is achievable. The phase transition for the thermodynamic stability is characterized by a discontinuity in the specific heat at $r = r_C$, with the stable (unstable) branch for $r < (>)r_C$.

1 Introduction

The gravity theory with higher-curvature term, the so-called Lovelock gravity, is one of the natural generalizations of Einstein's general relativity, introduced originally by Lanczos [1], and rediscovered by Lovelock [2]. The action of it contains higher-order curvature terms and that reduces to the Einstein–Hilbert action in four-dimensions, and its second-order term is the Gauss–Bonnet invariant. The Lovelock theories have some special characteristics, among the larger class of general higher-curvature theories, in having field equations involving not more than second derivatives of the metric. As higher-dimensional members of Einstein's general relativity family, the Lovelock theories allow us to explore several conceptual issues of gravity in a broader setup. Hence, these theories have received significant attention, especially when finding black hole solutions. Besides, the theory is well

known to be free of ghosts about other exact backgrounds [3–5]. The theory represents a very interesting scenario to study how higher-order curvature corrections to the black hole physics substantially change the qualitative features, as we know from our experience with black holes in general relativity. Since its inception, steadily attention has been given to black hole solutions, including their formation, stability, and thermodynamics. The spherically symmetric static black hole solution for the Einstein–Gauss–Bonnet theory was first obtained by Boulware and Deser [3–5], and later several authors explored exact black hole solutions and their thermodynamical properties [6–23]. The generalization of the Boulware-Desser solution has been obtained with a source as a cloud of strings, in Einstein–Gauss–Bonnet gravity [24,25], and also in Lovelock gravity [26–28].

The intense activity of studying black hole solutions in Einstein–Gauss–Bonnet theory of gravity is due to the fact that we have, besides theoretical results, cosmological evidence, e.g., dark matter and dark energy. Quintessence is a hypothetical form of dark energy postulated as an explanation of the observation for an acceleration of the Universe, rather than due to a true cosmological constant. If quintessence exists all over in the Universe, it can also be around a black hole. In this letter, we are interested in a solution to the Einstein equations with the assumption of spherical symmetry, with the quintessence matter obtained by Kiselev [29], and it was also rigorously analyzed by himself and others [29–34]. In particular, spherically symmetric quintessence black hole solutions [29] have been extended to higher dimensions [35], to include Narai solutions [36,37], and also charged black holes [38]. The black hole thermodynamics for the quintessence corrected solutions was obtained in [39–43] and quasinormal modes of such solutions are also discussed [44–47]. The generalization of the spherical quintessential solution [29] to the axially symmetric case, Kerr-like black hole, was also addressed, recently [48,49]. However, the

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solution of the Einstein–Gauss–Bonnet theory surrounded by quintessence matter is still not explored, i.e., the black holes surrounded by the quintessence matter in Einstein–Gauss–Bonnet theory is still unknown. It is the purpose of this letter to obtain an exact new five-dimensional (5D) spherically symmetric black holes solution for the Einstein–Gauss–Bonnet gravity surrounded by quintessence matter. In particular, we explicitly show how the effect of a background quintessence matter can alter black hole solutions and their thermodynamics. In turn, we analyze their thermodynamical properties and perform a thermodynamic stability analysis.

The letter is organized as follows. In Sect. 2, we derive a Einstein–Gauss–Bonnet solution to the 5D spherically symmetric static Einstein equations surrounded by the quintessence matter. In Sect. 3, we discuss the thermodynamics of the 5D Einstein–Gauss–Bonnet black holes surrounded by the quintessence matter. The letter ends with concluding remarks in Sect. 4.

We use units which fix the speed of light and the gravitational constant via $G = c = 1$, and use the metric signature $(-, +, +, +, +)$.

2 Quintessence matter surrounding black hole

Lovelock theory is an extension of the general relativity to higher-dimensions. In this theory the first- and second-order terms correspond to general relativity and Einstein–Gauss–Bonnet gravity, respectively. The action for 5D Einstein–Gauss–Bonnet gravity with a matter field reads

$$\mathcal{I}_G = \frac{1}{2} \int_{\mathcal{M}} \mathrm{d}^5 x \sqrt{-g} \left[\mathcal{L}_1 + \alpha \mathcal{L}_{\mathrm{GB}} \right] + \mathcal{I}_S, \tag{1}$$

with $\kappa_5 = 1$. $\mathcal{I}_S$ denotes the action associated with matter and α is coupling constant that we assume to be non-negative. The Einstein term is $\mathcal{L}_1 = R$, and the second-order Gauss–Bonnet term $\mathcal{L}_{\mathrm{GB}}$ is

$$\mathcal{L}_{\mathrm{GB}} = R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2. \tag{2}$$

Here, $R_{\mu\nu}$, $R_{\mu\nu\gamma\delta}$, and R are the Ricci tensor, Riemann tensor, and Ricci scalar, respectively. The variation of the action with respect to the metric $g_{\mu\nu}$ gives the Einstein–Gauss–Bonnet equations

$$G^{\mathrm{E}}_{\mu\nu} + \alpha G^{\mathrm{GB}}_{\mu\nu} = T^S_{\mu\nu}, \tag{3}$$

where $G^{\mathrm{E}}_{\mu\nu}$ is the Einstein tensor, while $G^{\mathrm{GB}}_{\mu\nu}$ is given explicitly by [50]

$$G^{\mathrm{GB}}_{\mu\nu} = 2\Big[- R_{\mu\sigma\kappa\tau} R^{\kappa\tau\sigma}{}_\nu - 2 R_{\mu\rho\nu\sigma} R^{\rho\sigma}$$
$$- 2 R_{\mu\sigma} R^\sigma{}_\nu + R R_{\mu\nu} \Big] - \frac{1}{2} \mathcal{L}_{\mathrm{GB}} g_{\mu\nu}, \tag{4}$$

and $T^S_{\mu\nu}$ is the energy-momentum tensor of the matter that we consider as a quintessence matter. We note that the divergence of the Einstein–Gauss–Bonnet tensor $G^{\mathrm{GB}}_{\mu\nu}$ vanishes. Here, we want to obtain 5D static spherically symmetric solutions of Eq. (3) surrounded by the quintessence matter and investigate its properties. We assume that the metric has the form [18,19,28]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2 \tilde{\gamma}_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j, \tag{5}$$

where $\tilde{\gamma}_{ij}$ is the metric of a 3D constant curvature space $k = -1, 0$, or 1. In this letter, we shall restrict ourselves to $k = 1$. Using this metric *ansatz*, the Einstein–Gauss–Bonnet equation (3) reduces to

$$T^t_t = T^r_r = \frac{3}{2r^2}[rf' + 3(f-1)] - \frac{6\alpha}{r^3}[(f-1)f'],$$
$$T^\theta_\theta = T^\phi_\phi = T^\psi_\psi = \frac{1}{2r^2}\left[r^2 f'' + 4rf' + 2(f-1) \right]$$
$$- \frac{\alpha}{r^2}\left[2(f-1)f'' + 2f'^2 \right]. \tag{6}$$

The energy-momentum tensor of the quintessence matter (see Ref. [29] for further details) gets modified to

$$T^t_t = \rho(r),$$
$$T^b_a = \rho(r)\beta\left[-(1 + 4B(r))\frac{r_a r^b}{r_n r^n} + B(r)\delta^b_a \right], \tag{7}$$

where $B(r)$ is a quintessential parameter; we have

$$\langle T^b_a \rangle = \rho(r)\frac{\beta}{4}\delta^b_a = -p(r)\delta^b_a \tag{8}$$

and

$$\left\langle r_a r^b = \frac{1}{4} r_n r^n \right\rangle.$$

Thus, we have the equation of state of the form

$$p = \omega\rho, \quad \omega = \frac{1}{4}\beta, \tag{9}$$

where for the quintessential matter $-1 < \omega < 0$, which implies $-4 < \beta < 0$ within this set up, the parameter B of the energy-momentum tensor reads [35]

$$B = -\frac{4\omega + 1}{\omega}. \tag{10}$$

Hence, the energy-momentum tensor for a quintessence matter takes the form

$$T^t_t = T^r_r = \rho,$$
$$T^\theta_\theta = T^\phi_\phi = T^\psi_\psi = -\frac{1}{3}\rho(4\omega + 1), \tag{11}$$

where ρ is the proper density of the quintessence matter. The range of the parameters ω and β gets modified in 5 dimensions.

The Gauss–Bonnet term is the only possibility for the leading correction to Einstein general relativity. The static spherically symmetric black hole solution of Einstein action, modified by the Einstein–Gauss–Bonnet term, was first obtained by Boulware and Deser [3–5] and demonstrated that the only spherically symmetric solution to the Einstein–Gauss–Bonnet theory is a Schwarzschild type solution. Later, Wiltshire [51] included the Maxwell field into the Einstein–Gauss–Bonnet action and found the charged black hole in this theory, which was a generalization of the Reissner–Nordström black hole. In general, it is difficult to find a solution of Einstein–Gauss–Bonnet field equations (3) with an equation of state. Here, we shall find a black hole solution surrounded by the quintessence matter with the equation of state (9). Making use of Eqs. (6) and (11), we deduce the master equation for the Einstein–Gauss–Bonnet gravity as

$$[r^2 f''(r) + (5 + 4\omega)r f'(r) + 2(2 + 4\omega)(f(r) - 1)]r$$
$$- \alpha[4r f''(r)(f(r) - 1)$$
$$+ 4[r f'(r) + (1 + 4\omega)(f(r) - 1)f'(r)]] = 0. \quad (12)$$

In general, Eq. (12) has one real and two complex solutions. It may have three real solutions as well under some conditions. Here, we consider only the real solution. Interestingly, Eq. (12), for the Einstein–Gauss–Bonnet case, admits a general solution

$$f_\pm(r) = 1 + \frac{r^2}{4\alpha}\left(1 \pm \sqrt{1 + \frac{8\alpha M}{r^4} + \frac{8\alpha q}{r^{4\omega+4}}}\right), \quad (13)$$

by appropriately relating M and q with integrating constants c_1 and c_2 [25]. Equation (13) is an exact solution of the field equation (12) for an equation of state (9), which in the case of there being no quintessence, $\omega = 0$; it reduces to the Boulware and Deser [3–5] Gauss–Bonnet black hole solution, and for $\omega = 1/2$ and $q = -4Q^2/3$ to a solution mathematically similar to the charged Gauss–Bonnet black hole due to Wiltshire [51]. When $\omega = -1$, $q = \Lambda/3$, Eq. (13) corresponds to a Gauss–Bonnet de Sitter solution. In the limit $\alpha \to 0$, the negative branch of the solution (13) reduces to the general relativity solution. To study the general structure of the solution (13), we take the limit $r \to \infty$ or $M = q = 0$ in the solution (13) to obtain

$$\lim_{r\to\infty} f_+(r) = 1 + \frac{r^2}{2\alpha}, \quad \lim_{r\to\infty} f_-(r) = 1; \quad (14)$$

this means the plus (+) branch of the solution (13) is asymptotically de Sitter (anti-de Sitter) depending on the sign of α ($\pm$), whereas the minus branch of the solution (13) is

asymptotically flat. In the large r limit (or $\alpha \to 0$), Eq. (13) reduces to the 5D Schwarzschild solution surrounded by the quintessence matter. Thus, the negative branch solution (13) is well behaved and it represents a short distance correction to the 5D black hole solution of general relativity. In a similar way, when $M = 0$, the solution (13) takes the form

$$f_\pm(r) = 1 + \frac{r^2}{4\alpha}\left(1 \pm \sqrt{1 + \frac{8\alpha q}{r^{4\omega+4}}}\right). \quad (15)$$

Obviously, by a proper choice of the constants M, q, and the parameter ω, one can generate many other known solutions. The above solutions include most of the known spherically symmetric solutions of the Einstein–Gauss–Bonnet field equations (3).

3 Thermodynamics

In this section, we shall discuss the thermodynamical properties of 5D quintessential black hole within the Einstein–Gauss–Bonnet framework. Henceforth, we shall restrict ourselves to the negative branch of the solution (13). By definition of a horizon, the value of $r = r_+$ is an event horizon when $f(r_+) = 0$. This is shown in Fig. 1, by plotting $f(r)$ as a function of r. It is interesting to note that the black holes admit only one horizon and the radius of the horizon increases with the value of the quintessence matter parameter ω. Next, we explore the thermodynamics of the black hole solution (13) surrounded by the quintessence matter in the Einstein–Gauss–Bonnet framework. The Einstein–Gauss–Bonnet black holes surrounded by the quintessence matter are characterized by their mass (M) and a quintessence matter parameter (ω). From Eq. (13), the mass of the black hole is obtained in terms of the horizon radius (r_+):

$$M_+^{\text{EGB}} = r_+^2\left(1 - \frac{q}{r_+^{4\omega+2}} + \frac{2\alpha}{r_+^2}\right). \quad (16)$$

The mass of the black hole is plotted in Fig. 2 for various values of the parameters ω and α, which shows an increase in the black hole mass with horizon radius r_+. To discuss the thermodynamics of the metric (5) with the function (13), we start with the Hawking temperature. The Hawking temperature associated with the black hole is defined by $T = \kappa/2\pi$, where κ is the surface gravity defined by [23,28],

$$\kappa^2 = -\frac{1}{4}g^{tt}g^{ij}g_{tt,i}\,g_{tt,j}. \quad (17)$$

Hence, the Hawking temperature for the Einstein–Gauss–Bonnet black hole surrounded by the quintessence matter can be calculated as
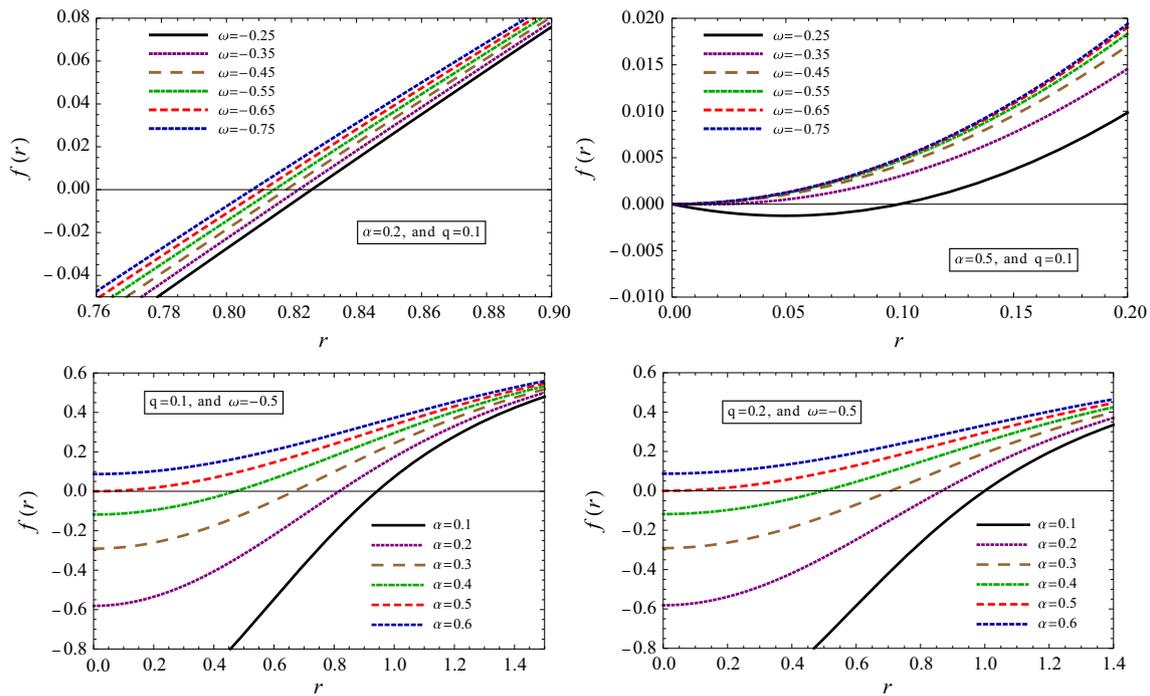
Fig. 1 Plot of metric function $f(r)$ vs. r for the 5D Einstein–Gauss–Bonnet black hole surrounded by the quintessence matter
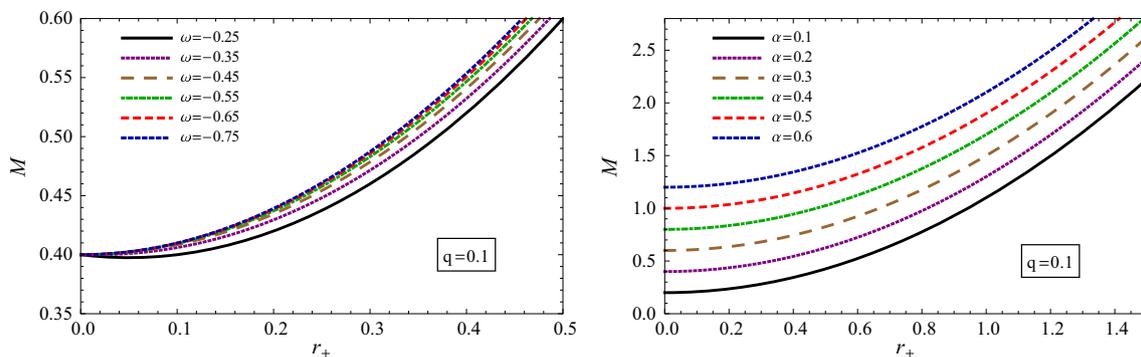

Fig. 2 Plot showing the behavior of mass M_+^{EGB} vs. horizon radius r_+ for 5D Einstein–Gauss–Bonnet black hole surrounded by the quintessence matter

$$T_+^{\text{EGB}} = \frac{1}{2\pi r_+}\left[\frac{1 + \frac{2q\omega}{r_+^{4\omega+2}}}{1 + \frac{4\alpha}{r_+^2}}\right]. \tag{18}$$

Note that the factor in the numerator of Eq. (18) modifies the Gauss–Bonnet black hole temperature [23,28], and taking the limit $q \rightarrow 0$, we recover the Gauss–Bonnet black hole temperature as

$$T_+^{\text{EGB}} = \frac{1}{2\pi r_+}\left[\frac{r_+^2}{r_+^2 + 4\alpha}\right]; \tag{19}$$

and when $\alpha \rightarrow 0$, it becomes the temperature given by $T_+ = \frac{1}{2\pi r_+}$. It is interesting to note that, for a particular radius of the horizon, the Hawking temperature vanishes.

The Hawking temperature diverges in general relativity as $r_+ \rightarrow 0$. However, in the Einstein–Gauss–Bonnet case it remains finite, as shown in Fig. 3. Also, when $q \neq 0$ and $\alpha \neq 0$, the Hawking temperature has a peak, which decreases and shifts as α increases or q increases (cf. Fig. 3).

A black hole behaves as a thermodynamical system; quantities associated with it must obey the first law of thermodynamics [21]:

$$dM = T_+ dS_+. \tag{20}$$

Hence, the entropy [52] can be obtained from the integration

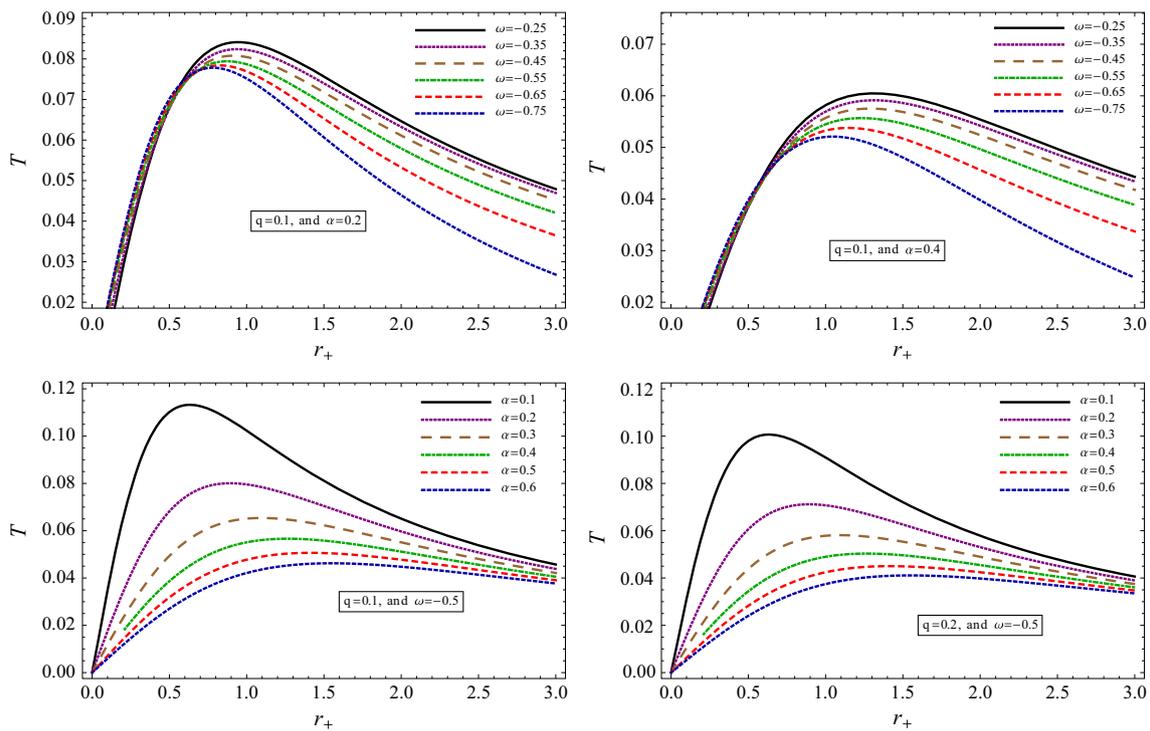$$S_+ = \int T_+^{-1} dM = \int T_+^{-1} \frac{\partial M}{\partial r_+} dr_+. \tag{21}$$

Fig. 3 The Hawking temperature (T_+^{EGB}) vs. horizon radius r_+ for different values of α ω with fixed $q = 0.1$ for the 5D Einstein–Gauss–Bonnet black hole surrounded by the quintessence matter

Now, the entropy of Einstein–Gauss–Bonnet gravity black holes surrounded by quintessence matter reads

$$S_+^{\text{EGB}} = \frac{4\pi r_+^3}{3} + 16\pi \alpha r_+. \tag{22}$$

However, it is interesting to note that the entropy of the black hole has no effect of the quintessence matter parameter.

Next, let us calculate the Wald entropy for the 5D black hole (5) with $f(r)$ is given by Eq. (13). Wald [53] showed that the black hole entropy can be calculated by

$$S_W = -\int_\Sigma \left(\frac{\partial \mathcal{L}}{\partial R_{abcd}} \right) \varepsilon_{ab}\varepsilon_{cd} dV_3^2, \tag{23}$$

where dV_3 is the volume element on Σ and the integral is performed on 3D space-like surface Σ. ε_{ab} is the binormal vector to the surface Σ normalized as $\varepsilon_{ab}\varepsilon^{ab} = -2$, and $\mathcal{L}$ is Lagrangian density as in (1). We note that the integrand can be calculated as

$$\frac{\partial \mathcal{L}}{\partial R_{abcd}} \varepsilon_{ab}\varepsilon_{cd} = -2 - \frac{24\alpha[1 - f_-(r_+)]}{r_+^2}. \tag{24}$$

On substituting Eq. (24) into Eq. (23), one obtains the Wald entropy of the 5D black hole (5) as

$$S_W = \left[1 + \frac{12\alpha[1 - f_-(r_+)]}{r_+^2} \right] \int_\Sigma dV_3^2,$$

$$= \frac{4\pi r_+^3}{3} \left[1 + \frac{12\alpha[1 - f_-(r_+)]}{r_+^2} \right],$$

$$= \frac{4\pi r_+^3}{3} + 16\pi \alpha r_+, \tag{25}$$

as $f_-(r_+) = 0$ and hence the Wald entropy equation (25) has exactly the same expression as obtained in Eq. (22). Furthermore, we verify that the 5D black hole (5) satisfies the first law of thermodynamics. The variation of the Wald entropy (25) with respect to the radius r_+ gives

$$dS_W = 4\pi(r_+^2 + 4\alpha), \tag{26}$$

and the variation of the mass (16) leads to

$$dM_+^{\text{EGB}} = 2r_+ \left(1 + \frac{2q\omega}{r_+^{4\omega+2}} \right). \tag{27}$$

Hence, with the help of Eqs. (18), (26) and (27), one can conclude that

$$dM_+^{\text{EGB}} = T_+^{\text{EGB}} dS_W. \tag{28}$$

Hence the first law of the black hole thermodynamics holds for a 5D black hole (5) with $f(r)$ is given by Eq. (13).
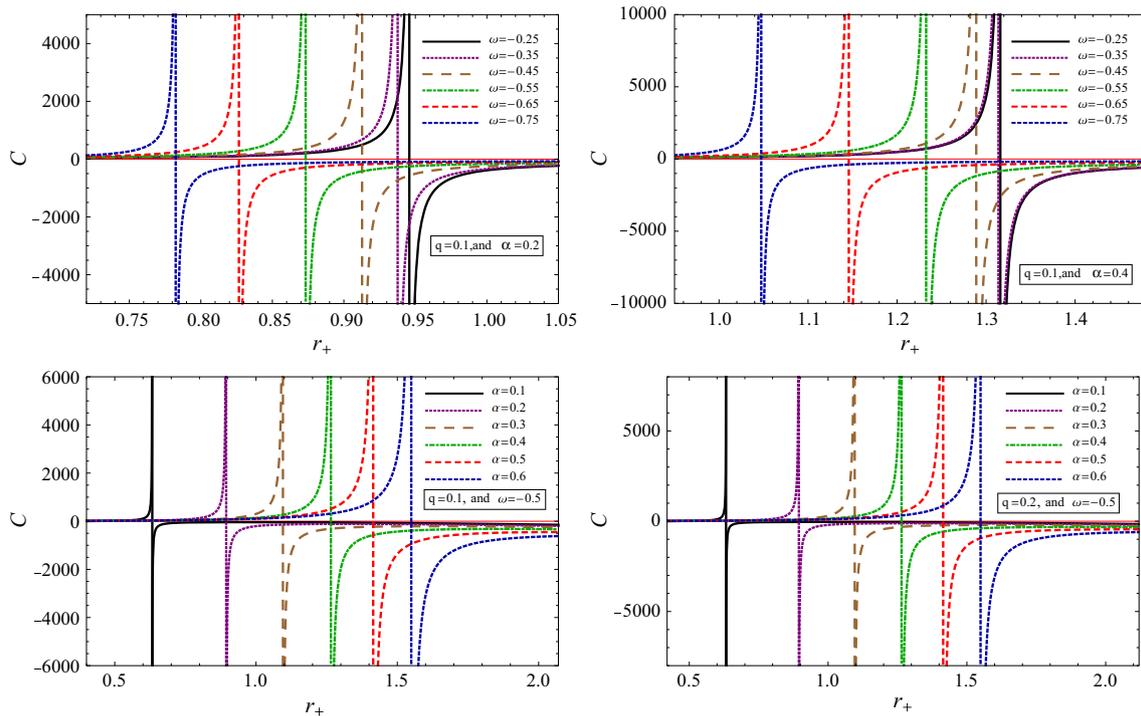
Fig. 4 The specific heat (C_+^{EGB}) vs. horizon radius r_+ for the 5D Einstein–Gauss–Bonnet black hole surrounded by the quintessence matter for different values of ω and α

Finally, we analyze how the quintessence matter influences the thermodynamic stability of the Einstein–Gauss–Bonnet black holes. The heat capacity of the black hole is defined as [21]

$$C_+ = \frac{\partial M}{\partial T_+} = \frac{\partial M}{\partial r_+}\frac{\partial r_+}{\partial T_+}. \tag{29}$$

The heat capacity of Einstein–Gauss–Bonnet black hole surrounded by quintessence matter, using Eqs. (16), (18), and (29), is given by

$$C_+^{\mathrm{EGB}} = \frac{-4\pi r_+^3 \left[1 + \frac{2q\omega}{r^{4\omega+2}}\right](r_+^2 + 4\alpha)^2}{-4\alpha r_+^2 \left[1 - \frac{2q\omega(1+4\omega)}{r_+^{4\omega+2}}\right] + r_+^4 \left[1 + \frac{2q\omega(3+4\omega)}{r^{4\omega+2}}\right]}. \tag{30}$$

It is clear that the heat capacity depends on both the Gauss–Bonnet coefficient α and the quintessence matter parameter ω. When $\alpha \to 0$, it reduces to the general relativity case. If in addition $q = 0$, it becomes

$$C_+ = \frac{4\pi r_+^3(r_+^2 + 4\alpha)^2}{4\alpha r_+^2 - r_+^4}, \tag{31}$$

which is exactly same as the Einstein–Gauss–Bonnet case [23,28]. We again recall that, for $C > 0$ ($C < 0$), the black hole is thermodynamically stable (unstable). It is difficult to analyze the heat capacity analytically hence, we plotted it in

Fig. 4, for different values of α and ω. Again, we note that there is a change of sign in the heat capacity around r_C, and C is discontinuous at $r_+ = r_C$. The heat capacity is positive for $r_+ < r_C$ and thereby suggests the thermodynamical stability of a black hole. On the other hand, the black hole is unstable for $r_+ > r_C$. Thus, the heat capacity of an Einstein–Gauss–Bonnet black hole, for different values of ω and α, is positive for $r_+ < r_C$, while for $r_+ > r_C$ it is negative. The phase transition occurs from a lower mass black hole with negative heat capacity to a higher mass black hole with positive heat capacity.

It may be noted that the critical radius r_C changes drastically in the presence of the quintessence matter, thereby affecting the thermodynamical stability. Indeed, the value of the critical radius r_C increases with the increase in the quintessence matter parameter ω for a given value of the Gauss–Bonnet coupling constant α. Further, r_C is also sensitive to the Gauss–Bonnet parameter α (cf. Fig. 4), and the critical parameter r_C also increases with α.

4 Conclusion

The Einstein–Gauss–Bonnet theory has a number of additional nice properties in addition to Einstein's general relativity that are not enjoyed by other higher-curvature theories. Hence, Einstein–Gauss–Bonnet theory has received signifi-

cant attention, especially when finding black hole solutions. We have obtained an exact 5D static spherically symmetric black hole solutions to Einstein–Gauss–Bonnet gravity surrounded by the quintessence matter. We then proceeded to find exact expressions, in Einstein–Gauss–Bonnet gravity, for the thermodynamical quantities like the black hole mass, Hawking temperature, entropy, specific heat and in turn also analyzed the thermodynamical stability of black holes. It turns out that due to the quintessence correction to the black hole solution, the thermodynamical quantities are also getting corrected except for the entropy, which does not depend on the background quintessence. The entropy of a black hole in Einstein–Gauss–Bonnet gravity does not obey the area law.

The phase transition is characterized by the divergence of the specific heat at a critical radius r_C, which is changing with Gauss–Bonnet parameter α as well as with w. In particular, the black hole is thermodynamically stable with a positive heat capacity for the range $0 < r < r_C$ and unstable for $r > r_C$. It would be important to understand how these black holes with positive specific heat ($C > 0$) would emerge from thermal radiation through a phase transition. We also discuss the phase transition of the black holes. The results presented here are a generalization of the previous discussions, on the Einstein–Gauss–Bonnet black hole, in a more general setting, and the possibility of a further generalization of these results to Lovelock gravity is an interesting problem for future research.

Appendix A: Exact solutions for general relativity

Making use of Eqs. (6) and (12), for $\alpha = 0$, we obtain

$$r^2 f''(r) + (5 + 4\omega)r f'(r) + 2(2 + 4\omega)(f(r) - 1) = 0,$$
$$\tag{A1}$$

in which a prime denotes a derivative with respect to r. Equation (A1) admits a general solution which describes a 5D black hole surrounded by the quintessence matter, and the corresponding metric for the spherically symmetric takes the form

$$ds^2 = -\left[1 - \frac{M}{r^2} - \frac{q}{r^{4\omega+2}}\right]dt^2 + \frac{1}{\left[1 - \frac{M}{r^2} - \frac{q}{r^{4\omega+2}}\right]}dr^2 + r^2 d\omega_3^2,$$
$$\tag{A2}$$

with $d\omega_3^2$ the metric on the 3-sphere. This solution for the d-dimensional case was found in Ref. [35]. In order to study the general structure of the solution (A2), we look for the essential singularity by calculating the Kretschmann scalar ($\mathcal{K} = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}$), which for the metric (5) after inserting Eq. (A2) reads

$$\mathcal{K} = \frac{18M}{r^8} + \frac{12(1+\omega)(4\omega+3)qM}{r^{4(\omega+2)}} + \frac{Bq^2}{r^{8(\omega+1)}},$$
$$\tag{A3}$$

with $B = 6\left(32\omega^4 + 80\omega^3 + 86\omega^2 + 42\omega + 9\right)$. The Kretschmann scalar diverges as $r \to 0$, indicating a scalar polynomial or essential singularity at $r = 0$ [54]. The energy density (ρ) and the pressure (P) for the quintessence matter can be expressed as

$$\rho = -\frac{6q\omega}{r^{4(\omega+1)}}, \quad P = \frac{2q\omega(1+4\omega)}{r^{4(\omega+1)}},$$
$$\tag{A4}$$

and

$$\rho + P = \frac{4q\omega(2\omega-1)}{r^{4(\omega+1)}}.$$
$$\tag{A5}$$

The weak energy condition is satisfied since $\rho_q \geq 0$ and $\rho_q + P_q \geq 0$, for $-1 < \omega < 0$. When $\omega = -1$, the metric (A2) takes the form of a 5D Schwarzschild–de Sitter black hole. The metric (A2) also reduces to a 5D Reissner–Nordström black hole when $\omega = 1/2$. The solution (A2) represents a general class of static, spherically symmetric solutions to the Einstein equations describing black holes with the energy-momentum tensor of the quintessence matter. The solution (A2) includes several well-known spherically symmetric solutions of the Einstein field equations, for instance, the 5D Schwarzschild solution for $q = 0$, including its generalization to the asymptotically de Sitter/Anti-de Sitter (dS/AdS) case for $q(v) \neq 0$, $\omega = -1$ [55]. Obviously, by a proper choice of the functions M and q, and of the ω parameter, one can generate many other solutions [55].

Next, we analyze the thermodynamics of the quintessence corrected black hole given by the metric (A2). The event horizon r_+ of the black hole, satisfy $g^{rr}(r_+) = 0$, i.e.,

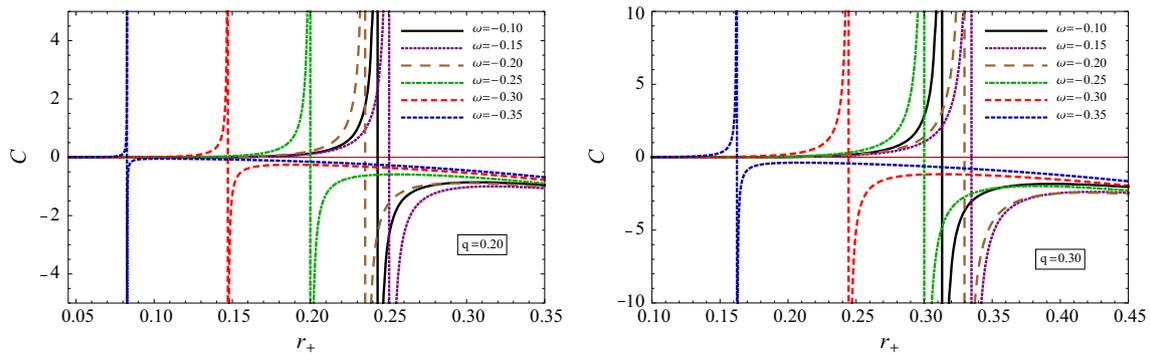$$r^{4\omega+2} - Mr^{4\omega} - q = 0.$$
$$\tag{A6}$$

Fig. 5 The specific heat (C_+) vs. horizon radius r_+ for the 5D general relativity black hole surrounded by the quintessence matter. The black hole thermodynamically unstable when $C_+ < 0$, and stable when $C_+ > 0$

On the other hand, the quintessence matter alone ($M = 0$) has a horizon placed at $r_+ = (q)^{1/4\omega+2}$. Obviously, in the limit $q \to 0$, the above solution will reduce to a 5D general relativity black hole, in which case $R = \mathcal{R} = 0$. Next, we shall discuss the thermodynamics of the 5D black hole surrounded by the quintessence matter. We note that the gravitational mass of a black hole is determined by $g^{rr}(r_+) = 0$, which, from Eq. (A2), reads

$$M_+ = r_+^2 \left[1 - \frac{q}{r_+^{4\omega+2}} \right]. \qquad (A7)$$

Equation (A7) takes the form of the 5D Schwarzschild black hole mass $M = r_+^2$, when $q \to 0$ [18,19]. Accordingly, the Hawking temperature of the black hole at outer horizon, r_+, reads

$$T_+ = \frac{\kappa}{2\pi} = \frac{1}{2\pi r_+} \left[1 + \frac{2q\omega}{r_+^{4\omega+2}} \right]. \qquad (A8)$$

Then we can easily see that the temperature is positive. Taking the limit $q \to 0$, we recover the temperature for 5D general relativity $T_+ = \frac{1}{2\pi r_+}$ [23,28], which shows that the Hawking temperature diverges as $r_+ \to 0$. Next, we turn to a calculation of the entropy associated with the black hole horizon from Eq. (25), and we arrive at

$$S = \frac{4\pi r_+^3}{3}. \qquad (A9)$$

Thus, we note that the quantity $4\pi r_+^3/3$ of Eq. (A9) is just the area of the black hole horizon. In proper units Eq. (A9) may be written as $S_+ = A/4G$ (see Appendix B). Thus, we conclude that the entropy of the 5D black hole obeys the area law, and the entropy of a black hole has no influence of the quintessence matter. Next, we turn our attention to the stability of the black holes by calculating the specific heat of the black hole solution (A2) and to discuss the effect of the quintessence matter. The Schwarzschild black hole and higher-dimensional Schwarzschild–Tangherlini case always

have negative heat capacity, indicating thermodynamic instability of these black holes [20]. Inserting Eqs. (A7) and (A8) in (29), we obtain

$$C_+ = \frac{-4\pi r_+^3 \left[1 + \frac{2q\omega}{r_+^{4\omega+2}} \right]}{\left[1 + \frac{2q\omega(3+4\omega)}{r_+^{4\omega+2}} \right]}. \qquad (A10)$$

It is well known that the thermodynamic stability of the system is related to the sign of the heat capacity (C). If the heat capacity is positive ($C > 0$), then the black hole is stable; when it is negative ($C < 0$), the black hole is said to be unstable. It is clear from Eq. (A10) that the heat capacity depends on the quintessence matter. When $q \to 0$, one gets $C = -4\pi r_+^3$, which means 5D general relativity black holes are thermodynamically unstable [21,28]. Next, we analyze the effect of the quintessence matter on thermodynamical stability of a black hole. We plot the specific heat (C) with radius r in Fig. 5 for different values of the parameter ω and q. It is seen that the heat capacity discontinuous at $r = r_C$, for each ω, and for a given q. We observe that the heat capacity is $C > 0$ ($C < 0$) for $r_+ < r_C$ ($r_+ > r_C$). Thus, the 5D black hole is thermodynamically stable for $r < r_C$, as the black hole has a positive heat capacity and is unstable for $r > r_C$ (cf. Fig. 5). The black hole mass increases with increasing r_+. Hence, a phase transition occurs from a lower mass black hole with negative heat capacity to a higher mass black hole with positive heat capacity.

Appendix B: Thermodynamics of *d*-dimensional black holes

The metric of *d*-dimensional spherically symmetric black hole surrounded by quintessence reads [35]

$$ds^2 = -\left[1 - \frac{16\pi G^{(d)} M}{(d-2)\Gamma^{(d-2)} c^4 r^{(d-3)}} - \frac{q}{r^{[(d-1)\omega+(d-3)]}}\right] dt^2$$
$$+ \left[1 - \frac{16\pi G^{(d)} M}{(d-2)\Gamma^{(d-2)} c^4 r^{(d-3)}} - \frac{q}{r^{[(d-1)\omega+(d-3)]}}\right]^{-1} dr^2$$
$$+ r^2 d\Omega^{(d-2)},$$

where $G^{(d)}$ is the d-dimensional Newton constant, $d\Omega^{(d-2)}$ is the line element on a unit $(d-2)$ sphere, and $\Gamma^{(d-2)}$ is the volume of the unit $(d-2)$ sphere. The black hole has a horizon at $r = r_+$, which is solution of $f(r_+) = 0$,

$$r_+^{[(d-1)\omega+d-3]} - \frac{16\pi G^{(d)} M}{(d-2)\Gamma^{(d-2)} c^4} r_+^{(d-1)\omega} - q = 0.$$

The black hole mass in terms of horizon r_+ reads

$$M = \frac{(d-2)\Gamma^{(d-2)} c^4 r_+^{(d-3)}}{16\pi G^{(d)}} \left[1 - \frac{q}{r_+^{[(d-1)\omega+(d-3)]}}\right].$$

The Hawking temperature is

$$T_+ = \frac{\hbar c(d-3)}{4\pi r_+} \left[1 - \frac{(d-1)}{(d-3)} \frac{q\omega}{r_+^{[(d-1)\omega+(d-3)]}}\right].$$

By applying the entropy formula (25), one can deduce the general expression for the entropy:

$$S = \frac{1}{4\hbar G^{(d)}} c^3 \Gamma^{(d-2)} r_+^{(d-2)} = \frac{A}{4G^{(d)}},$$

and it is seen to satisfy the area law.

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