Stable exponential cosmological solutions with zero variation of G and three different Hubble-like parameters in the Einstein–Gauss–Bonnet model with a Λ -term

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Received: 17 May 2017 / Accepted: 1 June 2017 / Published online: 16 June 2017

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Abstract We consider a D -dimensional gravitational model with a Gauss–Bonnet term and the cosmological term Λ . We restrict the metrics to diagonal cosmological ones and find for certain Λ a class of solutions with exponential time dependence of three scale factors, governed by three non-coinciding Hubble-like parameters $H > 0$, h_1 and h_2 , corresponding to factor spaces of dimensions $m > 2$, $k_1 > 1$ and $k_2 > 1$, respectively, with $k_1 \neq k_2$ and $D = 1 + m + k_1 + k_2$. Any of these solutions describes an exponential expansion of $3d$ subspace with Hubble parameter H and zero variation of the effective gravitational constant G . We prove the stability of these solutions in a class of cosmological solutions with diagonal metrics.

1 Introduction

In this paper we consider a D -dimensional gravitational model with Gauss–Bonnet term and cosmological term Λ . The so-called Gauss–Bonnet term appeared in string theory as a first order correction (in α') to the effective action [1–4].

We note that at present the Einstein–Gauss–Bonnet (EGB) gravitational model and its modifications, see [5–28] and the references therein, are intensively studied in cosmology, e.g. for possible explanation of accelerating expansion of the Universe which follow from supernova (type Ia) observational data [29–31].

In Ref. [28] we were dealing with the cosmological solutions with diagonal metrics governed by $n > 3$ scale factors depending upon one variable, which is the synchronous time variable. We have restricted ourselves by the solutions with exponential dependence of scale factors and have

presented a class of such solutions with two scale factors, governed by two Hubble-like parameters $H > 0$ and $h < 0$, which correspond to factor spaces of dimensions $m > 3$ and $l > 1$, respectively, with $D = 1 + m + l$ and $(m, l) \neq (6, 6), (7, 4), (9, 3)$. Any of these solutions describes an exponential expansion of $3d$ subspace with Hubble parameters $H > 0$ [32] and has a constant volume factor of $(m - 3 + l)$ -dimensional internal space, which implies zero variation of the effective gravitational constant G either in a Jordan or in an Einstein frame [33, 34]; see also [35–37] and the references therein. These solutions satisfy the most severe restrictions on variation of G [38]. We have studied the stability of these solutions in a class of cosmological solutions with diagonal metrics by using results of Refs. [24, 26] (see also approach of Ref. [22]) and have shown that all solutions, presented in Ref. [28], are stable. It should be noted that two special solutions for $D = 22, 28$ and $\Lambda = 0$ were found earlier in Ref. [21]; in Ref. [24] it was proved that these solutions are stable. Another set of six stable exponential solutions, five in dimensions $D = 7, 8, 9, 13$ and two for $D = 14$, were considered earlier in [27].

In this paper we extend the results of Ref. [28] to the case of solutions with three non-coinciding Hubble-like parameters. The structure of the paper is as follows. In Sect. 2 we present a setup. A class of exact cosmological solutions with diagonal metrics is found for certain Λ in Sect. 3. Any of these solutions describes an exponential expansion of 3-dimensional subspace with Hubble parameter H and zero variation of the effective gravitational constant G . In Sect. 4 we prove the stability of the solutions in a class of cosmological solutions with diagonal metrics. Certain examples are presented in Sect. 5.

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2 The cosmological model

The action of the model reads

$$S = \int_M d^D z \sqrt{|g|} \{ \alpha_1 (R[g] - 2\Lambda) + \alpha_2 \mathcal{L}_2[g] \}, \quad (2.1)$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric defined on the manifold M , $\dim M = D$, $|g| = |\det(g_{MN})|$, Λ is the cosmological term, $R[g]$ is scalar curvature,

$$\mathcal{L}_2[g] = R_{MNPQ} R^{MNPQ} - 4R_{MN} R^{MN} + R^2$$

is the standard Gauss–Bonnet term and α_1, α_2 are nonzero constants.

We consider the manifold

$$M = \mathbb{R} \times M_1 \times \cdots \times M_n \quad (2.2)$$

with the metric

$$g = -dt \otimes dt + \sum_{i=1}^n B_i e^{2v^i t} dy^i \otimes dy^i, \quad (2.3)$$

where $B_i > 0$ are arbitrary constants, $i = 1, \dots, n$, and $M_1, \dots, M_n$ are 1-dimensional manifolds (either $\mathbb{R}$ or S^1) and $n > 3$.

The equations of motion for the action (2.1) give us the set of polynomial equations [24]

$$E = G_{ij} v^i v^j + 2\Lambda - \alpha G_{ijkl} v^i v^j v^k v^l = 0, \quad (2.4)$$

$$Y_i = \left[2G_{ij} v^j - \frac{4}{3} \alpha G_{ijkl} v^j v^k v^l \right] \sum_{i=1}^n v^i - \frac{2}{3} G_{ij} v^i v^j + \frac{8}{3} \Lambda = 0, \quad (2.5)$$

$i = 1, \dots, n$, where $\alpha = \alpha_2/\alpha_1$. Here

$$G_{ij} = \delta_{ij} - 1, \quad G_{ijkl} = G_{ij} G_{ik} G_{il} G_{jk} G_{jl} G_{kl}, \quad (2.6)$$

are, respectively, the components of two metrics on $\mathbb{R}^n$ [16, 17]. The first one is a 2-metric and the second one is a Finslerian 4-metric. For $n > 3$ we get a set of fourth-order polynomial equations.

We note that for $\Lambda = 0$ and $n > 3$ the set of Eqs. (2.4) and (2.5) has an isotropic solution $v^1 = \cdots = v^n = H$ only if $\alpha < 0$ [16, 17]. This solution was generalized in [19] to the case $\Lambda \neq 0$.

It was shown in [16, 17] that there are no more than three different numbers among $v^1, \dots, v^n$ when $\Lambda = 0$. This is valid also for $\Lambda \neq 0$ if $\sum_{i=1}^n v^i \neq 0$ [26].

3 Solutions with constant G

In this section we present a class of solutions to the set of equations (2.4), (2.5) of the following form:

$$v = (\underbrace{H, H, H}_{\text{“our” space}}, \underbrace{H, \dots, H}_{m-3}, \underbrace{h_1, \dots, h_1}_{k_1}, \underbrace{h_2, \dots, h_2}_{k_2}). \quad (3.1)$$

where H is the Hubble-like parameter corresponding to an m -dimensional factor space with $m > 2$, h_1 is the Hubble-like parameter corresponding to an k_1 -dimensional factor space with $k_1 > 1$ and h_2 ($h_2 \neq h_1$) is the Hubble-like parameter corresponding to an k_2 -dimensional factor space with $k_2 > 1$. We split the m -dimensional factor space into the product of two subspaces of dimensions 3 and $m - 3$, respectively. The first one is identified with “our” 3d space, while the second one is considered as a subspace of $(m - 3 + k_1 + k_2)$ -dimensional internal space.

We put

$$H > 0 \quad (3.2)$$

for a description of an accelerated expansion of a 3-dimensional subspace (which may describe our Universe) and also put

$$(m - 3)H + k_1 h_1 + k_2 h_2 = 0 \quad (3.3)$$

for a description of a zero variation of the effective gravitational constant G .

We remind the reader that the effective gravitational constant $G = G_{eff}$ in the Brans–Dicke–Jordan (or simply Jordan) frame [33] (see also [34]) is proportional to the inverse volume scale factor of the internal space; see [35–37] and references therein.

Due to (3.1) “our” 3d space expands isotropically with Hubble parameter H , while the $(m - 3)$ -dimensional part of the internal space expands isotropically with the same Hubble parameter H too. Here, like in Ref. [28], we consider for cosmological applications (in our epoch) the internal space to be a compact one, i.e. we put in (2.2) $M_4 = \cdots = M_n = S^1$. We put the internal scale factors corresponding to present time t_0 : $a_j(t_0) = B_j^{1/2} \exp(v^j t_0)$, $j = 4, \dots, n$, (see (2.3)) to be small enough in comparison with the scale factor of “our” space for $t = t_0$: $a(t_0) = B^{1/2} \exp(H t_0)$, where $B_1 = B_2 = B_3 = B$.

According to the ansatz (3.1), the m -dimensional factor space is expanding with the Hubble parameter $H > 0$, while the k_i -dimensional factor space is contracting with the Hubble-like parameter $h_i < 0$, where i is either 1 or 2.

Now we consider the ansatz (3.1) with three Hubble parameters H, h_1 and h_2 which obey the following restrictions:

$$S_1 = mH + k_1h_1 + k_2h_2 \neq 0, \quad H \neq h_1, \\ H \neq h_2, \quad h_1 \neq h_2. \quad (3.4)$$

The first inequality in (3.4) is valid since $S_1 = 3H > 0$ due to (3.2) and (3.3).

In this case the set of $n + 1$ equations (2.4), (2.5) is equivalent to the set of three equations

$$E = 0, \quad Y_H = 0, \quad Y_{h_1} = 0, \quad Y_{h_2} = 0, \quad (3.5)$$

where

$$Y_H = Y_\mu, \quad Y_{h_1} = Y_\alpha, \quad Y_{h_2} = Y_a, \quad (3.6)$$

for all $\mu = 1, \dots, m$; $\alpha = m + 1, \dots, m + k_1$ and $a = m + k_1 + 1, \dots, n$. These relations follow from the definition of Y_i in (2.5) and the identities [16, 17]

$$v_i = G_{ij}v^j = v^i - S_1, \quad (3.7)$$

$$A_i = G_{ijkl}v^jv^kv^l = S_1^3 + 2S_3 - 3S_1S_2 \\ + 3(S_2 - S_1^2)v^i + 6S_1(v^i)^2 - 6(v^i)^3, \quad (3.8)$$

$i = 1, \dots, n$, where here and in what follows

$$S_k = \sum_{i=1}^n (v^i)^k. \quad (3.9)$$

Due to (2.5), (3.7), (3.8) we obtain

$$Y_{h_i} - Y_{h_j} = (h_i - h_j)S_1[2 + 4\alpha Q_{h_i, h_j}], \quad (3.10)$$

where

$$Q_{h_i, h_j} = S_1^2 - S_2 - 2S_1(h_i + h_j) + 2(h_i^2 + h_i h_j + h_j^2), \quad (3.11)$$

$i \neq j$; $i, j = 0, 1, 2$ and $h_0 = H$. Equations (3.4), (3.5) and (3.10) imply

$$Q_{h_i, h_j} = -\frac{1}{2\alpha}, \quad (3.12)$$

$i \neq j$ and $i, j = 0, 1, 2$.

Due to $S_1 = mH + k_1h_1 + k_2h_2 \neq 0$ the set of equations (3.5) is equivalent to the following set of equations:

$$E = 0, \quad Y_H - Y_{h_1} = 0, \quad Y_{h_1} - Y_{h_2} = 0, \\ mHY_H + k_1h_1Y_{h_1} + k_2h_2Y_{h_2} = 0. \quad (3.13)$$

The last relation in (3.13) may be omitted since $E = 0$ implies $Y_i h^i = mHY_H + k_1h_1Y_{h_1} + k_2h_2Y_{h_2} = 0$ [26]. Using this fact and Eqs. (3.4) and (3.10) we reduce the system (3.13) to the following one:

$$E = 0, \quad Q_{Hh_1} = -\frac{1}{2\alpha}, \quad Q_{h_1h_2} = -\frac{1}{2\alpha}. \quad (3.14)$$

Using the identity

$$Q_{Hh_1} - Q_{h_1h_2} = (H - h_2)(-S_1 + H + h_1 + h_2), \quad (3.15)$$

we reduce the set of equations (3.14) to the equivalent set

$$E = 0, \quad Q = -\frac{1}{2\alpha}, \quad H + h_1 + h_2 - S_1 = 0. \quad (3.16)$$

Here we put $Q = Q_{h_1h_2}$, though other choices, $Q = Q_{Hh_1}$ or $Q = Q_{Hh_2}$, give us equivalent sets of equations. Thus the set of $(n + 1)$ polynomial equations (2.4), (2.5) under ansatz (3.1) and restrictions (3.4) imposed is reduced to a set (3.16) of three polynomial equations (of fourth, second and first orders). This reduction is a special case of the more general prescription from Ref. [20].

Using the condition (3.3) of zero variation of G and the linear equation from (3.16) we obtain for $k_1 \neq k_2$,

$$h_1 = \frac{m + 2k_2 - 3}{k_2 - k_1}H, \quad h_2 = \frac{m + 2k_1 - 3}{k_1 - k_2}H. \quad (3.17)$$

For $k_1 = k_2$ we get $H = 0$, which is not appropriate for our consideration.

The substitution of (3.17) into relation $Q_{h_1h_2} = -\frac{1}{2\alpha}$ gives us the following relation:

$$\frac{P}{(k_2 - k_1)^2}H^2 = -\frac{1}{2\alpha}, \quad (3.18)$$

for $k_1 \neq k_2$, where

$$P = P(m, k_1, k_2) = -(m + k_1 + k_2 - 3)(m(k_1 + k_2 - 2) \\ + k_1(2k_2 - 5) + k_2(2k_1 - 5) + 6) \neq 0, \quad (3.19)$$

which implies

$$H = |k_1 - k_2|(-2\alpha P)^{-1/2}, \quad \alpha P < 0. \quad (3.20)$$

It may be readily verified that

$$P = P(m, k_1, k_2) < 0 \quad (3.21)$$

for all $m > 2$, $k_1 > 1$, $k_2 > 1$, $k_1 \neq k_2$ and hence our solutions take place for $\alpha > 0$.

The substitution of (3.17) into (3.5) gives us

$$2\Lambda = -F_1H^2 - F_2H^4 \quad (3.22)$$

where

$$F_1 = \frac{1}{(k_2 - k_1)^2}[(k_1 + k_2)m^2 + (k_1^2 + 6k_1k_2 + k_2^2 - 6k_1 - 6k_2)m \\ - 9(k_1^2 + k_2^2 - k_1 - k_2) + 2(2k_1 + 2k_2 - 3)k_1k_2] \quad (3.23)$$

and

$$F_2 = -\frac{3\alpha(m-3+k_1+k_2)}{(k_2-k_1)^4}[(k_1+k_2)(k_1+k_2-2)m^3 + (k_1+k_2)(k_1^2+k_2^2+10k_1k_2-15(k_1+k_2)+18)m^2 - (12(k_1^3+k_2^3)-63(k_1^2+k_2^2)+54(k_1+k_2) - 2(4(k_1^2+k_2^2) - 42(k_1+k_2+16k_1k_2+63)k_1k_2))m + 27(k_1^3+k_2^3)-81(k_1^2+k_2^2)+54(k_1+k_2) - (40(k_1^2+k_2^2)-16(k_1+k_2-6)k_1k_2+162 - 153(k_1+k_2))k_1k_2]. \quad (3.24)$$

Using Eqs. (3.20), (3.22), (3.23), (3.24) we obtain

$$\Lambda = \Lambda(m, k_1, k_2) = \frac{1}{8\alpha P^2}(m+k_1+k_2-3) \times [(k_1+k_2)(k_1+k_2-2)m^3 + (k_1^3+k_2^3+11(k_1^2k_2+k_1k_2^2)-19(k_1^2+k_2^2) - 22k_1k_2+18(k_1+k_2))m^2 - (8(k_1^3+k_2^3)-63(k_1+k_2)^2-8k_1^2(k_1-11)k_2 - 8k_2^2(k_2-11)k_1)-32k_1^2k_2^2+54(k_1+k_2))m - (9(k_1^3+k_2^3)+45(k_1^2+k_2^2)-54(k_1+k_2) + 8(k_1^2+k_2^2)k_1k_2 - 16(k_1+k_2-10)k_1^2k_2^2-9(21k_1+21k_2-26)k_1k_2)], \quad (3.25)$$

where $P = P(m, k_1, k_2)$ is defined in (3.19).

The function $\Lambda(m, k_1, k_2)$ in (3.25) is symmetric with respect to k_1 and k_2 , i.e.

$$\Lambda(m, k_1, k_2) = \Lambda(m, k_2, k_1). \quad (3.26)$$

For $k_2 = 0$ we get a function $\Lambda(m, k_1, 0) = \Lambda(m, k_1)$, where $\Lambda(m, k_1)$ was obtained in Ref. [28] for the case of two different Hubble-like parameters.

It may be readily verified that for $k_1(k) = n_1k + q_1$ and $k_2(k) = n_2k + q_2$, where $k, n_1 > 0, q_1, n_2 > 0, q_2$ are integer numbers, we get

$$\Lambda(m, k_1(k), k_2(k)) \rightarrow \frac{1}{8\alpha}, \quad (3.27)$$

as $k \rightarrow +\infty$ for any fixed $m \geq 3$. We note that the limit (3.27) is positive and does not depend upon m . For fixed integer $m > 2$ and $k_2 \geq 1$ we are led to the following limit:

$$\Lambda(m, k_1, k_2) \rightarrow \frac{1}{8\alpha(m+4k_2-5)^2} [m^2 - 8(1-k_2)m - 9 - 8k_2 + 16k_2^2] = \Lambda(m, \infty, k_2), \quad (3.28)$$

as $k_1 \rightarrow +\infty$ and there is an analogous relation (due to (3.26)) for fixed $m > 2, k_1 \geq 1$ and $k_2 \rightarrow +\infty$. It can

be easily verified that, for these values of m, k_1 we get $\Lambda(m, \infty, k_2) > 0$.

Equations (3.27) and (3.28) may be used in a context of $(1/D)$ -expansion for large D in the model under consideration; see [25] and the references therein.

4 The proof of stability

Here, as in [28], we have due to (3.3)

$$K = K(v) = \sum_{i=1}^n v^i = 3H > 0. \quad (4.1)$$

Let us put the restriction

$$\det(L_{ij}(v)) \neq 0 \quad (4.2)$$

on the matrix

$$L = (L_{ij}(v)) = (2G_{ij} - 4\alpha G_{ijks} v^k v^s). \quad (4.3)$$

We recall that, for a general cosmological setup with the metric

$$g = -dt \otimes dt + \sum_{i=1}^n e^{2\beta^i(t)} dy^i \otimes dy^i, \quad (4.4)$$

we have the set of equations [24]

$$E = G_{ij} h^i h^j + 2\Lambda - \alpha G_{ijkl} h^i h^j h^k h^l = 0, \quad (4.5)$$

$$Y_i = \frac{dL_i}{dt} + \left(\sum_{j=1}^n h^j \right) L_i - \frac{2}{3} (G_{sj} h^s h^j - 4\Lambda) = 0, \quad (4.6)$$

where $h^i = \dot{\beta}^i$,

$$L_i = L_i(h) = 2G_{ij} h^j - \frac{4}{3} \alpha G_{ijkl} h^j h^k h^l, \quad (4.7)$$

$i = 1, \dots, n$.

Due to the results of Ref. [26] a fixed point solution $(h^i(t)) = (v^i)$ ($i = 1, \dots, n; n > 3$) to Eqs. (4.5), (4.6) obeying restrictions (4.1), (4.2) is stable under perturbations,

$$h^i(t) = v^i + \delta h^i(t), \quad (4.8)$$

$i = 1, \dots, n$, as $t \rightarrow +\infty$.

In order to prove the stability of solutions we should prove Eq. (4.2). First, we show that for the vector v from (3.1), obeying Eqs. (3.4) the matrix L has a block-diagonal form,

$$(L_{ij}) = \text{diag}(L_{\mu\nu}, L_{\alpha\beta}, L_{ab}), \quad (4.9)$$

where here and in what follows: $\mu, \nu = 1, \dots, m$; $\alpha, \beta = m + 1, \dots, m + k_1$ and $a, b = m + k_1 + 1, \dots, n$.

Indeed, denoting $S_{ij} = G_{ijkl}v^k v^l$ we get from (3.8)

$$\begin{aligned} S_{ij} &= \frac{1}{3} \frac{\partial}{\partial v^j} (G_{iskl} v^s v^k v^l) \\ &= S_1^2 - S_2 + 2(v^i)^2 + 2(v^j)^2 + 2v^i v^j - 2S_1(v^i + v^j) \\ &\quad + \delta_{ij}(S_2 - S_1^2 + 4S_1 v^i - 6(v^i)^2). \end{aligned} \quad (4.10)$$

Here we use the notation $S_k = \sum_{i=1}^n (v^i)^k$ and the identity $\frac{\partial}{\partial v^j} S_k = k(v^j)^{k-1}$. It follows from (3.11) and (4.10) that

$$S_{HH_1} \equiv S_{\mu\alpha} = S_{\alpha\mu} = Q_{HH_1}, \quad (4.11)$$

$$S_{HH_2} \equiv S_{\mu a} = S_{a\mu} = Q_{HH_2}, \quad (4.12)$$

$$S_{h_1 h_2} \equiv S_{\alpha a} = S_{a\alpha} = Q_{h_1 h_2} \quad (4.13)$$

and hence $L_{\mu\alpha} = L_{\alpha\mu} = 0$, $L_{\mu a} = L_{a\mu} = 0$ and $L_{\alpha a} = L_{a\alpha} = 0$ due to Eq. (3.12). Thus, the matrix (L_{ij}) is block-diagonal.

For the other three blocks we have

$$L_{\mu\nu} = G_{\mu\nu}(2 + 4\alpha S_{HH}), \quad (4.14)$$

$$L_{\alpha\beta} = G_{\alpha\beta}(2 + 4\alpha S_{h_1 h_1}), \quad (4.15)$$

$$L_{ab} = G_{ab}(2 + 4\alpha S_{h_2 h_2}), \quad (4.16)$$

where

$$S_{h_i h_i} = S_1^2 - S_2 + 6h_i^2 - 4S_1 h_i, \quad (4.17)$$

$i = 0, 1, 2$ and $h_0 = H$. Here we denote $S_{HH} = S_{\mu\nu}$, $\mu \neq \nu$; $S_{h_1 h_1} = S_{\alpha\beta}$, $\alpha \neq \beta$ and $S_{h_1 h_1} = S_{ab}$, $a \neq b$.

Due to Eqs. (4.9), (4.14), (4.15), (4.16) the matrix (4.9) is invertible if and only if $m > 1$, $k_1 > 1$, $k_2 > 1$ and

$$S_{h_i h_i} \neq -\frac{1}{2\alpha}, \quad (4.18)$$

$i = 0, 1, 2$.

Now, we prove that inequalities (4.18) are satisfied for the solutions under consideration. Let us suppose that (4.18) is not satisfied for some $i_0 \in \{0, 1, 2\}$, i.e.

$$S_{h_{i_0} h_{i_0}} = S_1^2 - S_2 + 6h_{i_0}^2 - 4S_1 h_{i_0} = -\frac{1}{2\alpha}. \quad (4.19)$$

Let $i_1 \in \{0, 1, 2\}$ and $i_1 \neq i_0$. Then using Eqs. (3.11) and (3.12) we get

$$Q_{h_{i_0} h_{i_1}} - S_{h_{i_0} h_{i_0}} = 2(h_{i_1} - h_{i_0})(2h_{i_0} + h_{i_1} - S_1) = 0, \quad (4.20)$$

which implies

$$2h_{i_0} + h_{i_1} - S_1 = 0. \quad (4.21)$$

But due to (3.16)

$$h_{i_0} + h_{i_1} + h_{i_2} - S_1 = 0, \quad (4.22)$$

where $i_2 \in \{0, 1, 2\}$ and $i_2 \neq i_0$, $i_2 \neq i_1$. Subtracting (4.22) from (4.21) we obtain $h_{i_0} - h_{i_2} = 0$, i.e. $h_{i_0} = h_{i_2}$. But due to restrictions (3.4) we have $h_{i_0} \neq h_{i_2}$. We are led to a contradiction, which proves the inequalities (4.18) and hence the matrix L from (4.9) is invertible ($m > 2$, $k_1 > 1$, $k_2 > 1$), i.e. Eq. (4.2) is obeyed. Thus, the solutions under consideration are stable.

5 Examples

Here we present several examples of stable solutions under consideration.

5.1 The case $m = 3$

Let us consider the case $m = 3$. From (3.25) we get

$$\begin{aligned} \Lambda &= -\frac{1}{4\alpha} \frac{1}{(k_1 - 2k_1 k_2 + k_2)^2 (k_1 + k_1)} \\ &\quad \times (3(k_1^3 + k_2^3) - (2(k_1^2 + k_2^2) + (k_1 + k_2)(3 + 2k_1 k_2) \\ &\quad - 8k_1 k_2)k_1 k_2). \end{aligned} \quad (5.1)$$

For $(m, k_1, k_2) = (3, 3, 2)$ we have $P = -70$,

$$\Lambda = \frac{213}{980\alpha} \quad (5.2)$$

and

$$H = \frac{1}{\sqrt{140\alpha}}, \quad h_1 = -4H, \quad h_2 = 6H. \quad (5.3)$$

Now we put $(m, k_1, k_2) = (3, 4, 2)$. We obtain $P = -120$,

$$\Lambda = \frac{21}{100\alpha}, \quad (5.4)$$

and

$$H = \frac{1}{2\sqrt{15\alpha}}, \quad h_1 = -2H, \quad h_2 = 4H. \quad (5.5)$$

According to our analysis from the previous section both solutions are stable.

5.2 Examples for $m = 4$ and $m = 5$

Now we present other examples of stable solutions for $m = 4$ and $m = 5$.

First we put $(m, k_1, k_2) = (4, 3, 2)$. We find $P = -102$ and

$$\Lambda = \frac{123}{578\alpha}. \quad (5.6)$$

In this case we obtain

$$H = \frac{1}{\sqrt{204\alpha}}, \quad h_1 = -5H, \quad h_2 = 7H. \quad (5.7)$$

Now we enlarge the value of m by putting $(m, k_1, k_2) = (5, 3, 2)$. We find $P = -140$,

$$\Lambda = \frac{589}{2800\alpha}, \quad (5.8)$$

and

$$H = \frac{1}{\sqrt{280\alpha}}, \quad h_1 = -6H, \quad h_2 = 8H. \quad (5.9)$$

We note that in all examples above $\Lambda > 0$.

6 Conclusions

We have considered the D -dimensional Einstein–Gauss–Bonnet (EGB) model with the Λ -term and two constants α_1 and α_2 . By using the ansatz with diagonal cosmological metrics, we have found, for certain $\Lambda = \Lambda(m, k_1, k_2)$ and $\alpha = \alpha_2/\alpha_1 < 0$, a class of solutions with exponential time dependence of three scale factors, governed by three different Hubble-like parameters $H > 0$, h_1 and h_2 , corresponding to submanifolds of dimensions $m > 2$, $k_1 > 1$, $k_2 > 1$, respectively, with $k_1 \neq k_2$ and $D = 1 + m + k_1 + k_2$. Here $m > 2$ is the dimension of the expanding subspace.

Any of these solutions describes an exponential expansion of “our” 3-dimensional subspace with the Hubble parameter $H > 0$ and anisotropic behaviour of $(m - 3 + k_1 + k_2)$ -dimensional internal space: expanding in $(m - 3)$ dimensions (with Hubble-like parameter H) and either contracting in k_1 dimensions (with Hubble-like parameter h_1) and expanding in k_2 dimensions (with Hubble-like parameter h_2) for $k_1 > k_2$ or expanding in k_1 dimensions and contracting in k_2 dimensions for $k_1 < k_2$. Each solution has a constant volume factor of internal space and hence it describes zero variation of the effective gravitational constant G . By using the results of Ref. [26] we have proved that all these solutions are stable as $t \rightarrow +\infty$. We have presented several examples of stable solutions for $m = 3, 4, 5$.

Acknowledgements This paper was funded by the Ministry of Education and Science of the Russian Federation in the Program to increase the competitiveness of Peoples Friendship University (RUDN University) among the world’s leading research and education centers in the

2016–2020 and by the Russian Foundation for Basic Research, Grant No.16-02-00602.

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Funded by SCOAP³.

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