$SL(2, 7)$ representations and their relevance to neutrino physics

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Abstract The investigation of the rôle of finite groups in flavor physics and, particularly, in the interpretation of the neutrino data has been the subject of intensive research. Motivated by this fact, in this work we derive the three-dimensional unitary representations of the projective linear group $PSL_2(7)$. Based on the observation that the generators of the group exhibit a Latin square pattern, we use available computational packages on discrete algebra to determine the generic properties of the group elements. We present analytical expressions and discuss several examples which reproduce the neutrino mixing angles in accordance with the experimental data.

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1 Introduction

Abelian and non-abelian discrete symmetries have been extensively used to impose constraints on the Yukawa lagrangian. In model building they are often used to generate hierarchical structures in the fermion mass matrices, to eliminate proton decay operators and suppress other terms inducing unobserved processes. The structure of the neutrino mass matrix and the experimentally measured mixing angles in particular, hint to the existence of an underlying non-abelian flavor symmetry. In this context, the neutrino mass matrix is assumed to be invariant under certain transformations of the discrete group D_f (for reviews see [1–3]). Therefore, in unified theories of the fundamental gauge interactions the symmetry of the effective model is expected to contain a non-abelian gauge group G_{GUT} accompanied by a (non)-abelian discrete flavor symmetry D_f . In some string scenarios the total effective symmetry $G_{\text{GUT}} \times D_f$ is usually embedded in a higher unified group, such as E_8 [4]. For the most familiar GUT symmetries such as E_6 , $SO(10)$ and $SU(5)$, the discrete group is a subgroup of $SU(3)$. In the past, several of these cases have been considered, including those belonging to the chains S_n , A_n and possess triplet representations. In the present work we focus on some particular cases of a general class of discrete symmetries. These are the special linear groups $SL_2(p)$ [5,6] and their corresponding projective ones, $PSL_2(p)$, with p prime number. From this class of groups it is sufficient for our purposes to take $p \leq 7$, since these are the only cases where the resulting discrete groups contain triplet representations and at the same time are embedded in $SU(3)$. Moreover, from the physics point of view, the rather interesting case of $PSL_2(7)$ which is a simple subgroup of $SU(3)$, is less explored (see, however, [7–10]), and it is our main focus in the present work. Since we demand invariance of the neutrino mass matrix under certain group actions, a prerequisite for such an analysis is the explicit form of the group elements of $PSL_2(7)$. However, while finding

the representations of the elements and the multiplication tables of A_3 , A_4 , A_5 is relatively easy, this becomes an onerous process for $PSL_2(7)$.

In a previous work [11], using the automorphisms of the discrete and finite Heisenberg group, the three-dimensional representations of the $PSL_2(p)$ generators were constructed. It is our purpose here to systematically derive the structure of all the elements of the three-dimensional representations of $PSL_2(7)$. The explicit form of the three-dimensional unitary representations would be a very useful tool in many physics applications including their possible relevance to the structure of the neutrino mass matrix and lepton mixing angles. Focusing on neutrino physics, a usual approach is to construct neutrino and charged-lepton mass matrices invariant under certain elements of the assumed group. Hence, a systematic exploration of the specific properties and, in particular, the characteristic mixing matrix of the neutrino sector require such a methodology.

Finite groups were proposed a long time ago as a possible symmetry to the peculiar neutrino hierarchy and this is our basic motivation for the present construction. With many details found in reviews and other work, here, we only give a brief description of our assumptions. We consider a scenario where the charged-lepton and neutrino mass matrices are subject to constraints under the same or different subgroups of a covering parent discrete symmetry. This is compatible, for example, with model building in string theory framework. Taking for example the $SU(5)$ gauge theory, in some F-theory framework, the trilinear Yukawa couplings for the various types of fields are realized at different points of the internal manifold and they correspond to different symmetry enhancements of the $SU(5)$ singularity [12]. For example, the $10 \cdot 10 \cdot 5$ coupling is realized at a ‘point’ of the compact manifold associated with E_6 enhancement and the $10 \cdot \bar{5} \cdot \bar{5}$ at an $SO(12)$ enhancement. Similarly, the corresponding discrete symmetry associated with these points may differ, although they could be subgroups of the same covering discrete group. We assume that this is the case for m_ℓ and m_ν which are not realized at the same ‘point’. Then we consider that the neutrino mass matrix commutes with an element A of a given discrete group

$$[m_\nu, A] = 0, \quad (1)$$

while a similar relation holds for the charged-lepton mass matrix, as well. Then the vanishing of the commutator (1) implies that both m_ν and A have a common system of eigenvectors, hence they define the diagonalizing (mixing) matrix V_ν . In other words, we have $U_\nu^\dagger A U_\nu = A^{\text{diag.}}$ as well as $U_\nu^\dagger m_\nu U_\nu = m_\nu^{\text{diag.}}$.

The layout of the article is as follows. In Sect. 2 we summarize the basic steps of the generators construction [11] using the work of [13]. In Sect. 3, based on the observa-

tion that the $PSL_2(7)$ 3-d representations exhibit a specific structure, we calculate the elements of the three-dimensional representation. In Sect. 4 we discuss its possible relevance to neutrino physics and present several working examples. We summarize our results in Sect. 5.

2 The three-dimensional representations of the $PSL_2(7)$ group

In the present section we describe the basic steps for the construction of the unitary representations of the $PSL_2(p)$ group. This is defined by the 2×2 matrices α with elements integers modulo p , where p is a prime number, and determinant equal to one modulo p :

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad a, b, c, d \in \mathbb{Z}_p, \quad \det \alpha = 1 \bmod p \quad (2)$$

The elements of the group can be constructed from combinations of powers of two generators denoted here with α , β , which, in a specific representation are defined by the matrices

$$\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \quad (3)$$

These satisfy the relations

$$\alpha^2 = \beta^3 = -I \equiv -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4)$$

To define the projective linear group $PSL_2(p)$, we first observe that $SL_2(p)$ contains a normal subgroup of two elements, $Z_2 = \{I, -I\}$. The $PSL_2(p)$ is defined as the quotient subgroup, by identifying the unit matrix I with $-I$,

$$PSL_2(p) = SL_2(p)/\{I, -I\} \cong SL_2(p)/Z_2. \quad (5)$$

Next, we use Weyl’s metaplectic representation $U(A)$ derived long time ago by Balian and Itzykson [13] (see also [14]) to construct the p -dimensional unitary representations of $SL_2(p)$ groups. The explicit form of $U(A)$ is given in terms of the elements a, b, c, d of the 2×2 matrix (2) as follows: for $\delta = 2 - a - d \neq 0$

$$U(A) = \frac{\sigma(1)\sigma(\delta)}{p} \sum_{r,s} \omega^{[br^2+(d-a)rs-cs^2]/(2\delta)} J_{r,s}, \quad (6)$$

while if $\delta = 0$, we distinguish the following cases:

$$\begin{aligned} \delta = 0, \quad b \neq 0 : U(A) &= \frac{\sigma(-2b)}{\sqrt{p}} \sum_s \omega^{s^2/(2b)} J_{s(a-1)/b,s} \\ \delta = b = 0, \quad c \neq 0 : U(A) &= \frac{\sigma(2c)}{\sqrt{p}} \sum_r \omega^{-r^2/(2c)} P^r \\ \delta = b = 0 = c = 0 : U(1) &= I. \end{aligned} \quad (7)$$

A few clarifications on notation and definitions in the above formulas are needed.

The quantities $J_{r,s}$, $J_{s(a-1)/b,s}$ are defined as follows:

$$J_{n_1,n_2} \equiv J_{\vec{n}} = \omega^{\frac{n_1 n_2}{2}} P^{n_1} Q^{n_2}, \quad (8)$$

where $\omega = e^{2\pi i/p}$ is the p^{th} root of unity, while P , Q are position and momentum operators with elements $P_{kl} = \delta_{k-1,l}$ and, $Q_{kl} = \omega^k \delta_{kl}$, respectively. The generators (8) obey the ‘multiplication’ law

$$J_{\vec{m}} J_{\vec{n}} = \omega^{\frac{\vec{n} \times \vec{m}}{2}} J_{\vec{m} + \vec{n}}$$

and constitute a subset of the Heisenberg group [13].

The quantities $\sigma(a)$ and $\left(\frac{a}{p}\right)$, are the quadratic Gauss sum and the Legendre symbol, respectively. These are defined as follows:

$$\sigma(a) = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \omega^{ak^2} = \left(\frac{a}{p}\right) \times \begin{cases} 1 & \text{for } p = 4k + 1, \\ i & \text{for } p = 4k - 1, \end{cases} \quad (9)$$

and

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } a \text{ divides } p \\ +1 & \text{if } a = \mathcal{QR} \ p, \\ -1 & \text{if } a \neq \mathcal{QR} \ p, \end{cases} \quad (10)$$

where $\mathcal{QR}$ means quadratic residue.¹

The p -dimensional representation thus obtained decomposes into two irreducible unitary representations of dimensions $\frac{p+1}{2}$ and $\frac{p-1}{2}$. These are discrete subgroups of the unitary groups $SU(\frac{p+1}{2})$ and for $p = 7$ we obtain the three-dimensional representation of the discrete group $PSL_2(7)$ which is a subgroup of $SU(3)$. Smaller p values result to A_3 and A_5 groups which have been extensively studied, while the next value of $p = 11$ results in $PSL_2(11)$, which does not contain triplet representations in its decompositions ($p = 11 = 6 + 5$); therefore it is not of our primary interest.

The $SL_2(p)$ group with p prime has $p(p^2 - 1)$ elements and the corresponding projective $PSL_2(p)$ contains half of them. Therefore, the $PSL_2(7)$ has 168 elements and it is a simple discrete subgroup of $SU(3)$. Using the method described above we can construct [11] the three-dimensional representations of $PSL_2(7)$, satisfying the conditions

$$\mathbf{a}^2 = \mathbf{b}^3 = (\mathbf{ab})^7 = [\mathbf{a}, \mathbf{b}] = 1 \quad (11)$$

where the ‘commutator’ for the group elements is defined as usual: $[\mathbf{a}, \mathbf{b}] = \mathbf{a}^{-1} \mathbf{b}^{-1} \mathbf{ab}$. The generators of the three-dimensional unitary representation of the $PSL_2(7)$ group,

associated with $\mathbf{a}$ and $\mathbf{b}$ of (2) can be written in terms of the 7^{th} root of unity $\eta = e^{2\pi i/7}$, as follows:

$$A_{[3]} = \frac{i}{\sqrt{7}} \begin{pmatrix} \eta^2 - \eta^5 & \eta^6 - \eta & \eta^3 - \eta^4 \\ \eta^6 - \eta & \eta^4 - \eta^3 & \eta^2 - \eta^5 \\ \eta^3 - \eta^4 & \eta^2 - \eta^5 & \eta - \eta^6 \end{pmatrix} \quad (12)$$

and

$$B_{[3]} = \frac{i}{\sqrt{7}} \begin{pmatrix} \eta - \eta^4 & \eta^4 - \eta^6 & \eta^6 - 1 \\ \eta^5 - 1 & \eta^2 - \eta & \eta^5 - \eta \\ \eta^2 - \eta^3 & 1 - \eta^3 & \eta^4 - \eta^2 \end{pmatrix}. \quad (13)$$

It can be readily checked that these satisfy the required relations:

$$A_{[3]}^2 = B_{[3]}^3 = (A_{[3]} B_{[3]})^7 = [A_{[3]}, B_{[3]}]^4 = I. \quad (14)$$

Implementing the above method, we can find the $PSL_2(7)$ group elements, imposing the appropriate conditions on products of powers of 2×2 matrices given in (2) and then using the metaplectic representation to build the three-dimensional representations of $PSL_2(7)$. However, in this section we will follow a different approach, which, in our opinion, reveals some new and very interesting properties of the group elements.

For all practical purposes, however, it suffices to take the two generators and explicitly construct all the elements of the three-dimensional unitary representations of $PSL_2(7)$ using the GAP system for computational discrete algebra available in the web [15]. After some algebra it can be shown that the two generators can be written as

$$A_{[3]} = \begin{bmatrix} \rho_1 & -\rho_2 & -\rho_3 \\ -\rho_2 & \rho_3 & \rho_1 \\ -\rho_3 & \rho_1 & \rho_2 \end{bmatrix}, \quad B_{[3]} = \begin{bmatrix} \rho_1 \eta^{-1} & \rho_2 \eta^{\frac{3}{2}} & \rho_3 \eta^{-\frac{1}{2}} \\ \rho_2 \eta^{\frac{5}{2}} & \rho_3 \eta^{-2} & \rho_1 \eta^3 \\ \rho_3 \eta^{\frac{5}{2}} & \rho_1 \eta^{-2} & \rho_2 \eta^3 \end{bmatrix}, \quad (15)$$

where

$$\begin{aligned} \rho_1 &= -\frac{2}{\sqrt{7}} \cos \frac{\pi}{14}, & \rho_2 &= -\frac{2}{\sqrt{7}} \cos \frac{3\pi}{14}, \\ \rho_3 &= -1 - \rho_1 - \rho_2. \end{aligned} \quad (16)$$

The quantities ρ_1, ρ_2, ρ_3 satisfy the cubic equation

$$x^3 + x^2 - \frac{1}{7} = 0,$$

and the relation $\rho_2 = 7\rho_1^3 - 3\rho_1$.

We now observe that the moduli of the elements of both matrices follow a Latin square pattern [16]. The so found group matrices can be classified according to their conjugacy class as shown in Table 1.

¹ An integer q is called quadratic residue ($\mathcal{QR}$) iff $\exists x : x^2 = q \bmod p$.

Table 1 The order, character, and the number of elements of the conjugacy classes of $PSL_2(7)$

Order	Character	#	Tag
2	-1	21	el_2
3	0	56	el_3
4	+1	42	el_4
7	$-\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$	48	el_7

3 On the properties of the representation matrices

In this section we investigate useful properties of Latin square matrices in view of their relation to the $PSL_2(7)$ elements. We start with the 3×3 case with real entries. These are of the following two types:

$$M_1 = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_2 & r_3 & r_1 \\ r_3 & r_1 & r_2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_3 & r_1 & r_2 \\ r_2 & r_3 & r_1 \end{bmatrix}, \quad (17)$$

and their permutations. From these two, only the first type appears in $PSL_2(7)$. The orthogonality condition $M_1^2 = 1$ implies that

$$r_1^2 + r_2^2 + r_3^2 = 1, \quad r_1 r_2 + r_1 r_3 + r_2 r_3 = 0, \quad (18)$$

while requiring $\det M_1 = 1$ we get

$$r_1 + r_2 + r_3 = -1. \quad (19)$$

Thus, if the matrix M_1 is part of an irreducible group representation, it should belong to the conjugacy class el_2 with character -1. Notice that r_1, r_2, r_3 satisfy the algebraic equation

$$x^3 + x^2 - q = 0, \quad \text{where } q = r_1 r_2 r_3, \quad (20)$$

and the reality of the roots requires that $0 < q < \frac{4}{27}$. For the case of $PSL_2(7)$, $q = \frac{1}{7} < \frac{4}{27}$.

The obvious generalization includes complex elements and takes the form

$$M = \begin{bmatrix} r_1 e^{ic_1} & r_2 e^{ic_2} & r_3 e^{ic_3} \\ r_2 e^{ic_4} & r_3 e^{ic_5} & r_1 e^{ic_6} \\ r_3 e^{ic_7} & r_1 e^{ic_8} & r_2 e^{ic_9} \end{bmatrix}. \quad (21)$$

Unitarity and the condition $\det M = 1$ restrict the number of free parameters r_i, c_i . The conditions (18) and (19) still hold, and the final form is

$$M = \begin{bmatrix} r_1 e^{ic_1} & r_2 e^{ic_2} & r_3 e^{ic_3} \\ r_2 e^{i(c_1-c_2+c_5)} & r_3 e^{ic_5} & r_1 e^{i(c_3-c_2+c_5)} \\ r_3 e^{-i(c_3+c_5)} & r_1 e^{i(c_2-c_3-c_1-c_5)} & r_2 e^{-i(c_1+c_5)} \end{bmatrix}. \quad (22)$$

Looking more closely at the $PSL_2(7)$ elements constructed by GAP, we observe that a great number of them can be written in this form when we substitute ρ_1, ρ_2, ρ_3 given in (16) in place of r_1, r_2, r_3 and all the phases are closely related to the set of the seventh roots of unity. The secular equation for M reads

$$x^3 - \text{tr}(M)x^2 + \text{tr} M^* x - 1 = 0. \quad (23)$$

It can be readily checked that given the character of the conjugacy class $\text{Tr}(M)$, the secular Eq. (23) reproduces the correct eigenvalues of the representation matrices. For example, the order seven elements character is $\text{Tr} M = -\frac{1}{2} - i\frac{\sqrt{7}}{2}$ and solving the secular equation we find the eigenvalues

$$\exp\left[\frac{10\pi i}{7}\right], \quad \exp\left[\frac{6\pi i}{7}\right], \quad \exp\left[\frac{12\pi i}{7}\right], \quad (24)$$

in accordance to the group elements table. For the order two elements a trivial calculation implies that $c_1 = c_5 = 0$ giving the general form

$$M = \begin{bmatrix} r_1 & r_2 e^{ic_2} & r_3 e^{ic_3} \\ r_2 e^{-ic_2} & r_3 & r_1 e^{i(c_3-c_2)} \\ r_3 e^{-ic_3} & r_1 e^{i(c_2-c_3)} & r_2 \end{bmatrix}. \quad (25)$$

Next, for the general matrix M , information on the allowed values for c_1 and c_5 can be extracted by taking the system of equations

$$r_1 e^{ic_1} + r_2 e^{-i(c_1+c_5)} + r_3 e^{ic_5} = \text{Tr} M, \quad (26)$$

$$r_1 + r_2 + r_3 = -1, \quad (27)$$

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = 0, \quad (28)$$

and substituting the character $\text{Tr} M$ of the corresponding conjugacy class. Parametrizing the phases as

$$c_1 = \frac{2\pi}{7}n, \quad c_5 = \frac{2\pi}{7}m \quad (29)$$

and taking $q = r_1 r_2 r_3 = \frac{1}{7}$ we get only integer values for n, m . These values are symmetric under the interchange of m and n . Notice that the value, $q = \frac{1}{7}$ identifies r_1, r_2, r_3 with ρ_1, ρ_2, ρ_3 respectively.

Solving the system of Eqs. (26)–(28), for the order 3 elements we have

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \cos c_1 & \cos(c_1+c_5) & \cos c_5 \\ \sin c_1 & -\sin(c_1+c_5) & \sin c_5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad (30)$$

and the solutions found are

c_1	c_5	c_1	c_5	c_1	c_5	c_1	c_5
n	m	n	m	n	m	n	m
-5	-3	-4	-2	-2	-1	1	4
-5	1	-4	5	-2	3	1	2
-5	4	-3	1	-1	3	2	4
-4	-1	-3	2	-1	5	3	5

For the order 4 elements we have

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \cos c_1 & \cos(c_1 + c_5) & \cos c_5 \\ \sin c_1 & -\sin(c_1 + c_5) & \sin c_5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad (31)$$

which imply the following values for m, n and $c_{1,5}$:

c_1	c_5	c_1	c_5	c_1	c_5	c_1	c_5
n	m	n	m	n	m	n	m
-6	-3	-5	1	-4	5	-2	6
2	4	-6	-5	-5	4	-3	1
-1	3	3	6	-6	4	-4	-1
-3	2	-1	5	3	5	-6	2
-4	-2	-2	-1	1	4	5	6
-5	-3	-4	6	-2	3	1	2

For the order 7 elements we have

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \cos c_1 & \cos(c_1 + c_5) & \cos c_5 \\ \sin c_1 & -\sin(c_1 + c_5) & \sin c_5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -\frac{1}{2} \\ -\frac{\sqrt{7}}{2} \end{bmatrix}. \quad (32)$$

The solutions are

c_1	c_5	c_1	c_5	c_1	c_5
n	m	n	m	n	m
-6	-3	-5	-3	-3	2
-6	-5	-5	1	1	4
-6	4	-5	4	1	2
-6	2	-3	1	2	4

Note that all these solutions when applied to the system for r_1, r_2, r_3 just generate permutations of the root system ρ_1, ρ_2, ρ_3 . Concerning the possible values of the remaining phases c_2 and c_3 no further information can be extracted without using the group algebra.

4 $SL_2(7)$ invariance and the neutrino mixing matrix

Neutrino oscillations are associated with the existence of non-zero masses m_{ν_i} and non-zero θ_{ij} mixing angles in the lepton sector. In a standard parametrization the corresponding lepton mixing matrix is given by the expression

Table 2 Solutions for the charged-lepton mass matrix in terms of the order 3 elements

el_2	el_3	el_3	el_3	el_3	el_3	el_3	el_3	el_3
15	6	7	15	17	23	25	37	41
16	5	6	19	21	24	26	39	42
17	4	5	16	18	25	27	38	41
18	3	4	15	20	26	28	40	42
19	1	3	17	19	22	27	36	38
20	1	8	16	21	23	28	37	40
21	7	8	18	20	22	24	36	39

$$U = U_l^\dagger U_\nu$$

$$= \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{23}s_{12}s_{13} - c_{12}s_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix}, \quad (33)$$

where, to avoid clutter, we have denoted $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$. The 3σ range of the three mixing angles, in accordance with recent data, is given by

$$\sin^2 \theta_{12} = [0.25 - 0.35], \quad \sin^2 \theta_{23} = [0.38 - 0.62], \quad \sin^2 \theta_{13} = [0.0185 - 0.0246]. \quad (34)$$

We will confront our results on the masses and mixing matrices with the experimental data.

In order to construct the mixing matrices one has to first assume the symmetries of the neutrino and charged-lepton mass matrices. These symmetries are connected to the elements of $PSL_2(7)$ which leave the mass matrices invariant (i.e. vanishing commutator). Then we determine the diagonalizing matrices U for these elements. Note that the diagonalizing matrices are not uniquely defined since there are $3!$ ways to arrange the eigenvalues in the resulting diagonal matrices. Since $PSL_2(7)$ contains four conjugacy classes characterized by elements of order 2 (el_2), order 3 (el_3), order 4 (el_4), and 7 (el_7) (see the appendix for notation), in order to construct the mixing matrices one has to combine the diagonalizing matrices in all possible ways. This search of course can only be done numerically. In order to conform with experimental data we kept only the cases where $0.136 < |U_{13}| < 0.157$, $0.499 < |U_{12}| < 0.595$, $0.615 < |U_{23}| < 0.785$. It turns out that the only symmetry for the neutrino mass matrix that is compatible with data is connected to a number of order 2 elements. For the charged-lepton mass matrix the symmetry allowed is connected to both order 3 and order 7 elements. The results are shown in Tables 2 and 3. Obviously, these tables should also contain the inverse elements.

Since the order 2 elements equal their inverses and the rest can easily be calculated, the inverses are not shown for reasons of clarity.

Table 3 Solutions for the charged-lepton mass matrix in terms of the order 7 elements

el_2	el_7	el_7	el_7	el_7	el_7	el_7
8	3	5	15	16	21	24
9	1	2	17	18	22	25
10	4	6	12	13	23	26
11	3	7	14	15	20	24
12	1	5	16	17	21	25
13	2	4	12	18	22	26
14	6	7	13	14	20	23

Some comments concerning the order 2 elements are here in order. The eigenvalues of these matrices are $(1, -1, -1)$, respectively, i.e. there exists a degenerate two-dimensional subspace implying that the eigenvectors related to the degenerate eigenvalue cannot be uniquely defined. In fact, if v_1, v_2, v_3 are eigenvectors corresponding to the $(1, -1, -1)$ eigenvalues an equally good choice would be $v_1, \tilde{v}_2, \tilde{v}_3$ where

$$\begin{bmatrix} \tilde{v}_2 \\ \tilde{v}_3 \end{bmatrix} = \begin{bmatrix} e^{i\varphi_1} \cos \varphi & -e^{i\varphi_2} \sin \varphi \\ e^{i\varphi_1} \sin \varphi & e^{i\varphi_2} \cos \varphi \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \end{bmatrix}, \quad (35)$$

for arbitrary $\varphi, \varphi_1, \varphi_2$. This matrix defines a $U(2)$ rotation for v_2 and v_3 that leaves the representation matrices invariant.

4.1 Working examples

In the following we analyze a few working examples and show how to explicitly construct the mixing matrices.

4.1.1 The pair el_2 (16), el_3 (5)

The corresponding matrices are given by

$$\begin{aligned} el_2(16) &= \begin{bmatrix} r_3 & -r_1 & -r_2 \\ -r_1 & r_2 & r_3 \\ -r_2 & r_3 & r_1 \end{bmatrix}, \\ el_3(5) &= \begin{bmatrix} 0 & 0 & -e^{\frac{6\pi i}{7}} \\ e^{-\frac{2\pi i}{7}} & 0 & 0 \\ 0 & e^{-\frac{4\pi i}{7}} & 0 \end{bmatrix}. \end{aligned} \quad (36)$$

The normalized eigenvectors of el_2 (16) are given by

$$v_2[1] = \begin{bmatrix} s \\ \frac{1}{2}s - \frac{\sqrt{3}}{2}\sqrt{\frac{2}{3} - s^2} \\ \frac{1}{2}s + \frac{\sqrt{3}}{2}\sqrt{\frac{2}{3} - s^2} \end{bmatrix}, \quad v_2[2] = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

$$v_2[3] = \begin{bmatrix} \sqrt{\frac{2}{3} - s^2} \\ \frac{\sqrt{3}}{2}s + \frac{1}{2}\sqrt{\frac{2}{3} - s^2} \\ -\frac{\sqrt{3}}{2}s + \frac{1}{2}\sqrt{\frac{2}{3} - s^2} \end{bmatrix}$$

related to the eigenvalues $+1, -1, -1$, respectively, with s given by

$$s = \sqrt{2} \frac{(r_1 - r_2)}{\sqrt{(1 - 3r_3)^2 + 3(r_2 - r_1)^2}} \approx -0.815. \quad (37)$$

The normalized eigenvectors of el_3 (5) are

$$\begin{aligned} v_3[1] &= \frac{1}{\sqrt{3}} \begin{bmatrix} -e^{\frac{6\pi i}{7}} \\ e^{\frac{4\pi i}{7}} \\ 1 \end{bmatrix}, \quad v_3[2] = \frac{1}{\sqrt{3}} \begin{bmatrix} -e^{\frac{4\pi i}{21}} \\ -e^{\frac{5\pi i}{21}} \\ 1 \end{bmatrix}, \\ v_3[3] &= \frac{1}{\sqrt{3}} \begin{bmatrix} e^{\frac{11\pi i}{21}} \\ -e^{\frac{19\pi i}{21}} \\ 1 \end{bmatrix} \end{aligned} \quad (38)$$

related to the eigenvalues $1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$ correspondingly. Further manipulation shows that a mixing matrix compatible with data occurs only for the following two combinations:

$$\begin{aligned} U_1 &= [v_3[3], v_3[2], v_3[1]]^\dagger \cdot [\tilde{v}_2[2], v_2[1], \tilde{v}_2[3]], \\ U_2 &= [v_3[3], v_3[1], v_3[2]]^\dagger \cdot [\tilde{v}_2[2], v_2[1], \tilde{v}_2[3]] \end{aligned} \quad (39)$$

where $\tilde{v}_2[2], \tilde{v}_2[3]$, as in (35). Explicit calculations show that the modulus of the second column elements for both matrices U_1 and U_2 is $1/\sqrt{3}$. This is due to the fact that

$$\begin{aligned} \|v_3[2]^\dagger \cdot v_2[1]\| &= \|v_3[1]^\dagger \cdot v_2[1]\| = \|v_3[3]^\dagger \cdot v_2[1]\| \\ &= \frac{1}{\sqrt{3}} \approx 0.5773 \end{aligned} \quad (40)$$

only for the given value of s (37) and, surprisingly, is not obviously related to the existence of the degenerate subspace. For the pair el_2 (16) and el_3 (5) the mixing matrices for $\varphi = 0$ are given by

$$U_1 = \begin{bmatrix} 0.80217e^{0.5667i} & 0.57735e^{2.3948i} & 0.152283e^{-1.27039i} \\ 0.36647e^{0.106487i} & 0.57735e^{-0.8735i} & 0.729634e^{-0.3499i} \\ 0.471405e^{-1.6582i} & 0.57735e^{3.05416i} & 0.666667e^{0.635302i} \end{bmatrix}, \quad (41)$$

$$U_2 = \begin{bmatrix} 0.80217e^{0.5667i} & 0.57735e^{2.3948i} & 0.152283e^{-1.27039i} \\ 0.471405e^{-1.6582i} & 0.57735e^{3.05416i} & 0.666667e^{0.635302i} \\ 0.36647e^{0.106487i} & 0.57735e^{-0.8735i} & 0.729634e^{-0.3499i} \end{bmatrix}. \quad (42)$$

More generally, using the degeneracy of the subspace (35) we can determine the range of φ in accordance with the experimental data. This is depicted in Fig. 1.

Fig. 1 Case el_2 (16), el_3 (5): the range of $\sin \theta_{13}$ as a function of the angle ϕ parametrizing the mixing (35) of the degenerate subspace. The orange line defines the upper experimental bound and the blue the lower one on $\sin \theta_{13}$

4.1.2 The pair el_2 (16), el_3 (19)

The relevant matrix is

$$el_3(19) = \begin{bmatrix} r_1 e^{\frac{2\pi i}{7}} & r_2 e^{-\frac{5\pi i}{7}} & r_3 e^{-\frac{5\pi i}{7}} \\ r_2 e^{-\frac{3\pi i}{7}} & r_3 e^{\frac{4\pi i}{7}} & r_1 e^{\frac{4\pi i}{7}} \\ r_3 e^{\frac{\pi i}{7}} & r_1 e^{-\frac{6\pi i}{7}} & r_2 e^{-\frac{6\pi i}{7}} \end{bmatrix}. \quad (43)$$

The eigenvectors which correspond to the eigenvalues 1, $e^{\frac{2\pi i}{3}}$, $e^{\frac{4\pi i}{3}}$ are

$$v_3[1] = N_1 \left\{ r_2 \left(\eta^{-\frac{5}{2}} + \eta^{\frac{1}{2}} \right), \eta^3 - r_1 \eta^4 + r_3 \eta - r_2, r_1 (1 + \eta) \right\} \quad (44)$$

$$v_3[2] = N_2 \left\{ r_2 \left(\eta^{-\frac{5}{2}} + \eta^{\frac{17}{6}} \right), \eta^{\frac{23}{3}} - \eta^{\frac{7}{3}} (r_2 + r_1 \eta^4) + r_3 \eta, r_1 \left(\eta^{\frac{7}{3}} + \eta \right) \right\} \quad (45)$$

$$v_3[3] = N_3 \left\{ r_2 \left(\eta^{-\frac{5}{2}} + \eta^{\frac{31}{6}} \right), \eta^{\frac{37}{3}} - \eta^{\frac{14}{3}} (r_2 + r_1 \eta^4) + r_3 \eta, r_1 \left(\eta^{\frac{14}{3}} + \eta \right) \right\} \quad (46)$$

where N_1 , N_2 and N_3 are normalization factors. We find that a mixing matrix compatible with data occurs only for the two combinations

$$U_1 = [v_3[2], v_3[1], v_3[3]]^\dagger \cdot [\tilde{v}_2[3], v_2[1], \tilde{v}_2[2]], \quad (47)$$

$$U_2 = [v_3[2], v_3[3], v_3[1]]^\dagger \cdot [\tilde{v}_2[3], v_2[1], \tilde{v}_2[2]]. \quad (48)$$

In this case also, we find that the modulus of the second column elements for both matrices U_1 and U_2 is $1/\sqrt{3}$. This is due to the fact that

$$\begin{aligned} \|v_3[1]^\dagger \cdot v_2[1]\| &= \|v_3[2]^\dagger \cdot v_2[1]\| \\ &= \|v_3[3]^\dagger \cdot v_2[1]\| = \frac{1}{\sqrt{3}} \end{aligned} \quad (49)$$

only for the given value of s as in the previous case. For the pair el_2 (16) and el_3 (19) the mixing matrices for $\varphi = 0$ are given by

$$U_1 = \begin{bmatrix} 0.80217e^{-1.82071i} & 0.57735e^{-0.868576i} & 0.152283e^{-3.1252i} \\ 0.36647e^{2.60604i} & 0.57735e^{0.0831114i} & 0.729634e^{-0.0791203i} \\ 0.471405e^{-3.09347i} & 0.57735e^{1.25755i} & 0.666667e^{-2.24541i} \end{bmatrix},$$

$$U_2 = \begin{bmatrix} 0.80217e^{-1.82071i} & 0.57735e^{-0.868576i} & 0.152283e^{-3.1252i} \\ 0.471405e^{-3.09347i} & 0.57735e^{1.25755i} & 0.666667e^{-2.24541i} \\ 0.36647e^{2.60604i} & 0.57735e^{0.0831114i} & 0.729634e^{-0.0791203i} \end{bmatrix}.$$

Again, making use of the degenerate subspace we can find the range of φ values in accordance with the experimental findings. For the two examples above the range of φ compatible with the experiment is between $[-0.0196, 0.123]$ and is plotted in Fig. 2.

4.1.3 The pair el_2 (10), el_7 (23)

We proceed now to an example which involves seventh-order elements of $PSL_2(7)$. We take the pair el_2 (10), el_7 (23) which is represented by the matrices

$$li_2(10) = \begin{bmatrix} r_1 & -r_2 & -r_3 \\ -r_2 & r_3 & r_1 \\ -r_3 & r_1 & r_2 \end{bmatrix},$$

$$li_7(23) = \begin{bmatrix} r_1 e^{\frac{4\pi i}{7}} & r_2 e^{\frac{\pi i}{7}} & r_3 e^{-\frac{5\pi i}{7}} \\ r_2 e^{-\frac{3\pi i}{7}} & r_3 e^{-\frac{6\pi i}{7}} & r_1 e^{\frac{2\pi i}{7}} \\ r_3 e^{-\frac{3\pi i}{7}} & r_1 e^{-\frac{6\pi i}{7}} & r_2 e^{\frac{2\pi i}{7}} \end{bmatrix}.$$

Fig. 2 The acceptable range of $\sin \theta_{13}$ vs. the angle ϕ for the pair e_2 (16) e_3 (19)

The eigenvectors of li_2 (10) are

$$v_2 [1] = \begin{bmatrix} \sqrt{\frac{2}{3} - s^2} \\ \frac{\sqrt{3}s}{2} + \frac{1}{2}\sqrt{\frac{2}{3} - s^2} \\ -\frac{\sqrt{3}s}{2} + \frac{1}{2}\sqrt{\frac{2}{3} - s^2} \end{bmatrix}, \quad v_2 [2] = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

$$v_2 [3] = \begin{bmatrix} s \\ \frac{1}{2}s - \frac{\sqrt{3}}{2}\sqrt{\frac{2}{3} - s^2} \\ \frac{1}{2}s + \frac{\sqrt{3}}{2}\sqrt{\frac{2}{3} - s^2} \end{bmatrix}$$

corresponding to the eigenvalues 1, -1 , -1 , respectively. Here s is given by

$$s = \sqrt{\frac{2}{3}} \frac{(2 + 3r_3)}{\sqrt{3(1 + r_1 - r_2)^2 + (2 + 3r_3)^2}} \approx 0.732. \quad (50)$$

The normalized eigenvectors of li_7 (23) are given by

$$v_7 [1] = \begin{bmatrix} r_3 e^{\frac{4\pi i}{7}} \\ r_1 e^{-\frac{5\pi i}{7}} \\ r_2 e^{\frac{5\pi i}{7}} \end{bmatrix}, \quad v_7 [2] = \begin{bmatrix} r_2 e^{\frac{6\pi i}{7}} \\ r_3 e^{-\frac{3\pi i}{7}} \\ -r_1 \end{bmatrix},$$

$$v_7 [3] = \begin{bmatrix} -r_1 \\ r_2 e^{-\frac{2\pi i}{7}} \\ r_3 e^{-\frac{6\pi i}{7}} \end{bmatrix},$$

and they correspond to the eigenvalues $e^{\frac{6\pi i}{7}}$, $e^{\frac{10\pi i}{7}}$, $e^{\frac{12\pi i}{7}}$ respectively. It turns out that the diagonalizing matrices of all order seven elements for some unclear reason can be written as latin square matrices which, however, do not constitute elements of the group. The mixing matrices compatible with data are

$$U_1 = [v_7 [1]^\dagger, v_7 [2]^\dagger, v_7 [3]^\dagger] \cdot [v_2 [1], \tilde{v}_2 [3], \tilde{v}_2 [2]] \quad (51)$$

$$U_2 = [v_7 [1]^\dagger, v_7 [3]^\dagger, v_7 [2]^\dagger] \cdot [v_2 [1], \tilde{v}_2 [3], \tilde{v}_2 [2]] \quad (52)$$

Note that for $\phi = 0$ the mixing matrices are completely out. However, for $\phi = \frac{2\pi}{7}$ we get

$$U_1 = \begin{bmatrix} 0.814857e^{-\frac{3\pi i}{7}} & 0.558406e^{\frac{\pi i}{14}} & 0.15532e^{\frac{\pi i}{14}} \\ 0.362646e^{\frac{4\pi i}{7}} & 0.700416e^{\frac{\pi i}{14}} & 0.614741e^{-\frac{13\pi i}{14}} \\ 0.452212e^{-\frac{3\pi i}{7}} & 0.444523e^{-\frac{13\pi i}{14}} & 0.773242e^{-\frac{13\pi i}{14}} \end{bmatrix},$$

$$U_2 = \begin{bmatrix} 0.814857e^{-\frac{3\pi i}{7}} & 0.558406e^{\frac{\pi i}{14}} & 0.15532e^{\frac{\pi i}{14}} \\ 0.452212e^{-\frac{3\pi i}{7}} & 0.444523e^{-\frac{13\pi i}{14}} & 0.773242e^{-\frac{13\pi i}{14}} \\ 0.362646e^{\frac{4\pi i}{7}} & 0.700416e^{\frac{\pi i}{14}} & 0.614741e^{-\frac{13\pi i}{14}} \end{bmatrix}. \quad (53)$$

The range of $\delta\varphi \equiv \varphi - \frac{2\pi}{7}$ that falls within the experimental bounds is very narrow, and it is given by (see also Fig. 3)

$$-0.0028 < \delta\varphi < +0.03. \quad (54)$$

Before closing this section, we would like to comment on a few issues related to the symmetry left over in the neutrino sector. It is evident that, in the absence of any Yukawa terms, fermion masses are zero and since there are three families of fermions f_j , the term $\sum_{j=1}^3 \bar{f}_j \gamma^\mu \partial_\mu f_j$ in the lagrangian, exhibits a $U(3)$ symmetry (unitarity is required for the invariance of the kinetic term). The MSSM lagrangian possesses several global symmetry factors $U(3)^n$, ($n_{\max} = 6$) but when unified gauge symmetries are employed, the symmetry reduces and $n < n_{\max}$. Focusing on the leptonic sector, in the massless limit neutrino and charged leptons possess a $U(3)^2$ global symmetry. Of course, we may assume that our model can be embedded in a unified symmetry, such as the $SO(10)$ for example, where all fermions are embedded in the

Fig. 3 Plots show the experimentally compatible range of $\sin \theta_{13}$ as a function of φ , for example 3 involving $PSL_2(7)$ elements of order 7. Orange and blue lines define the experimental bounds

representation $\underline{16} \in SO(10)$ and the global symmetry is simply $U(3)$ [18]. We assume that the GUT symmetry is broken by fluxes close to the compactification scale and so do the global ones to some discrete subgroup $D \in U(3)$ [19,20]. Alternatively, the breaking of a continuous group to a discrete subgroup by means of Higgs fields can in general be achieved in two ways: i) explicit breaking through a Higgs polynomial invariant under the action of the discrete group, or ii) through VEVs of higher than fundamental representations of Higgs fields. Of course, these two mechanisms are interrelated. For the breaking $SU(3) \rightarrow PSL(2, 7)$ in particular that concerns us in the present work, the explicit breaking is realized by means of the Klein quartic, given by

$$\phi_1^3 \phi_3 + \phi_3^3 \phi_2 + \phi_2^3 \phi_1 \quad (55)$$

for a suitable Higgs field representation (ϕ_1, ϕ_2, ϕ_3) [21]. The corresponding VEVs can be found by means of the appropriate computational packages [21].

5 Summary and conclusions

The last couple of decades, a substantial amount of research in physics beyond the Standard Model has been devoted to an interpretation of the lepton mixing matrix, and in particular, the neutrino data. A rather established approach to this task is to postulate invariance of the Yukawa lagrangian under some suitable finite group. Remarkably, such symmetries appear naturally in a wide class of extensions of the Standard Model emerging in the framework of string and F-theory constructions. Given these facts and the continuing interest on these issues as well as the considerably wide applications of the discrete groups in phenomenological models, in this article, we focused our investigations on the projective linear group $PSL_2(7)$. This group is a simple discrete subgroup

of $SU(3)$, and the largest one possessing three-dimensional unitary representations. Therefore it is a suitable candidate for a discrete flavor symmetry of the effective theory. However, despite its interesting features, its implications in low energy phenomenology have not been widely explored, partially because of the apparently complicated structure of the representations of its elements.

In this work we generate viable textures of charged-lepton neutrino mixing matrices by assuming that both types of matrices commute with some element of the $PSL_2(7)$ group. A tedious calculation reveals that the basic hypothesis is correct and it is valid for a number of group elements which are given in the context. It turns out that the neutrino mass matrix can only commute with order 2 group elements while the charged-lepton mass matrix can commute with both order 3 and order 7 elements. The results indicate there are only two types of matrices for the (el_3, el_2) combinations and also only two for the (el_7, el_2) combination. The eigenvector degeneracy of the el_2 elements (parametrized by a single parameter, the angle φ) may change somehow the values of the mixing angles. It appears, though, that the allowed values of the free parameter φ are very strongly centered around a fixed value which is $\varphi = 0$ for the (el_3, el_2) combination and $\varphi = \frac{2\pi}{7}$ for the (el_7, el_2) . The value $\varphi = 0$ suggests that nature prefers the eigenvector $\frac{1}{\sqrt{3}}(-1, 1, 1)$, which is independent of the values ρ_1, ρ_2, ρ_3 , which characterize the group $PSL_2(7)$. In the more general case, this eigenvector is replaced by $\frac{1}{\sqrt{3}}(e^{ic_3}, e^{i(c_3-c_2)}, 1)$ which is again independent of the values ρ_1, ρ_2, ρ_3 and the results obtained are similar to those presented here. Note that the algebra of the pairs (el_3, el_2) generates various A_4 subgroups of $PSL_2(7)$. Probably this is the reason for the appearance of the resulting mixing matrix as a generalization of the tri-bi-maximal mixing [17], i.e. the moduli of the middle column elements are equal to $\frac{1}{\sqrt{3}}$ —a phenomenon which is known to be con-

nected with an A_4 symmetry. Here, this simplification occurs non-trivially because the form of the elements is very complicated. As for the (el_7, el_2) pairs, no $PSL_2(7)$ subgroup is generated, a fact that leaves more space to the middle column elements to arrange themselves. Note that the allowed values of φ are centered around the central value $\varphi = \frac{2\pi}{7}$, which is the phase associated with the basic seventh root of unity $e^{i\frac{2\pi}{7}}$.

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Appendix A: Correspondence between $PSL_2(7)$ elements and GAP enumeration

The following tables depict the correspondence between the $PSL_2(7)$ elements as calculated and enumerated by GAP and their corresponding distribution among conjugacy classes used in the text.

• Character -1

el_2	1	2	3	4	5	6	7
GAP	119	120	121	122	123	124	125
el_2	8	9	10	11	12	13	14
GAP	140	141	142	143	144	145	146
el_2	15	16	17	18	19	20	21
GAP	161	162	163	164	165	166	167

• Character 0

el_3	1	2	3	4	5	6	7	8
GAP	1	2	3	4	5	6	7	8
el_3	9	10	11	12	13	14	15	16
GAP	9	10	11	12	13	14	51	52
el_3	17	18	19	20	21	22	23	24
GAP	53	54	55	56	57	58	59	60
el_3	25	26	27	28	29	30	31	32
GAP	61	62	63	64	72	73	74	75
el_3	33	34	35	36	37	38	39	40
GAP	76	77	78	97	98	99	100	101
el_3	41	42	43	44	45	46	47	48
GAP	102	103	126	127	128	129	130	131
el_3	49	50	51	52	53	54	55	56
GAP	132	133	134	135	136	137	138	139

• Character $+1$

el_4	1	2	3	4	5	6	7
GAP	16	17	18	19	20	21	22
el_4	8	9	10	11	12	13	14
GAP	23	24	25	26	27	28	29
el_4	15	16	17	18	19	20	21
GAP	30	31	32	33	34	35	36
el_4	22	23	24	25	26	27	28
GAP	37	38	39	40	41	42	43
el_4	29	30	31	32	33	34	35
GAP	44	45	46	47	48	49	50
el_4	36	37	38	39	40	41	42
GAP	79	80	81	82	83	84	85

• Character $-\frac{1}{2} - i\frac{\sqrt{7}}{2}$

el_7	1	2	3	4	5	6	7	8
GAP	65	66	67	68	69	70	71	86
el_7	9	10	11	12	13	14	15	16
GAP	87	88	89	90	91	92	93	94
el_7	17	18	19	20	21	22	23	24
GAP	95	96	104	105	106	107	108	109
el_7	25	26	27	28	29	30	31	32
GAP	110	111	112	113	114	115	116	117
el_7	33	34	35	36	37	38	39	40
GAP	118	147	148	149	150	151	152	153
el_7	41	42	43	44	45	46	47	48
GAP	154	155	156	157	158	159	160	168

Appendix B: The general eigenvectors of the matrix M

Given ρ , the three distinct eigenvalues of the matrix M , the non-normalized eigenvectors are given by the expression

$$v = \left(e^{ic_3} r_2 \left(1 + \rho e^{i(c_1+c_5)} \right), e^{i(c_3-c_2)} \left[\rho^2 e^{i(c_1+c_5)} - \rho \left(r_2 + r_1 e^{i(2c_1+c_5)} \right) \right] + r_3 e^{ic_1}, r_1 \left(\rho + e^{ic_1} \right) \right).$$

When normalized, while the el_2 and el_3 elements do not produce anything worth mentioning, the el_7 eigenvectors produce diagonalizing matrices which are Latin squares. All the phases can be exactly calculated, however, the trace of the resulting matrix to the best of our knowledge does not correspond to a known group character so this tantalizing result must remain a curiosity for the time being.

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