Hamiltonian dynamics and Faddeev–Jackiw formulation of 3D gravity with a Barbero–Immirzi like parameter

Alberto Escalante^a, J. Manuel-Cabrera

Instituto de Física, Universidad Autónoma de Puebla, Apartado Postal J-48, 72570 Puebla, Pue., Mexico

Received: 8 November 2016 / Accepted: 27 April 2017 / Published online: 12 May 2017
© The Author(s) 2017. This article is an open access publication

Abstract A detailed Dirac and Faddeev–Jackiw formulation of Bonzom–Livine model describing gravity in three dimensions is performed. The full structure of the constraints, the gauge transformations and the generalized Faddeev–Jackiw brackets are found. In addition, we show that the Faddeev–Jackiw and Dirac brackets coincide.

1 Introduction

It is well known that three-dimensional gravity is a good test theory for trying to understand the difficulties that emerge in the quantization of four-dimensional gravity. Since the work by Achúcarro, Towsend, Witten and other authors [1, 2], there has been a huge effort for understanding the classical and quantum connection between gravity and gauge theories, such as three-dimensional Chern–Simons theory [3]; then it is expected that the insight gained in the three-dimensional case could be useful for constructing better tools and apply them in the quantization of the four-dimensional theory. In this respect, it is common to obtain in three dimensions a relation between the Palatini and the Chern–Simons theories, which are equivalent at the Lagrangian level up to a total derivative [2–4]. Such a relation is not the only one, since the so-called exotic action with a Barbero–Immirzi like parameter [we call it from now on the Bonzom–Livine action (BL)] turns out to be classically equivalent to Palatini’s theory as well [5]. In fact, the BL model describes a set of actions sharing the equations of motion with Palatini’s theory; however, the symplectic structure is different. The symplectic structure in the BL model depends on a Barbero–Immirzi like parameter, from which one expects that the quantum theories will be different [4]. In this respect, something similar happens in the four-dimensional case with the Holst action [6], which provides a set of actions classically equivalent to Einstein’s theory; it depends on the so-called Barbero–Immirzi (γ) parameter

whose contribution can be appreciated at the classical level in the symplectic structure of the theory and the coupling of fermionic matter with gravity. In fact, it determines the coupling constant of a four-fermion interaction [7]. From the quantum point of view, the parameter gives a contribution in the quantum spectra of the area and volume operators in the context of loop quantum gravity [8–10]. Furthermore, the term added by Holst to Palatini’s action facilitates the canonical description of General Relativity, and depending of the values of that parameter we can reproduce the different scenarios found in canonical gravity. For instance, it is possible to obtain the ADM, Ashtekar and Barbero formulations in a straightforward way [6]. Nevertheless, in spite of the Holst action providing a general action for gravity, the γ parameter is still controversial [8–10]. In this sense, the BL action comes to be the three-dimensional equivalent model of Holst’s action. In fact, the equivalence is not given only with the presence of a γ parameter, but at the classical level if we perform a partial gauge fixing in the canonical description of BL, it is possible obtain a full Ashtekar connection dynamics [4]. In this respect, the analysis of the symmetries of the BL action has been performed in [4, 5]; in this work the canonical analysis was performed by using the Dirac method. However, the analysis was developed on a smaller phase space and the complete structure of the constraints on the full phase space was not reported.

It is important to remark that if some of the Dirac steps is omitted, then it is possible to obtain incomplete results [11, 12]. In this manner, an analysis developed on the full phase space and following all the Dirac steps is mandatory. Nevertheless, in some cases, to develop the Dirac method for gauge theories is a large and tedious task, hence, it is necessary to use alternative formulations that could give us a complete canonical description of the theory, in this sense, there is a different approach for studying gauge theories, the so-called Faddeev–Jackiw [FJ] formalism [13]. The FJ method is a symplectic approach, namely, all relevant information of

^a e-mail: aescalan@ifuap.buap.mx

the theory can be obtained through an invertible symplectic tensor, which is constructed by means of the symplectic variables identified as the degrees of freedom. Because of the theory is singular there will be constraints, and the FJ framework has the advantage that all constraints of the theory are to be treated on the same footing, namely, it is not necessary to perform the classification of the constraints in primary, secondary, first class or second class as in Dirac's method [14, 15]. When a symplectic tensor is obtained, then its components are identified with the FJ generalized brackets; at the end Dirac's brackets and FJ brackets coincide.

In this paper a complete Dirac and FJ analysis of the BL model is performed. In fact, in order to compare the two approaches, it is necessary to work out a complete Dirac analysis. Hence, we need to know the complete structure of the constraints over the full phase space for constructing the Dirac brackets and to compare these brackets with the generalized FJ ones [16]. Furthermore, we shall prove that the FJ approach is more economic than Dirac's one.

It is important to comment that our results have not been reported in the literature, and as a special case those results reported in [4, 5] are reproduced. In addition, we would also remark that for BL theory we shall construct the Dirac brackets by eliminating the second class constraints and there will remain first class ones. Furthermore, at the end of the paper, we have added an appendix where the analysis of an Abelian BL theory is performed; in that appendix, we construct the Dirac brackets by fixing the gauge and also we reproduce all those results by means of FJ formalism.

The paper is organized as follows; in Sect. 2 a detailed canonical analysis of BL is performed. We report the complete structure of the constraints defined on the full phase space, then we eliminate the second class constraints for constructing the Dirac brackets. In Sect. 3, we study the relation between BL and Chern–Simons theory. We reproduce the results of the previous section by performing a pure Dirac analysis of a generalized Chern–Simons theory. In Sect. 4, a detailed FJ analysis of BL action is developed. In order to reproduce all the Dirac results, we work with the configuration space fields as symplectic variables, and identifying all the constraints of the theory we show that the FJ generalized and Dirac brackets coincide. In Sect. 5 we add some remarks and conclusions.

2 Hamiltonian dynamics for three-dimensional BL gravity

In this section, we will study the Hamiltonian dynamics of the action proposed by BL [5]. We will perform our analysis by using a pure Dirac method, namely, we will find all constraints defined on the full phase space. As was commented on above, there is an analysis of the BL action developed

on a smaller phase space reported in [4, 5]. However, in this work the structure of the constraints is not complete, thus, in order to compare the FJ method with the Dirac one it is mandatory to perform the Dirac analysis on the full phase space by following all Dirac steps.

It is well known that three-dimensional gravity with a cosmological constant can be written as a Chern–Simons theory [1–5]. In fact, if the principal gauge bundle G over M is given by $G = \text{SU}(2)$ for 3d Euclidean gravity, then we can enlarge the group G to $\tilde{G}$, where $\tilde{G}$ could be $\text{SO}(4)$, $\text{ISO}(3)$ or $\text{SO}(3, 1)$, depending on the sign of the cosmological constant Λ , it being positive, zero or negative respectively. Hence, the algebra of the generators of $\tilde{G}$ will satisfy the following commutation relations [4, 5]:

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ij}^k J_k & [J_i, K_j] &= \epsilon_{ij}^k K_k \\ [K_i, K_j] &= s \epsilon_{ij}^k J_k, \end{aligned} \quad (1)$$

where $s = -1, 0, 1$, corresponding to the sign of the cosmological constant $i, j, k = 1, 2, 3$ and J_i, K_i are rotations and boosts respectively. In order to construct a Chern–Simons theory being equivalent to standard Einstein's action of gravity, we choose the following non-degenerate invariant bilinear form:

$$\langle J_i, K_j \rangle = \delta_{ij}, \quad \langle J_i, J_j \rangle = \langle K_i, K_j \rangle = 0, \quad (2)$$

in this manner, the 3d Palatini's action with cosmological constant can be written as

$$S'_{\text{Palatini}} = \frac{1}{\sqrt{|\Lambda|}} \int_M \epsilon^{\mu\nu\rho} \left(\langle A_\mu, \partial_\nu A_\rho \rangle + \frac{1}{3} \langle A_\mu, [A_\nu, A_\rho] \rangle \right). \quad (3)$$

On the other hand, if $\Lambda \neq 0$, then there is another invariant non-degenerate bilinear form, given by

$$(J_i, J_j) = \delta_{ij} \quad (K_i, K_j) = s \delta_{ij} \quad (J_i, K_j) = 0, \quad (4)$$

in this case, we can obtain from the following Chern–Simons action:

$$\begin{aligned} \tilde{S}_{\text{Exotic}} &= \frac{1}{\sqrt{|\Lambda|}} \int_M \epsilon^{\mu\nu\rho} \left((A_\mu, \partial_\nu A_\rho) \right. \\ &\quad \left. + \frac{1}{3} (A_\mu, [A_\nu, A_\rho]) \right), \end{aligned} \quad (5)$$

the so-called exotic action for gravity,

$$\begin{aligned} \tilde{S}[A, e] &= \frac{1}{\sqrt{|\Lambda|}} \left[\int_M A^i \wedge dA_i + \frac{1}{3} \epsilon_{ijk} A^i \wedge A^j \wedge A^k \right] \\ &\quad + s \sqrt{|\Lambda|} \int_M e^i \wedge dA e_i, \end{aligned} \quad (6)$$

where the 1-form $A^i = A_\mu^i dx^\mu$, $(dA v)^i = dv^i + [A, v]^i = dv^i + \epsilon^i_{jk} A^j \wedge v^k$ and $F^i = dA^i + \frac{1}{2} \epsilon^i_{jk} A^j \wedge A^k$ is the strength 2-form. In this manner, the BL model consists of considering the combination of Palatini's and exotic action

through the γ parameter, this being a kind of Barbero–Immirzi parameter,

$$S_\gamma[A, e] = S'_{\text{Palatini}}[A, e] + \frac{1}{\gamma} \tilde{S}_{\text{Exotic}}[A, e]. \quad (7)$$

In fact, from the action (7) we obtain a family of theories classically equivalent to 3d gravity in the sense that Palatini's theory with cosmological constant and BL actions share the equations of motion, which can be seen from the variation of the action (7),

$$\begin{aligned} \frac{\delta S_\gamma[A, e]}{\delta e_{\mu i}} : \quad & \epsilon^{\mu\nu\rho} \left[F^i{}_{\nu\rho}[A] + s \frac{|\Lambda|}{2} \epsilon^i{}_{jk} e_\nu{}^j e_\rho{}^k \right] \\ & + s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{\mu\nu\rho} D_\nu e^i{}_\rho = 0, \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{\delta S_\gamma[A, e]}{\delta A_{\mu i}} : \quad & \epsilon^{\mu\nu\rho} D_\nu e^i{}_\rho + \frac{1}{\gamma \sqrt{|\Lambda|}} \epsilon^{\mu\nu\rho} \\ & \times \left[F^i{}_{\nu\rho}[A] + s \frac{|\Lambda|}{2} \epsilon^i{}_{jk} e_\nu{}^j e_\rho{}^k \right] = 0, \end{aligned} \quad (9)$$

and Eqs. (8) and (9) are equivalent to Einstein's equations. Hence, in order to develop the Hamiltonian analysis, we perform the 2+1 decomposition of the action (7) obtaining

$$\begin{aligned} S_\gamma[e, A] = \int d^3x \left[2\epsilon^{0ab} \delta_{ij} \left(e_0^i + \frac{1}{\gamma \sqrt{|\Lambda|}} A_0^i \right) \right. \\ \times \left(F^j{}_{ab} + s \frac{|\Lambda|}{2} \epsilon^j{}_{kl} e_a^k e_b^l \right) + 2\epsilon^{0ab} \delta_{ij} D_a e_b^i \\ \times \left(A_0^j + s \frac{\sqrt{|\Lambda|}}{\gamma} e_0^j \right) + 2\epsilon^{0ab} \delta_{ij} \left(e_b^i \partial_0 A_a^j \right. \\ \left. + \frac{1}{2\gamma \sqrt{|\Lambda|}} A_b^i \partial_0 A_a^j + s \frac{\sqrt{|\Lambda|}}{2\gamma} e_b^i \partial_0 e_a^j \right) \Big], \end{aligned} \quad (10)$$

where $a, b, c = 1, 2$. The definition of the momenta $(\pi^\alpha{}_i, \Pi^\alpha{}_i)$ canonically conjugate to $(e^i{}_\alpha, A_\alpha{}^i)$ is given by

$$\Pi^\alpha{}_i = \frac{\delta \mathcal{L}}{\delta \dot{A}_\alpha{}^i}, \quad \pi^\alpha{}_i = \frac{\delta \mathcal{L}}{\delta \dot{e}^i{}_\alpha}. \quad (11)$$

The matrix elements of the Hessian

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu e^i{}_\alpha) \partial(\partial_\mu e^j{}_\beta)}, \quad & \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu e^i{}_\alpha) \partial(\partial_\mu A_\beta{}^i)}, \\ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu A_\alpha{}^i) \partial(\partial_\mu A_\beta{}^i)}, \end{aligned} \quad (12)$$

are identically zero, thus we expect 18 primary constraints. From the definition of the momenta (11) we identify the following 18 primary constraints:

$$\begin{aligned} \phi_i{}^0 &:= \pi_i{}^0 \approx 0, \\ \phi_i{}^a &:= \pi_i{}^a - s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{0ab} \delta_{ij} e_b^j \approx 0, \\ \Phi_i{}^0 &:= \Pi_i{}^0 \approx 0, \\ \Phi_i{}^a &:= \Pi_i{}^a - 2\epsilon^{0ab} \delta_{ij} \left(e_b^j + \frac{1}{2\gamma \sqrt{|\Lambda|}} A_b^j \right) \approx 0. \end{aligned} \quad (13)$$

The canonical Hamiltonian takes the form

$$\begin{aligned} H_c = \int d^3x \left[-2\epsilon^{0ab} \delta_{ij} D_a e_b^i \left(A_0^i + s \frac{\sqrt{|\Lambda|}}{\gamma} e_0^i \right) \right. \\ \left. - 2\epsilon^{0ab} \delta_{ij} \left(e_0^i + \frac{1}{\gamma \sqrt{|\Lambda|}} A_0^i \right) \right. \\ \left. \times \left(F^j{}_{ab} + s \frac{|\Lambda|}{2} \epsilon^j{}_{kl} e_a^k e_b^l \right) \right], \end{aligned} \quad (14)$$

and the primary Hamiltonian is given by

$$H_P = H_c + \int d^3x \left[\lambda^i{}_\alpha \phi_i{}^\alpha + \xi^i{}_\alpha \Phi_i{}^\alpha \right], \quad (15)$$

where $\lambda^i{}_\alpha, \xi^i{}_\alpha$ are Lagrange multipliers enforcing the constraints $(\phi_i{}^\alpha, \Phi_i{}^\alpha)$. The fundamental Poisson brackets of the theory are given

$$\begin{aligned} \{e_\alpha{}^i(x), \pi^\beta{}_j(y)\} &= \delta^\beta{}_\alpha \delta^i{}_j \delta^2(x-y), \\ \{A_\alpha{}^i(x), \Pi^\beta{}_j(y)\} &= \delta^\beta{}_\alpha \delta^i{}_j \delta^2(x-y), \end{aligned} \quad (16)$$

where we can observe that in these fundamental brackets there is not any contribution of the γ parameter; in the Dirac brackets, however, there will be a non-trivial contribution. In order to determine the presence of more constraints, we calculate the following 18×18 matrix whose entries are the Poisson brackets among the constraints (13):

$$\begin{aligned} \{\phi_i{}^a(x), \phi_j{}^b(y)\} &= -2s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\ \{\phi_i{}^a(x), \Phi_j{}^b(y)\} &= -2\epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\ \{\Phi_i{}^a(x), \Phi_j{}^b(y)\} &= -2 \frac{1}{\gamma \sqrt{|\Lambda|}} \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \end{aligned} \quad (17)$$

we appreciate that this matrix has rank = 12 and six null vectors. By using the six null vectors and consistency conditions we obtain the following six secondary constraints:

$$\begin{aligned} \gamma_i{}^0 &:= \pi_i{}^0 \approx 0, \\ \tilde{\gamma}_i{}^0 &:= \Pi_i{}^0 \approx 0, \\ \dot{\phi}_i{}^0 &= \{\phi_i{}^0(x), H_P\} \approx 0 \Rightarrow \psi_i := 2\epsilon^{0ab} s \frac{\sqrt{|\Lambda|}}{\gamma} D_a e_{ib} \\ &+ 2\epsilon^{0ab} \left(F_{iab} + \frac{s|\Lambda|}{2} \epsilon_{ijk} e_a^j e_b^k \right) \approx 0, \end{aligned}$$

$$\begin{aligned}\dot{\Phi}_i^0 &= \{\Phi_i^0(x), H_P\} \approx 0 \Rightarrow \Psi_i := 2\epsilon^{0ab} D_a e_{ib} \\ &+ 2\epsilon^{0ab} \frac{1}{\gamma \sqrt{|\Lambda|}} \left(F_{iab} + \frac{s|\Lambda|}{2} \epsilon_{ijk} e^j{}_a e^k{}_b \right) \approx 0,\end{aligned}\quad (18)$$

and the rank allows us to fix the following values for the Lagrangian multipliers:

$$\begin{aligned}\dot{\Phi}_i^a &= \{\Phi_i^a, H_P\} \approx 0 \\ &\Rightarrow 2\epsilon^{0ab} \frac{s - \gamma^2}{\gamma} \sqrt{|\Lambda|} (-\lambda_{bi} + D_b e_{0i} + \epsilon_{lim} e_b^m e_0^l) \approx 0, \\ \dot{\Phi}_i^a &= \{\Phi_i^a, H_P\} \approx 0 \\ &\Rightarrow 2\epsilon^{0ab} \frac{s - \gamma^2}{\gamma^2} (-\xi_{bi} + D_b A_{0i} + s|\Lambda| \epsilon_{lim} e_b^m e_0^l) \approx 0.\end{aligned}\quad (19)$$

Consistency requires conservation in time of the secondary constraints, however, for this theory there are no tertiary constraints. At this point, we need to identify from the primary and secondary constraints which ones correspond to first and second class. To this aim, we need to calculate the rank and the null vectors of the 24×24 matrix whose entries will be the Poisson brackets between primary and secondary constraints[this is

$$\begin{aligned}\{\phi_i^a(x), \phi_j^b(y)\} &= -2s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{\phi_i^a(x), \Phi_j(y)\} &= -2\epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{\phi_i^a(x), \psi_j(y)\} &= -2\epsilon^{0ab} \left[\frac{s\sqrt{|\Lambda|}}{\gamma} \delta_{ij} \partial_b - \epsilon_{ijk} \right. \\ &\quad \times \left. \left(\frac{s\sqrt{|\Lambda|}}{\gamma} A^k{}_b + s|\Lambda| e^k{}_b \right) \right] \delta^2(x - y), \\ \{\phi_i^a(x), \Psi_j(y)\} &= -2\epsilon^{0ab} \left[\delta_{ij} \partial_b - \epsilon_{ijk} \left(A^k{}_b \right. \right. \\ &\quad \left. \left. + \frac{s\sqrt{|\Lambda|}}{\gamma} e^k{}_b \right) \right] \delta^2(x - y), \\ \{\Phi_i^a(x), \Phi_j^b(y)\} &= -2 \frac{1}{\gamma \sqrt{|\Lambda|}} \epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{\Phi_i^a(x), \psi_j(y)\} &= -2\epsilon^{0ab} \left[\delta_{ij} \partial_b - \epsilon_{ijk} \right. \\ &\quad \times \left. \left(A^k{}_b + \frac{s\sqrt{|\Lambda|}}{\gamma} e^k{}_b \right) \right] \delta^2(x - y), \\ \{\Phi_i^a(x), \Psi_j(y)\} &= -2\epsilon^{0ab} \left[\frac{1}{\gamma \sqrt{|\Lambda|}} \delta_{ij} \partial_b - \epsilon_{ijk} \right. \\ &\quad \times \left. \left(\frac{1}{\gamma \sqrt{|\Lambda|}} A^k{}_b + e^k{}_b \right) \right] \delta^2(x - y), \\ \{\psi_i(x), \psi_j(y)\} &= 0, \\ \{\Psi_i(x), \Psi_j(y)\} &= 0, \\ \{\psi_i(x), \Psi_j(y)\} &= 0.\end{aligned}\quad (20)$$

This matrix has a rank = 12 and 12 null vectors, thus the theory presents a set of 12 first class constraints and 12 second class

constraints. By using the contraction of the null vectors with the constraints (13) and (18), we identify the following 12 first class constraints:

$$\begin{aligned}\gamma^0{}_i &= \pi_i^0 \approx 0, \\ \Gamma^0{}_i &= \Pi_i^0 \approx 0, \\ \omega_i &= D_a \chi_i^a - s|\Lambda| \epsilon_{ijk} e^k{}_a \Xi_j^a + 2\epsilon^{0ab} s \frac{\sqrt{|\Lambda|}}{\gamma} D_a e_{ib} \\ &+ 2\epsilon^{0ab} \left(F_{iab} + \frac{s|\Lambda|}{2} \epsilon_{ijk} e^j{}_a e^k{}_b \right) \approx 0, \\ \Gamma_i &= D_a \Xi_i^a - \epsilon_{ijk} e^k{}_a \chi_j^a + 2\epsilon^{0ab} D_a e_{ib} \\ &+ 2\epsilon^{0ab} \frac{1}{\gamma \sqrt{|\Lambda|}} \left(F_{iab} + \frac{s|\Lambda|}{2} \epsilon_{ijk} e^j{}_a e^k{}_b \right) \approx 0,\end{aligned}\quad (21)$$

and the following 12 second class constraints:

$$\begin{aligned}\chi_i^a &= \pi_i^a - s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{0ab} \delta_{ij} e_b^j \approx 0, \\ \Xi_i^a &= \Pi_i^a - 2\epsilon^{0ab} \delta_{ij} \left(e_b^j + \frac{1}{2\gamma \sqrt{|\Lambda|}} A_b^j \right) \approx 0.\end{aligned}\quad (22)$$

It is important to remark that these constraints have not been reported in the literature, and their complete structure defined on the full phase space will be relevant in order to know the fundamental gauge transformations and for constructing the Dirac brackets. On the other hand, the constraints will play a key role to make the progress in the quantization. We have commented above that the structure of the constraints (21) is obtained by means of the null vectors, for instance, a set of null vectors of the matrix (20) are given by

$$\begin{aligned}V_i^1 &= (0, -\delta_j^i \partial_a \delta^2(x - y) - \epsilon_{jk}^i A_a^k \delta^2(x - y), 0, \\ &-s|\Lambda| \epsilon_{ijk} e_a^k \delta^2(x - y), \delta_j^i \delta^2(x - y), 0),\end{aligned}$$

hence, by performing the contraction of these null vectors with the primary and secondary constraints, the first class constraint ω_i given in (21) is obtained. Now, we will calculate the algebra of the constraints

$$\begin{aligned}\{\chi_i^a(x), \chi_j^b(y)\} &= -2s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{\chi_i^a(x), \Xi_j^b(y)\} &= -2\epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{\Xi_i^a(x), \Xi_j^b(y)\} &= -2 \frac{1}{\gamma \sqrt{|\Lambda|}} \epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{\chi_i^a(x), \omega_j(y)\} &= s|\Lambda| \epsilon_{ijk} \Phi^a{}_k \delta^2(x - y) \approx 0, \\ \{\Xi_i^a(x), \omega_j(y)\} &= \epsilon_{ijk} \phi^a{}_k \delta^2(x - y) \approx 0, \\ \{\chi_i^a(x), \Gamma_j(y)\} &= \epsilon_{ijk} \phi^a{}_k \delta^2(x - y) \approx 0, \\ \{\Xi_i^a(x), \Gamma_j(y)\} &= \epsilon_{ijk} \Phi^a{}_k \delta^2(x - y) \approx 0, \\ \{\omega_i(x), \omega_j(y)\} &= s|\Lambda| \epsilon_{ijk} \Gamma^k \delta^2(x - y) \approx 0,\end{aligned}\quad (23)$$

$$\{\Gamma_i(x), \Gamma_j(y)\} = \epsilon_{ijk} \Gamma^k \delta^2(x-y) \approx 0,$$

$$\{\omega_i(x), \Gamma_j(y)\} = \epsilon_{ijk} \omega^k \delta^2(x-y) \approx 0.$$

where we can observe that the algebra is closed. Furthermore, with all the information obtained until now, we can construct the Dirac brackets. In fact, there are two options for constructing them. The first one is by eliminating the second class constraints and keeping on the first class, and the second option is by fixing the gauge and converting the first class constraints into second class ones. In this section we will eliminate the second class constraints, the first class ones remaining; in the FJ approach we will discuss both. Hence, in order to construct the Dirac brackets, we shall use the matrix whose elements are only the Poisson brackets among second class constraints, namely $C_{\alpha\beta}(u, v) = \{\zeta^\alpha(u), \zeta^\beta(v)\}$, given by

$$[C_{(\alpha\beta)}(x, x')] = -2 \begin{pmatrix} \frac{s\sqrt{|\Lambda|}}{\gamma} & 1 \\ 1 & \frac{1}{\gamma\sqrt{|\Lambda|}} \end{pmatrix} \delta_{ij} \epsilon^{0ab} \delta^2(x-x'), \quad (24)$$

its inverse is given by

$$[C^{-1}_{(\alpha\beta)}(x, x')] = \frac{\gamma^2}{2(s-\gamma^2)} \begin{pmatrix} \frac{1}{\gamma\sqrt{|\Lambda|}} & -1 \\ -1 & \frac{s\sqrt{|\Lambda|}}{\gamma} \end{pmatrix} \times \delta^{ij} \epsilon_{0ba} \delta^2(x-x'). \quad (25)$$

The Dirac brackets for two functionals A, B are expressed by

$$\{A(x), B(y)\}_D = \{A(x), B(y)\}_P - \int du dv \{A(x), \zeta^\alpha(u)\} C^{-1}_{\alpha\beta}(u, v) \{\zeta^\beta(v), B(y)\}, \quad (26)$$

where $\{A(x), B(y)\}_P$ is the usual Poisson bracket between the functionals A, B and $\zeta^\alpha(u) = (\chi_i^a, \Xi_i^a)$ is the set of second class constraints. Hence, by using (25) and (26) we obtain the following Dirac brackets of the theory:

$$\{e^i_a(x), \pi^b_j(y)\}_D = \frac{s}{2(s-\gamma^2)} \delta^b_a \delta^i_j \delta^2(x-y),$$

$$\{e^i_a(x), e^j_b(y)\}_D = \frac{1}{2\sqrt{|\Lambda|}} \frac{\gamma}{s-\gamma^2} \delta^{ij} \epsilon_{0ab} \delta^2(x-y),$$

$$\{\pi^a_i(x), \pi^b_j(y)\}_D = \frac{s^2 \sqrt{|\Lambda|}}{2\gamma s-\gamma^2} \delta_{ij} \epsilon^{0ab} \delta^2(x-y),$$

$$\{A^i_a(x), \Pi^b_j(y)\}_D = \frac{s-2\gamma^2}{2(s-\gamma^2)} \delta^b_a \delta^i_j \delta^2(x-y),$$

$$\{A^i_a(x), A^j_b(y)\}_D = \frac{s\sqrt{|\Lambda|}}{2} \frac{\gamma}{s-\gamma^2} \delta^{ij} \epsilon_{0ab} \delta^2(x-y),$$

$$\{\pi^a_i(x), \Pi^b_j(y)\}_D = \frac{s}{2\gamma\sqrt{|\Lambda|}} \frac{1}{s-\gamma^2} \delta_{ij} \epsilon^{0ab} \delta^2(x-y),$$

$$\{A^i_a(x), e^j_b(y)\}_D = \frac{1}{2} \frac{\gamma^2}{\gamma^2-s} \delta^{ij} \epsilon_{0ab} \delta^2(x-y),$$

$$\{e^i_a(x), \Pi^b_j(y)\}_D = \frac{1}{2\sqrt{|\Lambda|}} \frac{\gamma}{s-\gamma^2} \delta^b_a \delta^i_j \delta^2(x-y),$$

$$\{A^i_a(x), \pi^b_j(y)\}_D = -\frac{s\sqrt{|\Lambda|}}{2} \frac{\gamma}{s-\gamma^2} \delta^b_a \delta^i_j \delta^2(x-y),$$

$$\{\pi^a_i(x), \Pi^b_j(y)\}_D = \frac{1}{2} \frac{s}{s-\gamma^2} \delta_{ij} \epsilon^{0ab} \delta^2(x-y). \quad (27)$$

We can observe that the Dirac brackets depend on the constants (s, γ) , and we can reproduce several scenarios depending of specific values. Hence, for $s = 1$ and the limit $\gamma \rightarrow \infty$, we recover the Dirac canonical structure of Palatini's action [11]. It is important to remark that in the BL model the fields e, A and their canonical momenta have become non-commutative, while in Palatini's action they are commutative. This is a classical difference between the BL model and Palatini's theory. Moreover, at the quantum level this difference is fundamental for constructing the Ashtekar representation of the BL model [4].

Now, we can calculate the gauge transformations on the full phase space. In fact, the correct gauge symmetry is obtained according to Dirac's conjecture by constructing a gauge generator using the first class constraints, and the structure of the constraints defined on the full phase space will give us the fundamental gauge transformations. To this aim, we will apply the Castellani algorithm for constructing the gauge generator; we define the generator of gauge transformations as

$$G = \int_{\Sigma} \left[D_0 \varepsilon^i{}_0 \gamma^0{}_i + D_0 \tau_0^i \Gamma^0{}_i + \varepsilon^i \omega_i + \tau^i \Gamma_i \right]. \quad (28)$$

Therefore, we find that the gauge transformations on the phase space are

$$\delta_0 e^i{}_0 = D_0 \varepsilon^i{}_0,$$

$$\delta_0 e^i{}_a = -D_a \varepsilon^i + \epsilon^i{}_{jk} e_a{}^k \tau^j,$$

$$\delta_0 A^0{}_i = D_0 \tau_0^i,$$

$$\delta_0 A^i{}_a = -D_a \tau^i + s |\Lambda| \epsilon^i{}_{jk} e^k{}_a \varepsilon^j,$$

$$\delta_0 \pi^0{}_i = 0,$$

$$\delta_0 \pi^a{}_i = -\Omega \epsilon^{0ab} D_b \varepsilon_i + \epsilon_{ij}{}^k \pi^a{}_k \tau^j + s |\Lambda| \epsilon_{ij}{}^k \Xi^a{}_k,$$

$$\delta_0 \Pi^0{}_i = -\epsilon_{ij}{}^k \left(\pi^0{}_k \varepsilon^j - \Pi^0{}_k \tau^j \right),$$

$$\delta_0 \Pi^a{}_i = -2\epsilon^{0ab} D_b \varepsilon_i + \epsilon_{ij}{}^k \chi^a{}_k \varepsilon^j$$

$$- \frac{1}{\gamma\sqrt{|\Lambda|}} \epsilon^{0ab} D_b \tau_i + \epsilon_{ij}{}^k \Xi^a{}_k \tau^j$$

$$+ 2\epsilon^{0ab} \epsilon_{ijk} e^k{}_b \tau^j + \Omega \epsilon^{0ab} \epsilon_{ijk} e^k{}_b \varepsilon^j. \quad (29)$$

We realize that the fundamental gauge transformations of the BL action are given by (29) and they are Λ -deformed ISO(3) transformations. It is important to remark that the internal group of the theory is SU(2) (or SO(3)), however, the fundamental gauge symmetry corresponds to Λ -deformed ISO(3) transformations, all these results were not reported in [4,5]. On the other hand, any theory with background

independence is diffeomorphisms covariant, and this symmetry must be obtained from the fundamental gauge transformations. Hence, the diffeomorphisms can be found by redefining the gauge parameters as $\varepsilon_0^i = -\varepsilon^i = \xi^\rho e^i_\rho$, $\tau_0^i = -\tau^i = \xi^\rho A_\rho^i$, and the gauge transformation (29) takes the following form:

$$\begin{aligned} e^i_\alpha &\rightarrow e^i_\alpha + \mathcal{L}_\xi e^i_\alpha + \xi^\rho \left[D_\alpha e^i_\rho - D_\rho e^i_\alpha \right], \\ A^i_\alpha &\rightarrow A^i_\alpha + \mathcal{L}_\xi A^i_\alpha + \xi^\rho \left[\partial_\alpha A^i_\rho - \partial_\rho A^i_\alpha \right. \\ &\quad \left. + \epsilon_{ijk} A^j_\alpha A^k_\rho + s \mid \Lambda \mid \epsilon^i_{jk} e^j_\alpha e^k_\rho \right]. \end{aligned} \quad (30)$$

Therefore, the diffeomorphisms are obtained (on shell) from the fundamental gauge transformations as an internal symmetry of the theory. With the correct identification of the constraints, we can carry out the counting of degrees of freedom in the following form: there are 36 canonical variables (e^i_α , A^i_α , π^{α}_i , Π^{α}_i), 12 first class constraints (γ_i^0 , Γ_i^0 , ω_i , Γ_i) and 12 second class constraints (χ_i^a , Ξ_i^a) and one concludes that the $S_\gamma[A, e]$ action for gravity in three dimensions is devoid of degrees of freedom; therefore, the theory is topological.

As a conclusion of this part, we have obtained the extended action, the extended Hamiltonian, the complete structure of the constraints on the full phase space and the algebra among them. The price to pay for using the complete phase space is that the theory presents a set of first and second class constraints; by using the second class constraints we have constructed the Dirac brackets and they will be useful in the quantization of the theory [4].

3 Relation with Chern–Simons theory

We have seen in previous sections that either Palatini's theory or an exotic action for gravity can be expressed as a Chern–Simons theory, however, it will be interesting to express the BL action as a Chern–Simons theory as well; is this possible? The answer is yes. In fact, the action analyzed in the previous section can be written in an elegant form in terms of a Chern–Simons theory. By introducing the variables $\omega^{\pm i} = A^i \pm \sigma \sqrt{|\Lambda|} e^i$ [5], where $\sigma^2 = s$ with $\sigma = 1$ for $\Lambda > 0$ and $\sigma = i$ for $\Lambda < 0$, we see that the action (7) can be written as

$$\begin{aligned} S_\gamma &= \frac{\sqrt{|\Lambda|}}{2} \left(\gamma^{-1} + \frac{\sigma}{s} \right) S_{CS}(\omega^+) \\ &\quad + \frac{\sqrt{|\Lambda|}}{2} \left(\gamma^{-1} - \frac{\sigma}{s} \right) S_{CS}(\omega^-), \end{aligned} \quad (31)$$

where

$$S_{CS}(\omega^\pm) = \int_M \omega^{\pm i} \wedge d\omega_i^\pm + \frac{1}{3} \epsilon_{ijk} \omega^{\pm i} \wedge \omega^{\pm j} \wedge \omega^{\pm k}.$$

The equations of motion obtained from (31) imply that the connections $\omega^{\pm i}$ are flat, and it is easy to prove that these flatness conditions are equivalent to the equations of motion given in (8) and (9). On the other hand, if we develop a pure Dirac analysis of the action (31) we will reproduce the results given in the previous section, in particular we will reproduce the Dirac brackets. In fact, in summary, by performing a pure Dirac method we obtain the following results.

We have the following first class constraints:

$$\begin{aligned} \mathfrak{G}_i^\pm &= \frac{s \pm \sigma \gamma}{s \gamma \sqrt{|\Lambda|}} \epsilon^{0ab} \left(\partial_a \omega_{bi}^{(\pm)} + \frac{1}{2} \epsilon_{ijk} \omega_b^{(\pm)j} \omega_c^{(\pm)k} \right) \\ &\quad + D_a^{(\pm)} \chi_i^{(\pm)a} \approx 0, \end{aligned} \quad (32)$$

and the following second class constraints:

$$\chi_i^{(\pm)a} = \pi_i^{(\pm)a} - \frac{1}{2} \frac{s \pm \sigma \gamma}{s \gamma \sqrt{|\Lambda|}} \epsilon^{0ab} \omega_{(\pm)bi} \approx 0, \quad (33)$$

here, $\pi_i^{(\pm)a}$ are the conjugate canonical momenta of the connection $\omega_a^{\pm i}$ and $D_a^{\pm} \lambda^i = \partial_a \lambda^i + \epsilon^i_{jk} \omega_a^{\pm j} \lambda^k$. Furthermore, by using the second class constraints (33) the following Dirac brackets are obtained:

$$\begin{aligned} \left\{ \omega_a^{(\pm)i}(x), \omega_b^{(\pm)j}(y) \right\}_D &= \frac{s \gamma \sqrt{|\Lambda|}}{s \pm \sigma \gamma} \delta^{ij} \epsilon_{0ab} \delta^2(x - y), \\ \left\{ \omega_a^{(\pm)i}(x), \pi_j^{(\pm)b}(y) \right\}_D &= \frac{1}{2} \delta_j^i \delta_a^b \delta^2(x - y), \\ \left\{ \pi_i^{(\pm)a}(x), \pi_j^{(\pm)b}(y) \right\}_D &= \frac{1}{4} \frac{s \pm \sigma \gamma}{s \gamma \sqrt{|\Lambda|}} \delta_{ij} \epsilon^{0ab} \delta^2(x - y); \end{aligned} \quad (34)$$

we can observe that the Dirac brackets between the dynamical variables given in (34) depend on the constants (s , γ , σ). This fact will be important because, by using the definition of $\omega_a^{\pm i}$ given in terms of A_a^i and e_a^i in (34), the Dirac brackets given in (27) are reproduced. It is important to comment that our results are given in a general form and contain the cases for $\Lambda > 0$ and $\Lambda < 0$, thus in particular we can reproduce the results given in [17] where the case $\Lambda < 0$ was studied. On the other hand, in the limit $\gamma \rightarrow \infty$ the action (31) is reduced to $S_{CS}(\omega)$ and the Dirac brackets are reduced to those reported in [11], where Palatini's theory was analyzed. Finally, we can observe that in this section we have proved the equivalence between the BL model and the Chern–Simons theory, thus the standard quantization procedure can be performed. In the following section we will study the action (7) by using the FJ approach and we will obtain all the Dirac results in an alternative way.

4 Faddeev–Jackiw analysis for BL theory

By rewriting the action (10) in the following form:

$$\mathcal{L} = 2\epsilon^{0ab}\delta_{ij}e_b^i\dot{A}_a^j + \beta\epsilon^{0ab}\delta_{ij}A_b^i\dot{A}_a^j + \Omega\epsilon^{0ab}\delta_{ij}e_b^i\dot{e}_a^j - V^{(0)}, \quad (35)$$

where $V^{(0)} = -2\epsilon^{0ab}\delta_{ij}\left[(e_0^i + \beta A_0^i)(F_{ab}^j + s\frac{|\Lambda|}{2}\epsilon^{jkl}e_a^ke_b^l) + (\Omega e_0^i + A_0^i)D_a e_b^j\right]$ is called the symplectic potential and we have introduced the following constants Ω and β :

$$\Omega = \frac{s\sqrt{|\Lambda|}}{\gamma}, \quad \beta = \frac{1}{\gamma\sqrt{|\Lambda|}}. \quad (36)$$

In the FJ framework, the Euler–Lagrange equations of motion are given by [13]

$$f_{ab}^{(0)}\dot{\xi}^b = \frac{\partial V^{(0)}(\xi)}{\partial \xi^a}, \quad (37)$$

where the symplectic matrix $f_{ab}^{(0)}$ takes the form

$$f_{ab}^{(0)}(x, y) = \frac{\delta a_b(y)}{\delta \xi^a(x)} - \frac{\delta a_a(x)}{\delta \xi^b(y)}, \quad (38)$$

with $\xi^{(0)a}$ and $a^{(0)}_a$ are representing a set of symplectic variables. It is important to comment that in the FJ framework we are free to choose the symplectic variables; we can choose the field configuration variables or the phase space variables. In previous sections, we have constructed the Dirac brackets by eliminating the second class constraints, hence, in order to obtain these results by means of the FJ method we will use the configuration space as symplectic variables [16]. To this aim, we choose from the symplectic Lagrangian (35) the symplectic variables $\xi^{(0)a}(x) = \{e_a^i, e_0^i, A_a^i, A_0^i\}$, and the components of the symplectic 1-forms are $a^{(0)}_a(x) = \{\Omega\epsilon^{0ab}e_{bi}, 0, 2\epsilon^{0ab}e_{bi} + \beta\epsilon^{0ab}A_{bi}, 0\}$. Hence, by using our set of symplectic variables, the symplectic matrix (38) takes the form

$$f_{ab}^{(0)}(x, y) = \begin{pmatrix} -2\Omega\epsilon^{0ab}\delta_{ij} & 0 & -2\epsilon^{0ab}\delta_{ij} & 0 \\ 0 & 0 & 0 & 0 \\ -2\epsilon^{0ab}\delta_{ij} & 0 & -2\beta\epsilon^{0ab}\delta_{ij} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x - y). \quad (39)$$

The symplectic matrix $f_{ab}^{(0)}$ is of dimension $[18 \times 18]$ and it is a singular matrix. In fact, in the FJ method this means that there are constraints present. In order to obtain these constraints, we calculate the zero modes of the symplectic

matrix; the modes are given by $(v_a^{(0)})_1^T = (0, v^{e_0^i}, 0, 0)$ and $(v_a^{(0)})_2^T = (0, 0, 0, v^{A_0^i})$, where $v^{e_0^i}$ and $v^{A_0^i}$ are arbitrary functions. In this manner, by using the zero modes and the symplectic potential $V^{(0)}$ we obtain

$$\begin{aligned} \Omega_i^{(0)} &= \int d^2x (v^{(0)})_a^T(x) \frac{\delta}{\delta \xi^{(0)a}(x)} \int d^2y V^{(0)}(\xi) \\ &= - \int d^2x v^{e_0^i}(x) 2\epsilon^{0ab}\delta_{ij} \\ &\quad \times \left[\left(F_{ab}^j + s\frac{|\Lambda|}{2}\epsilon^{jkl}e_a^ke_b^l \right) + \Omega D_a e^j_b \right] \\ &\rightarrow -2\epsilon^{0ab}\delta_{ij} \left[\left(F_{ab}^j + s\frac{|\Lambda|}{2}\epsilon^{jkl}e_a^ke_b^l \right) + \Omega D_a e^j_b \right] = 0, \end{aligned} \quad (40)$$

$$\begin{aligned} \beta_i^{(0)} &= \int d^2x (v^{(0)})_a^T(x) \frac{\delta}{\delta \xi^{(0)a}(x)} \int d^2y V^{(0)}(\xi) \\ &= - \int d^2x v^{A_0^i}(x) 2\epsilon^{0ab}\delta_{ij} \\ &\quad \times \left[\beta \left(F_{ab}^j + s\frac{|\Lambda|}{2}\epsilon^{jkl}e_a^ke_b^l \right) + D_a e^j_b \right] \\ &\rightarrow -2\epsilon^{0ab}\delta_{ij} \left[\beta \left(F_{ab}^j + s\frac{|\Lambda|}{2}\epsilon^{jkl}e_a^ke_b^l \right) + D_a e^j_b \right] = 0. \end{aligned} \quad (41)$$

thus we identify the following constraints:

$$\Omega_i^{(0)} = 2\epsilon^{0ab}\delta_{ij} \left[\left(F_{ab}^j + s\frac{|\Lambda|}{2}\epsilon^{jkl}e_a^ke_b^l \right) + \Omega D_a e^j_b \right] = 0, \quad (42)$$

$$\beta_i^{(0)} = 2\epsilon^{0ab}\delta_{ij} \left[\beta \left(F_{ab}^j + s\frac{|\Lambda|}{2}\epsilon^{jkl}e_a^ke_b^l \right) + D_a e^j_b \right] = 0, \quad (43)$$

these constraints are the secondary constraints found by means of Dirac's method in the previous sections. In order to observe if there are more constraints, we calculate [18–21]

$$f_{cb}^{(1)}\dot{\xi}^b = Z_c(\xi), \quad (44)$$

where

$$Z_c(\xi) = \begin{pmatrix} \frac{\partial V^{(0)}(\xi)}{\partial \xi^a} \\ 0 \\ 0 \end{pmatrix} \quad (45)$$

and

$$f_{cb}^{(1)} = \begin{pmatrix} f_{ab}^{(0)} \\ \frac{\partial \Omega^{(0)}}{\partial \xi^b} \\ \frac{\partial \beta^{(0)}}{\partial \xi^b} \end{pmatrix} = \begin{pmatrix} -2\Omega\epsilon^{0ab}\delta_{ij} & 0 & -2\epsilon^{0ab}\delta_{ij} & 0 \\ 0 & 0 & 0 & 0 \\ -2\epsilon^{0ab}\delta_{ij} & 0 & -2\beta\epsilon^{0ab}\delta_{ij} & 0 \\ 0 & 0 & 0 & 0 \\ 2\epsilon^{0ab}(\Omega\delta_{ij}\partial_a - \epsilon_{ijk}(\Omega A_a^k + s | \Lambda | e_a^k)) & 0 & 2\epsilon^{0ab}(\delta_{ij}\partial_a - \epsilon_{ijk}(A_a^k + \Omega e_a^k)) & 0 \\ 2\epsilon^{0ab}(\delta_{ij}\partial_a - \epsilon_{ijk}(A_a^k + \Omega e_a^k)) & 0 & 2\epsilon^{0ab}(\beta\delta_{ij}\partial_a - \epsilon_{ijk}(\beta A_a^k + e_a^k)) & 0 \end{pmatrix} \delta^2(x-y). \quad (46)$$

We can observe that the matrix (A9) is not a square matrix as expected, however, it has linearly independent modes

ables and 1-forms, we can calculate the following symplectic matrix:

$$f_{ab}^{(1)}(x, y) = \begin{pmatrix} -2\delta_{ij} & -2(\Omega\delta_{ij}\partial_b + \epsilon_{ijk}(\Omega A_b^k + s | \Lambda | e_b^k)) & -2\delta_{ij} & -2(\delta_{ij}\partial_b + \epsilon_{ijk}(A_b^k + \Omega e_b^k)) \\ -2(\Omega\delta_{ij}\partial_a - \epsilon_{ijk}(\Omega A_a^k + s | \Lambda | e_a^k)) & 0 & -2(\Omega\delta_{ij}\partial_a - \epsilon_{ijk}(A_a^k + \Omega e_a^k)) & 0 \\ -2\delta_{ij} & -2(\delta_{ij}\partial_b + \epsilon_{ijk}(A_b^k + \Omega e_b^k)) & -2\beta\delta_{ij} & -2(\beta\delta_{ij}\partial_b + \epsilon_{ijk}(\beta A_b^k + e_b^k)) \\ -2(\delta_{ij}\partial_a - \epsilon_{ijk}(A_a^k + \Omega e_a^k)) & 0 & -2(\beta\delta_{ij}\partial_a - \epsilon_{ijk}(\beta A_a^k + e_a^k)) & 0 \end{pmatrix} \times \epsilon^{0ab}\delta^2(x-y) \quad (49)$$

given by $(v^{(1)})_1^T = (\delta_j^i \partial_a v^\lambda + \epsilon_{kj}^i A_a^k v^\lambda, \delta_j^i v^{e_0}, -s | \Lambda | \epsilon^i_{jk} e_a^k, 0, \delta_j^i v^\lambda, 0)$ and $(v^{(1)})_2^T = (-\epsilon^i_{jk} e_a^k, 0, \delta_j^i \partial_a v^\beta + \epsilon^i_{kj} A_a^k v^\beta, \delta_j^i v^{A_0}, 0, \delta_j^i v^\beta)$; these modes are used in order to obtain further constraints. In fact, by calculating the following contraction [18–21]:

$$(v^{(1)})_c^T Z_c = 0, \quad (47)$$

where $c = 1, 2$, we find that (47) is an identity, thus in the FJ formalism there are not more constraints for the theory under study.

Now, we will construct a new symplectic Lagrangian with the information of the constraints obtained in (42) and (43). In order to archive this aim, we introduce $e_0^i = \dot{\lambda}^i$ and $A_0^i = \dot{\theta}^i$, as Lagrange multipliers associated to those constraints obtaining the following symplectic Lagrangian:

$$\mathcal{L}^{(1)} = 2\epsilon^{0ab}\delta_{ij}e_b^i\dot{A}_a^j + \beta\epsilon^{0ab}\delta_{ij}A_b^i\dot{A}_a^j + \Omega\epsilon^{0ab}\delta_{ij}e_b^i\dot{e}_a^j + \Omega_i^{(0)}\dot{\lambda}^i + \beta_i^{(0)}\dot{\theta}^i - V^{(1)}, \quad (48)$$

where $V^{(1)} = V^{(0)}|_{\Omega_i^{(0)}=0, \beta_i^{(0)}=0} = 0$, the symplectic potential vanishes reflecting the general covariance of the theory. In this manner, from (48) we identify the following new symplectic variables: $\xi^{(1)a}(x) = \{e_a^i, \lambda^i, A_a^i, \theta^i\}$ and the new symplectic 1-forms $a^{(0)}_a(x) = \{\Omega\epsilon^{0ab}e_{bi}, \Omega_i^{(0)} + \phi_i, 0, 2\epsilon^{0ab}e_{bi} + \beta\epsilon^{0ab}A_{bi}, \beta_i^{(0)} + \alpha_i, 0\}$. Furthermore, by using these symplectic variables we find that the symplectic matrix is given by

The symplectic matrix $f_{ab}^{(1)}$ represents a $[18 \times 18]$ singular matrix. However, we have commented that there are not more constraints; the noninvertibility of (49) means that the theory has a gauge symmetry. In order to invert the symplectic matrix we choose the following gauge fixing as constraints:

$$A_0^i(x) = 0, \\ e_0^i(x) = 0,$$

then we introduce the Lagrange multipliers ϕ_i and α_i associated with the above gauge fixing for constructing a new symplectic Lagrangian. The symplectic Lagrangian is given by

$$\mathcal{L}^{(2)} = 2\epsilon^{0ab}\delta_{ij}e_b^i\dot{A}_a^j + \beta\epsilon^{0ab}\delta_{ij}A_b^i\dot{A}_a^j + \Omega\epsilon^{0ab}\delta_{ij}e_b^i\dot{e}_a^j + (\Omega_i^{(0)} + \phi_i)\dot{\lambda}^i + (\beta_i^{(0)} + \alpha_i)\dot{\theta}^i, \quad (50)$$

thus we identify the following set of symplectic variables: $\xi^{(2)a}(x) = \{e_a^i, \lambda^i, \phi_i, A_a^i, \theta^i, \alpha_i\}$ and the symplectic 1-forms $a^{(0)}_a(x) = \{\Omega\epsilon^{0ab}e_{bi}, \Omega_i^{(0)} + \phi_i, 0, 2\epsilon^{0ab}e_{bi} + \beta\epsilon^{0ab}A_{bi}, \beta_i^{(0)} + \alpha_i, 0\}$. Furthermore, by using these symplectic variables we find that the symplectic matrix is given by

$$f_{ab}^{(2)}(x, y) = \begin{pmatrix} -2\Omega\delta_{ij} & -2(\Omega\delta_{ij}\partial_b + \epsilon_{ijk}(\Omega A_b^k + s|\Lambda|e_b^k)) & 0 & -2\delta_{ij} & -2(\delta_{ij}\partial_b + \epsilon_{ijk}(A_b^k + \Omega e_b^k)) & 0 \\ -2(\Omega\delta_{ij}\partial_a - \epsilon_{ijk}(\Omega A_a^k + s|\Lambda|e_a^k)) & 0 & -\delta_j^i & -2(\Omega\delta_{ij}\partial_a - \epsilon_{ijk}(A_a^k + \Omega e_a^k)) & 0 & 0 \\ 0 & \delta_j^i & 0 & 0 & 0 & 0 \\ -2\delta_{ij} & -2(\delta_{ij}\partial_b + \epsilon_{ijk}(A_b^k + \Omega e_b^k)) & 0 & -2\beta\delta_{ij} & -2(\beta\delta_{ij}\partial_b + \epsilon_{ijk}(\beta A_b^k + e_b^k)) & 0 \\ -2(\delta_{ij}\partial_a - \epsilon_{ijk}(A_a^k + \Omega e_a^k)) & 0 & 0 & -2(\beta\delta_{ij}\partial_a - \epsilon_{ijk}(\beta A_a^k + e_a^k)) & 0 & -\delta_j^i \\ 0 & 0 & 0 & 0 & \delta_j^i & 0 \end{pmatrix} \times \epsilon^{0ab}\delta^2(x-y). \quad (51)$$

The symplectic matrix $f_{ab}^{(2)}$ represents a $[24 \times 24]$ non-singular matrix, hence, it is a symplectic tensor. After a long calculation, the inverse is given by

the generalized FJ brackets (54) take the form

$$\{e^i_a(x), \pi^b_j(y)\}_{FD} = \frac{s}{2(s-\gamma^2)}\delta^b_a\delta^i_j\delta^2(x-y),$$

$$[f_{ab}^{(2)}(x, y)]^{-1} = \begin{pmatrix} \frac{\gamma}{2\sqrt{|\Lambda|}(s-\gamma^2)}\epsilon_{0ab}\delta^{ij} & 0 & \delta_j^i\partial_a - \epsilon^i_{jk}A_a^k & -\frac{\gamma^2}{2(s-\gamma^2)}\epsilon_{0ab}\delta^{ij} & 0 & -\epsilon^i_{jk}e_a^k \\ 0 & 0 & \delta_j^i & 0 & 0 & 0 \\ \delta_j^i\partial_b - \epsilon^j_{ik}A_b^k & -\delta_j^i & 0 & -s|\Lambda| & \epsilon^j_{ik}e_b^k & 0 \\ -\frac{\gamma^2}{2(s-\gamma^2)}\epsilon_{0ab}\delta^{ij} & 0 & -s|\Lambda| & \epsilon^i_{jk}e_a^k & \frac{s\gamma\sqrt{|\Lambda|}}{2(s-\gamma^2)}\epsilon_{0ab}\delta^{ij} & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_j^i \\ -\epsilon^j_{ik}e_b^k & 0 & 0 & \delta_j^i\partial_b - \epsilon^j_{ik}A_b^k & -\delta_j^i & 0 \end{pmatrix} \delta^2(x-y). \quad (52)$$

Therefore, by means of

$$\{\xi_i^{(2)}(x), \xi_j^{(2)}(y)\}_{FD} = [f_{ij}^{(2)}(x, y)]^{-1} \quad (53)$$

from (52) it is possible to identify the following FJ generalized brackets:

$$\begin{aligned} \{e^i_a(x), e^j_b(y)\}_{FD} &= \frac{\gamma}{2\sqrt{|\Lambda|}(s-\gamma^2)}\epsilon_{0ab}\delta^{ij}\delta^2(x-y), \\ \{A^i_a(x), e^j_b(y)\}_{FD} &= -\frac{\gamma^2}{2(s-\gamma^2)}\epsilon_{0ab}\delta^{ij}\delta^2(x-y), \\ \{A^i_a(x), A^j_b(y)\}_{FD} &= \frac{s\sqrt{|\Lambda|}\gamma}{2(s-\gamma^2)}\epsilon_{0ab}\delta^{ij}\delta^2(x-y), \\ \{e^i_a(x), \phi_j(y)\}_{FD} &= (\delta_j^i\partial_a - \epsilon^i_{jk}A_a^k)\delta^2(x-y), \\ \{A^i_a(x), \alpha_j(y)\}_{FD} &= (\delta_j^i\partial_a - \epsilon^i_{jk}A_a^k)\delta^2(x-y), \\ \{e^i_a(x), \alpha_j(y)\}_{FD} &= -\epsilon^i_{jk}e_a^k, \\ \{A^i_a(x), \phi_j(y)\}_{FD} &= -s|\Lambda|\epsilon^i_{jk}e_a^k, \\ \{\lambda^i(x), \phi_j(y)\}_{FD} &= \delta_j^i\delta^2(x-y), \\ \{\theta^i(x), \alpha_j(y)\}_{FD} &= \delta_j^i\delta^2(x-y). \end{aligned} \quad (54)$$

It is important to comment that the generalized FJ brackets coincide with those obtained by means of the Dirac method reported in the previous section. In fact, if we perform a redefinition of the fields introducing the momenta given by

$$\begin{aligned} \pi_i^a &= s\frac{\sqrt{|\Lambda|}}{\gamma}\epsilon^{0ab}\delta_{ij}e_b^j, \\ \Pi_i^a &= 2\epsilon^{0ab}\delta_{ij}\left(e_b^j + \frac{1}{2\gamma\sqrt{|\Lambda|}}A_b^j\right), \end{aligned} \quad (55)$$

$$\{e^i_a(x), e^j_b(y)\}_{FD} = \frac{1}{2\sqrt{|\Lambda|}}\frac{\gamma}{s-\gamma^2}\delta^{ij}\epsilon_{0ab}\delta^2(x-y),$$

$$\{\pi^a_i(x), \pi^b_j(y)\}_{FD} = \frac{s^2\sqrt{|\Lambda|}}{2\gamma s-\gamma^2}\delta_{ij}\epsilon^{0ab}\delta^2(x-y),$$

$$\{A^i_a(x), \Pi^b_j(y)\}_{FD} = \frac{s-2\gamma^2}{2(s-\gamma^2)}\delta^b_a\delta^i_j\delta^2(x-y),$$

$$\{A^i_a(x), A^j_b(y)\}_{FD} = \frac{s\sqrt{|\Lambda|}}{2}\frac{\gamma}{s-\gamma^2}\delta^{ij}\epsilon_{0ab}\delta^2(x-y),$$

$$\{\Pi^a_i(x), \Pi^b_j(y)\}_{FD} = \frac{s}{2\gamma\sqrt{|\Lambda|}}\frac{1}{s-\gamma^2}\delta_{ij}\epsilon^{0ab}\delta^2(x-y),$$

$$\{A^i_a(x), e^j_b(y)\}_{FD} = \frac{1}{2}\frac{\gamma^2}{\gamma^2-s}\delta^{ij}\epsilon_{0ab}\delta^2(x-y),$$

$$\{e^i_a(x), \Pi^b_j(y)\}_{FD} = \frac{1}{2\sqrt{|\Lambda|}}\frac{\gamma}{s-\gamma^2}\delta^b_a\delta^i_j\delta^2(x-y),$$

$$\{A^i_a(x), \pi^b_j(y)\}_{FD} = -\frac{s\sqrt{|\Lambda|}}{2}\frac{\gamma}{s-\gamma^2}\delta^b_a\delta^i_j\delta^2(x-y),$$

$$\{\pi^a_i(x), \Pi^b_j(y)\}_{FD} = \frac{1}{2}\frac{s}{s-\gamma^2}\delta_{ij}\epsilon^{0ab}\delta^2(x-y), \quad (56)$$

where we can observe that they coincide with the full Dirac brackets found in (27).

Furthermore, we have commented above that in the FJ approach it is not necessary to classify the constraints in first class and second class, but all the constraints are at the same footing. Thus, we can carry out the counting of physical degrees of freedom in the following form. There are 12 dynamical variables (e_a^i, A_a^i) and 12 constraints ($\Omega_i^{(0)}, \beta_i^{(0)}, A_0^i, e_0^i$), therefore, the theory lacks physical degrees of freedom.

We finish this section by calculating the gauge transformations of the theory, by determining the modes of the matrix

(49)

$$(w^{(1)})_1^T = (-\delta_j^i \partial_a \delta^2(x-y) - \epsilon^i{}_{jk} A_a^k \delta^2(x-y), \delta_j^i \delta^2(x-y), -s | \Lambda | \epsilon^i{}_{jk} e_a^k \delta^2(x-y), 0), \quad (57)$$

$$(w^{(1)})_2^T = (-\epsilon^i{}_{jk} e_a^k \delta^2(x-y), 0, -\delta_j^i \partial_a \delta^2(x-y) - \epsilon^i{}_{jk} A_a^k \delta^2(x-y), \delta_j^i \delta^2(x-y)). \quad (58)$$

In agreement with the FJ symplectic formalism, the zero modes $(w^{(1)})_1^T$ and $(w^{(1)})_2^T$ are the generators of infinitesimal gauge transformations of the action (35) and are given by

$$\begin{aligned} \delta e_a^i(x) &= -D_a \varepsilon^i + \epsilon^i{}_{jk} e_a^k \tau^j, \\ \delta e_0^i(x) &= \partial_0 \varepsilon^i, \\ \delta A_a^i(x) &= -D_a \tau^i + s | \Lambda | \epsilon^i{}_{jk} e_a^k \varepsilon^j, \\ \delta A_0^i(x) &= \partial_0 \tau^i. \end{aligned}$$

In fact, the mode (57) is the generator of translations and the mode (58) is the generator of rotations. In this manner, by using the FJ symplectic framework we have reproduced the Λ -deformed ISO(3) gauge transformations reported by means of Dirac's method. Finally, in order to complete our work, in Appendix A the Dirac analysis for the Abelian case is developed. In that appendix, we report the full constraints program and we have constructed the Dirac brackets by fixing the gauge, then in Appendix B we reproduce all Dirac's results in a more economical way by using FJ framework.

5 Conclusions and prospects

In this paper a pure Dirac formalism and a full FJ approach for the BL model have been performed. With respect to Dirac's method, the complete structure of the constraints was found and the algebra between the constraints defined on the full phase space was developed. Furthermore, we observed that the internal group is SU(2) (or SO(3)), however, the fundamental gauge symmetry correspond to an Λ -deformed ISO(3) transformations. In addition, we have eliminated the second class constraints by introducing the Dirac brackets, and we observed that the Dirac brackets depend on the γ parameter and this fact makes the BL model classically different from Palatini's theory. On the other hand, we have reproduced all relevant Dirac results by performing within the FJ framework. In fact, we have found the FJ constraints and we have showed that there are less constraints than those found with Dirac's method. Moreover, we have showed that the generalized FJ brackets and the Dirac ones coincide. In this manner, we have reproduced all relevant Dirac results by using the FJ method, in particular we can see that the FJ method is more economical than Dirac's method. In addition, we proved the

equivalence between the BL model and Chern–Simons theory; from a pure Dirac analysis of the Chern–Simons theory, all relevant Dirac results obtained using the connection and the frame fields as dynamical variables were reproduced. Finally, we comment that our FJ analysis is generic and we can reproduce all the results reported in [4] where Dirac's method was employed. In fact, it is straightforward to observe that using the axial gauge in the matrix (49), the BL theory can be written in terms of a SO(3) Ashtekar connection. This result is obtained in more economical way by using the FJ framework. Hence, our results extend those results found in the literature.

Acknowledgements This work was supported by CONACyT under Grant No. CB-2014-01/ 240781. We would like to thank R. Cartas-Fuentevilla for discussion of the subject and reading of the manuscript.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. Funded by SCOAP³.

Appendix A: Canonical analysis of the BL Abelian theory

In this appendix, we shall resume the canonical analysis of the Abelian version of BL action given by

$$\begin{aligned} S^{\text{Abelian}}[A, e] &= \int 2e^i \wedge F_i[A] \\ &+ \frac{1}{\sqrt{|\Lambda|}} \int A^i \wedge dA_i + s\sqrt{|\Lambda|} e^i \wedge de_i \\ &= \int \epsilon^{0ab} \left[\left(\frac{A_0^i}{\gamma\sqrt{|\Lambda|}} + e_0^i \right) F_{abi} + \left(\frac{A_b^i}{\gamma\sqrt{|\Lambda|}} + e_b^i \right) \dot{A}_{ai} \right. \\ &\quad \left. + \left(\frac{s\sqrt{|\Lambda|}}{\gamma} e_0^i + A_0^i \right) T_{abi} + \left(\frac{s\sqrt{|\Lambda|}}{\gamma} e_b^i + A_b^i \right) \dot{e}_{ai} \right], \end{aligned} \quad (A1)$$

where A_μ^a and e_μ^a are a set of three $U(1)$ vector fields, $F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i$, $T_{ab} = \partial_a e_b^i - \partial_b e_a^i$. By introducing the canonical momenta defined by

$$\pi_i^\lambda := \frac{\partial \mathcal{L}}{\partial \dot{A}_\lambda^i} = \epsilon^{0\lambda\rho} \left[\frac{1}{\sqrt{|\Lambda|}\gamma} A_{\rho i} + e_{\rho i} \right], \quad (A2)$$

$$p_i^\lambda := \frac{\partial \mathcal{L}}{\partial \dot{e}_\lambda^i} = \epsilon^{0\lambda\rho} \left[A_{\rho i} + \frac{s\sqrt{|\Lambda|}}{\gamma} e_{\rho i} \right]. \quad (A3)$$

and performing the canonical analysis, we obtain the following results. We find four first class constraints:

$$\begin{aligned}\gamma^1 &= p_i^0 \approx 0, \\ \gamma^2 &= 2\partial_a p_i^a - \partial_a \phi_i^a \approx 0, \\ \gamma^3 &= \pi_i^0 \approx 0, \\ \gamma^4 &= 2\partial_a \pi_i^a - \partial_a \Phi_i^a \approx 0,\end{aligned}\quad (\text{A4})$$

and four second class constraints:

$$\begin{aligned}\chi_{1i}^a &= p_i^a - \epsilon^{0ab} [A_{bi} + \Omega e_{bi}] \approx 0, \\ \chi_{2i}^a &= \pi_i^a - \epsilon^{0ab} [\beta A_{bi} + e_{bi}] \approx 0.\end{aligned}\quad (\text{A5})$$

Now, the algebra between the second class constraints is given by

$$\begin{aligned}\{\chi_{1i}^a(x), \chi_{1j}^b(y)\} &= -2\Omega \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\ \{\chi_{1i}^a(x), \chi_{2j}^b(y)\} &= -2\epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\ \{\chi_{2i}^a(x), \chi_{2j}^b(y)\} &= -2\beta \epsilon^{0ab} \delta_{ij} \delta^2(x-y).\end{aligned}\quad (\text{A6})$$

Hence, the Dirac brackets will be constructed by eliminating the second class constraints. For archive this aim, we write in matrix form the Poisson brackets among second class constraints, namely

$$C^{ab} = \begin{pmatrix} -2\Omega & -2 \\ -2 & -2\beta \end{pmatrix} \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \quad (\text{A7})$$

and we calculate its inverse given by

$$[C^{ab}]^{-1} = \frac{\gamma^2}{2(s-\gamma^2)} \begin{pmatrix} \beta & -1 \\ -1 & \Omega \end{pmatrix} \epsilon_{0ab} \delta^{ij} \delta^2(x-y). \quad (\text{A8})$$

Hence, by using the matrix (A8) we obtain the following Dirac brackets of the theory:

$$\begin{aligned}\{e_a^i(x), e_b^j(y)\}_D &= \frac{\gamma}{2\sqrt{|\Lambda|}(s-\gamma^2)} \epsilon_{0ab} \delta^{ij} \delta^2(x-y), \\ \{A_a^i(x), e_b^j(y)\}_D &= -\frac{\gamma^2}{2(s-\gamma^2)} \epsilon_{0ab} \delta^{ij} \delta^2(x-y), \\ \{e_a^i(x), p_b^j(y)\}_D &= \frac{1}{2} \delta_a^b \delta_j^i \delta^2(x-y),\end{aligned}$$

$$\begin{aligned}\{p_i^a(x), p_j^b(y)\}_D &= \frac{s\sqrt{|\Lambda|}}{2\gamma} \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\ \{A_a^i(x), A_b^j(y)\}_D &= \frac{s\sqrt{|\Lambda|}\gamma}{2(s-\gamma^2)} \epsilon_{0ab} \delta^{ij} \delta^2(x-y), \\ \{A_a^i(x), \pi_j^b(y)\}_D &= \frac{1}{2} \delta_a^b \delta_j^i \delta^2(x-y), \\ \{\pi_i^a(x), \pi_j^b(y)\}_D &= \frac{\beta}{2} \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\ \{p_i^a(x), \pi_j^b(y)\}_D &= \frac{1}{2} \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\ \{e_0^i(x), p_j^0(y)\}_D &= \delta_j^i \delta^2(x-y), \\ \{A_0^i(x), \pi_j^0(y)\}_D &= \delta_j^i \delta^2(x-y),\end{aligned}\quad (\text{A9})$$

hence, the Dirac brackets for Abelian and non-Abelian theory coincide. In the following lines, we will construct the Dirac brackets by fixing the gauge, then we will reproduce these results by means FJ framework.

A.1: Dirac's brackets by fixing the gauge

In order to construct the Dirac brackets it is necessary to convert the first class constraints into second class by fixing the gauge, and we will work with the temporal and Coulomb gauge

$$\begin{aligned}\Omega_1 &= e_0^i \approx 0, \\ \Omega_2 &= \partial^a e_a^i \approx 0, \\ \Omega_3 &= A_0^i \approx 0, \\ \Omega_4 &= \partial^a A_a^i \approx 0, \\ \Omega_5 &= p_i^0 \approx 0, \\ \Omega_6 &= 2\partial_a p_i^a - \partial_a \chi_{1i}^a \approx 0, \\ \Omega_7 &= \pi_i^0 \approx 0, \\ \Omega_8 &= 2\partial_a \pi_i^a - \partial_a \chi_{2i}^a \approx 0, \\ \Omega_9 &= p_i^a - \epsilon^{0ab} [A_{bi} + \Omega e_{bi}] \approx 0, \\ \Omega_{10} &= \pi_i^a - \epsilon^{0ab} [\beta A_{bi} + e_{bi}] \approx 0.\end{aligned}\quad (\text{A10})$$

in this manner, the matrix whose entries are the Poisson brackets between the constraints, namely G , is given by

$$G(x, y) = \begin{pmatrix} -2\Omega \epsilon^{0ab} \delta_{ij} & 0 & 0 & 0 & \delta_i^j \partial_x^a & -2\epsilon^{0ab} \delta_{ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta_i^j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_j^i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_i^j \nabla_x^2 & 0 & 0 & 0 & 0 & 0 \\ \delta_j^i \partial_x^b & 0 & 0 & -\delta_j^i \nabla_x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\epsilon^{0ab} \delta_{ij} & 0 & 0 & 0 & 0 & -2\beta \epsilon^{0ab} \delta_{ij} & 0 & 0 & 0 & \delta_i^j \partial_x^a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_i^j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_j^i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_i^j \nabla_x^2 \\ 0 & 0 & 0 & 0 & 0 & \delta_j^i \partial_x^b & 0 & 0 & -\delta_j^i \nabla_x^2 & 0 \end{pmatrix} \times \delta^2(x-y). \quad (\text{A11})$$

Hence, the inverse of G becomes

$$[G(x, y)]^{-1} = \begin{pmatrix} \epsilon_{0ab}\delta^{ij}\frac{\beta}{2\theta} & 0 & 0 & \epsilon_{0ab}\delta^i_j\frac{\beta\partial_x^b}{2\theta\nabla_x^2} & 0 & -\epsilon_{0ab}\delta^{ij}\frac{1}{2\theta} & 0 & 0 & -\epsilon_{0ab}\delta^i_j\frac{\partial_x^b}{2\theta\nabla_x^2} & 0 \\ 0 & 0 & \delta^j_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\delta^i_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon_{0ba}\delta^j_i\frac{\beta\partial_x^a}{2\theta\nabla_x^2} & 0 & 0 & 0 & -\frac{\delta^j_i}{\nabla_x^2} & -\epsilon_{0ba}\delta^j_i\frac{\partial_x^a}{2\theta\nabla_x^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\delta^j_i}{\nabla_x^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\epsilon_{0ab}\delta^{ij}\frac{1}{2\theta} & 0 & 0 & -\epsilon_{0ab}\delta^i_j\frac{\partial_x^b}{2\theta\nabla_x^2} & 0 & \epsilon_{0ab}\delta^{ij}\frac{\Omega}{2\theta} & 0 & 0 & \epsilon_{0ab}\delta^i_j\frac{\Omega\partial_x^b}{2\theta\nabla_x^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta^j_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\delta^i_j & 0 & 0 & 0 \\ -\epsilon_{0ba}\delta^j_i\frac{\partial_x^a}{2\theta\nabla_x^2} & 0 & 0 & 0 & 0 & \epsilon_{0ba}\delta^j_i\frac{\Omega\partial_x^a}{2\theta\nabla_x^2} & 0 & 0 & 0 & -\frac{\delta^j_i}{\nabla_x^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\delta^j_i}{\nabla_x^2} & 0 \end{pmatrix} \delta^2(x-y),$$

here we have defined $\theta = \beta\Omega - 1$. Finally, we use the inverse matrix G^{-1} and we find the following Dirac brackets:

$$\begin{aligned} \{e_a^i(x), p_j^b(y)\}_D &= \delta^i_j \left(\delta^b_a - \frac{\partial_a \partial^b}{\nabla^2} \right) \delta(x-y), \\ \{e_a^i(x), e_b^j(y)\}_D &= 0, \\ \{p_i^a(x), p_j^b(y)\}_D &= 0, \\ \{A_a^i(x), \pi_j^b(y)\}_D &= \delta^i_j \left(\delta^b_a - \frac{\partial_a \partial^b}{\nabla^2} \right) \delta(x-y), \\ \{A_a^i(x), A_b^j(y)\}_D &= 0, \\ \{\pi_i^a(x), \pi_j^b(y)\}_D &= 0. \end{aligned} \quad (A12)$$

In the following section, we will reproduce these results by means of FJ formalism.

Appendix B: Faddeev–Jackiw analysis of BL Abelian theory using the phase space

Now, we shall study the action (A1) by means FJ formalism, we shall work with the phase space as symplectic variables [18]. Hence, the Lagrangian density can be written as

$$\mathcal{L} = \epsilon^{0ab} \left[\left(\frac{A_b^i}{\gamma\sqrt{|\Lambda|}} + e_b^i \right) \dot{A}_{ai} + \left(\frac{s\sqrt{|\Lambda|}}{\gamma} e_b^i + A_b^i \right) \dot{e}_{ai} \right] - V^{(0)}, \quad (B1)$$

where the symplectic potential is given by

$$V^{(0)} = -\epsilon^{0ab} \left[\left(\frac{A_0^i}{\gamma\sqrt{|\Lambda|}} + e_0^i \right) F_{abi} + \left(\frac{s\sqrt{|\Lambda|}}{\gamma} e_0^i + A_0^i \right) T_{abi} \right]. \quad (B2)$$

By introducing the canonical momenta

$$\begin{aligned} p_i^a &= \epsilon^{0ab} (A_{bi} + \Omega e_{bi}), \\ \pi_i^a &= \epsilon^{0ab} (e_{bi} + \beta A_{bi}), \end{aligned} \quad (B3)$$

and writing the fields in the following form:

$$\begin{aligned} \epsilon^{0ab} e_{bi} &= \frac{s}{s-\gamma^2} (\beta p_j^a - \pi_j^a), \\ \epsilon^{0ab} A_{bi} &= \frac{s}{s-\gamma^2} (\Omega \pi_j^a - p_j^a), \end{aligned}$$

the first-order symplectic Lagrangian density takes the form

$$\mathcal{L}^{(0)} = \pi_i^a \dot{A}_a^i + p_i^a \dot{e}_a^i - V^{(0)}, \quad (B4)$$

where the symplectic potential $V^{(0)}$ is given by

$$V^{(0)} = -2A_0^i \partial_a \pi_i^a - 2e_0^i \partial_a p_i^a. \quad (B5)$$

In this manner, we can identify the corresponding symplectic variables $\xi^{(0)a}(x) = \{e_a^i, p_i^a, e_0^i, A_a^i, \pi_i^a, A_0^i\}$ and the symplectic 1-form $a^{(0)}_a(x) = \{p_i^a, 0, 0, \pi_i^a, 0, 0\}$; thus, by using the symplectic variables, the symplectic matrix takes the form

$$f_{ab}^{(0)}(x, y) = \begin{pmatrix} 0 & -\delta^a_b \delta^j_i & 0 & 0 & 0 & 0 \\ \delta^b_a \delta^i_j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta^a_b \delta^j_i & 0 \\ 0 & 0 & 0 & \delta^a_b \delta^i_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x-y). \quad (B6)$$

This matrix is singular, this means that the theory has constraints. The zero modes of this matrix are given by $(v_a^{(0)})_1^T = (0, 0, 0, v_0^i, 0, 0, 0)$ and $(v_a^{(0)})_2^T = (0, 0, 0, 0, 0, 0, v^{A_0^i})$, where v_0^i and $v^{A_0^i}$ are arbitrary functions. Now, by using the zero

modes we can get the following constraints:

$$\begin{aligned} 0 &= \int d^2x (v^{(0)})_a^T(x) \frac{\delta}{\delta \xi^{(0)i}(x)} \int d^2y V^{(0)}(\xi) \\ &= \int d^2x v^{e_0^i}(x) [-2\partial_a p_i^a] \\ &\rightarrow \Omega_i^{(0)} := -2\partial_a p_i^a = 0, \end{aligned} \quad (\text{B7})$$

$$\begin{aligned} 0 &= \int d^2x (v^{(0)})_a^T(x) \frac{\delta}{\delta \xi^{(0)i}(x)} \int d^2y V^{(0)}(\xi) \\ &= \int d^2x v^{A_0^i}(x) [-2\partial_a \pi_i^a] \\ &\rightarrow \Theta_i^{(0)} := -2\partial_a \pi_i^a = 0. \end{aligned} \quad (\text{B8})$$

On the other hand, we will see if there are more constraints by calculating the following contraction [19]:

$$f_{cd}^{(1)} \xi^d = Z_c(\xi), \quad (\text{B9})$$

where

$$\begin{aligned} f_{cd}^{(1)} &= \begin{pmatrix} f_{ab}^{(0)} \\ \frac{\partial \Omega^{(0)}}{\partial \xi^a} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\delta^a_b \delta^j_i & 0 & 0 & 0 & 0 \\ \delta^b_a \delta^i_j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta^a_b \delta^j_i & 0 \\ 0 & 0 & 0 & \delta^b_a \delta^j_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\partial_b \delta^j_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\partial_b \delta^j_i & 0 \end{pmatrix} \delta^2(x-y), \end{aligned} \quad (\text{B10})$$

and

$$Z_c(\xi) = \begin{pmatrix} \frac{\partial V^{(0)}(\xi)}{\partial \xi^a} \\ 0 \end{pmatrix}. \quad (\text{B11})$$

Although the matrix (B10) is not a square matrix, it still has linearly independent modes given by $(v^{(1)})_1^T = (2\partial_a v^\lambda, 0, v^{e_0^i}, 0, 0, 0, v^\lambda, 0)$ and $(v^{(1)})_2^T = (0, 0, 0, 2\partial_a v^\alpha, 0, v^{A_0^i}, 0, v^\alpha)$. Multiplication of $(f_{cd}^{(1)})$ by $(v^{(1)})_c^T$ from the

left side gives zero. The contraction of these modes reads

$$(v^{(1)})_c^T Z_c|_{\Omega^{(0)}=0} = 0, \quad (\text{B12})$$

which is an identity, hence, there is not new constraints.

Furthermore, enforcing the constraints (B7) and (B8) in order to construct a new symplectic Lagrangian we use the Lagrangian multipliers (λ^i, ρ^i) :

$$\mathcal{L}^{(1)} = \pi_i^a \dot{A}_a^i + p_i^a \dot{e}_a^i + (2\partial_a p_i^a) \dot{\lambda}^i + (2\partial_a \pi_i^a) \dot{\rho}^i - V^{(1)}, \quad (\text{B13})$$

where $V^{(1)} = V^{(0)}|_{\partial_a \pi_i^a=0, \partial_a p_i^a=0} = 0$, is the symplectic potential. From (B13) we identify the following symplectic variables: $\xi^{(1)a}(x) = \{e_a^i, p_i^a, \lambda^i, A_a^i, \pi_i^a, \rho^i\}$, and the 1-forms $a^{(1)}_a(x) = \{p_i^a, 0, 2\partial_a p_i^a, \pi_i^a, 0, 2\partial_a \pi_i^a\}$, thus the corresponding symplectic matrix is given by

$$\begin{aligned} f_{ab}^{(1)}(x, y) &= \begin{pmatrix} 0 & -\delta^a_b \delta^j_i & 0 & 0 & 0 & 0 \\ \delta^b_a \delta^i_j & 0 & -2\delta^i_j \partial_a & 0 & 0 & 0 \\ 0 & -2\delta^j_i \partial_b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta^a_b \delta^j_i & 0 \\ 0 & 0 & 0 & \delta^b_a \delta^i_j & 0 & -2\delta^i_j \partial_a \\ 0 & 0 & 0 & 0 & -2\delta^j_i \partial_b & 0 \end{pmatrix} \\ &\delta^2(x-y), \end{aligned} \quad (\text{B14})$$

the matrix is still singular, but we have proved, however, that there are not new constraints. Therefore this system has a gauge symmetry. In order to obtain a symplectic tensor, we will fix the gauge, let us fixing the Coulomb gauge $\partial^a e_a^i = 0$, $\partial^a A_a^i = 0$ and we will introduce this information by constructing a new symplectic Lagrangian adding new Lagrange multiples, namely ϕ_i and θ_i , enforcing the gauge fixing, we obtain

$$\begin{aligned} \mathcal{L}^{(2)} &= \pi_i^a \dot{A}_a^i + p_i^a \dot{e}_a^i + (2\partial_a p_i^a) \dot{\lambda}^i \\ &\quad + (2\partial_a \pi_i^a) \dot{\rho}^i + (\partial^a e_a^i) \dot{\phi}_i + (\partial^a A_a^i) \dot{\theta}_i, \end{aligned} \quad (\text{B15})$$

now the symplectic variables are given by $\xi^{(2)}(x) = \{e_a^i, p_i^a, \lambda^i, \phi_i, A_a^i, \pi_i^a, \rho^i, \theta_i\}$ and the 1-forms $a^{(2)}_a(x) = \{p_i^a, 0, 2\partial_a p_i^a, \partial^a e_a^i, \pi_i^a, 0, 2\partial_a \pi_i^a, \partial^a A_a^i\}$. In this manner, the symplectic matrix takes the form

$$f_{ab}^{(2)}(x, y) = \begin{pmatrix} 0 & -\delta^a_b \delta^j_i & 0 & -\delta^j_i \partial_a & 0 & 0 & 0 & 0 \\ \delta^b_a \delta^i_j & 0 & -2\delta^i_j \partial_a & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\delta^j_i \partial_b & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta^i_j \partial_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta^a_b \delta^j_i & 0 & -\delta^j_i \partial_a \\ 0 & 0 & 0 & 0 & \delta^b_a \delta^i_j & 0 & -2\delta^i_j \partial_a & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\delta^j_i \partial_b & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta^i_j \partial_b & 0 & 0 & 0 \end{pmatrix} \delta^2(x-y), \quad (\text{B16})$$

where we can observe that $f_{ab}^{(2)}(x, y)$ is a symplectic tensor and therefore is invertible. The inverse matrix of $f_{ab}^{(2)}(x, y)$ is given by

$$[f_{ab}^{(2)}(x, y)]^{-1} = \begin{pmatrix} 0 & \delta^i_j(\delta^b_a - \frac{\partial_a \partial^b}{\nabla^2}) & 0 & -\delta^i_j \frac{\partial_a}{\nabla^2} & 0 & 0 & 0 & 0 \\ -\delta^j_i(\delta^a_b - \frac{\partial_b \partial^a}{\nabla^2}) & 0 & -\frac{1}{2}\delta^j_i \frac{\partial^a}{\nabla^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\delta^i_j \frac{\partial^b}{\nabla^2} & 0 & -\delta^i_j \frac{1}{2} \frac{1}{\nabla^2} & 0 & 0 & 0 & 0 \\ -\delta^j_i \frac{\partial_b}{\nabla^2} & 0 & \frac{1}{2}\delta^j_i \frac{1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta^i_j(\delta^b_a - \frac{\partial_a \partial^b}{\nabla^2}) & 0 & -\delta^i_j \frac{\partial_a}{\nabla^2} \\ 0 & 0 & 0 & 0 & -\delta^j_i(\delta^a_b - \frac{\partial_b \partial^a}{\nabla^2}) & 0 & -\frac{1}{2}\delta^j_i \frac{\partial^a}{\nabla^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\delta^i_j \frac{\partial^b}{\nabla^2} & 0 & -\frac{1}{2}\delta^i_j \frac{1}{\nabla^2} \\ 0 & 0 & 0 & 0 & -\delta^j_i \frac{\partial_b}{\nabla^2} & 0 & \frac{1}{2}\delta^j_i \frac{1}{\nabla^2} & 0 \end{pmatrix} \times \delta^2(x - y), \quad (B17)$$

where it is possible identify the FJ generalized brackets as

$$\{\xi_a^{(2)}(x), \xi_b^{(2)}(y)\}_{FD} = [f_{ab}^{(2)}(x, y)]^{-1}, \quad (B18)$$

thus we find the following brackets:

$$\begin{aligned} \{e_a^i(x), p_j^b(y)\}_{FD} &= [f_{12}^{(2)}(x, y)]^{-1} \\ &= \delta^i_j \left(\delta^b_a - \frac{\partial_a \partial^b}{\nabla^2} \right) \delta^2(x - y), \\ \{A_a^i(x), \pi_j^b(y)\}_{FD} &= [f_{56}^{(2)}(x, y)]^{-1} \\ &= \delta^i_j \left(\delta^b_a - \frac{\partial_a \partial^b}{\nabla^2} \right) \delta^2(x - y), \\ \{e_a^i(x), e_b^j(y)\}_{FD} &= [f_{11}^{(2)}(x, y)]^{-1} = 0, \\ \{A_a^i(x), A_b^j(y)\}_{FD} &= [f_{55}^{(2)}(x, y)]^{-1} = 0, \\ \{p_i^a(x), p_j^b(y)\}_{FD} &= [f_{22}^{(2)}(x, y)]^{-1} = 0, \\ \{\pi_i^a(x), \pi_j^b(y)\}_{FD} &= [f_{66}^{(2)}(x, y)]^{-1} = 0, \\ \{p_i^a(x), \lambda^j(y)\}_{FD} &= [f_{23}^{(2)}(x, y)]^{-1} = -\frac{1}{2}\delta_i^j \frac{\partial^a}{\nabla^2} \delta^2(x - y), \\ \{\pi_i^a(x), \rho^j(y)\}_{FD} &= [f_{67}^{(2)}(x, y)]^{-1} = -\frac{1}{2}\delta_i^j \frac{\partial^a}{\nabla^2} \delta^2(x - y), \\ \{\lambda^i(x), \phi_j(y)\}_{FD} &= [f_{34}^{(2)}(x, y)]^{-1} = -\frac{1}{2}\delta_j^i \frac{1}{\nabla^2} \delta^2(x - y), \\ \{\rho^i(x), \theta_j(y)\}_{FD} &= [f_{78}^{(2)}(x, y)]^{-1} = -\frac{1}{2}\delta_j^i \frac{1}{\nabla^2} \delta^2(x - y), \\ \{e_a^i(x), \phi_j(y)\}_{FD} &= [f_{14}^{(2)}(x, y)]^{-1} = -\delta_j^i \frac{\partial_a}{\nabla^2} \delta^2(x - y), \\ \{A_a^i(x), \theta_j(y)\}_{FD} &= [f_{58}^{(2)}(x, y)]^{-1} = -\delta_j^i \frac{\partial_a}{\nabla^2} \delta^2(x - y). \end{aligned} \quad (B19)$$

We can observe that the Dirac brackets given in (A12) and the FJ generalized brackets given in (B19) coincide.

References

1. A. Achucarro, P.K. Townsend, Phys. Lett. B **180**, 89 (1996)
2. E. Witten, Nucl. Phys. B **311**, 46 (1988)
3. H.J. Matschull, Class. Quantum Gravity **16**, 2599–2609 (1999)
4. R.M.S. Barbosa, C.P. Constantinidis, Z. Oporto, O. Piguet, Class. Quantum Gravity **29**, 155011 (2012)
5. V. Bonzom, E.R. Livine, Class. Quantum Gravity **25**, 195024 (2008)
6. S. Holst, Phys. Rev D. **53**, 5966–5969 (1996)
7. A. Perez, C. Rovelli, Phys. Rev. D **73**, 044013 (2006)
8. C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, 2004)
9. T. Thiemann, *Modern Canonical General Relativity* (Cambridge University Press, Cambridge, 2005)
10. A. Perez, Introduction to loop quantum gravity and spin foams. [arXiv:gr-qc/0409061](https://arxiv.org/abs/gr-qc/0409061)
11. A. Escalante, O. Rodríguez Tzompantzi, JHEP **05**, 073 (2014)
12. A. Escalante, J. Manuel-Cabrera, Ann. Phys. **343**, 27–39 (2014)
13. L.D. Faddeev, R. Jackiw, Phys. Rev. Lett. **60**, 1692 (1988)
14. D.M. Gitman, I.V. Tyutin, *Quantization of Fields with Constraints, Springer Series in Nuclear and Particle Physics* (Springer, Berlin, Germany, 1990)
15. A. Hanson, T. Regge, C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Roma, 1978)
16. A. Escalante, J. Manuel Cabrera, Ann. Phys. **361**, 585–604 (2015)
17. R. Basu, S.K. Paul, Class. Quantum Gravity **27**, 125003 (2010)
18. J. Barcelos-Neto, C. Wotzasek, Mod. Phys. Lett A **7**, 1737 (1992)
19. J. Barcelos-Neto, C. Wotzasek, Int. J. Mod. Phys. A **7**, 4981 (1992)
20. Y.C. Huang, J.L. Yang, Phys. Lett. B **662**, 438–441 (2008)
21. Y. Jin-Long, H. Yong-Chang, Chin. Phys. C (HEP and NP) **32**(10), 788 (2008)