

# Higher-order gravity in higher dimensions: geometrical origins of four-dimensional cosmology?

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**Abstract** Determining the cosmological field equations is still very much debated and led to a wide discussion around different theoretical proposals. A suitable conceptual scheme could be represented by gravity models that naturally generalize Einstein theory like higher-order gravity theories and higher-dimensional ones. Both of these two different approaches allow one to define, at the effective level, Einstein field equations equipped with source-like energy-momentum tensors of geometrical origin. In this paper, the possibility is discussed to develop a five-dimensional fourth-order gravity model whose lower-dimensional reduction could provide an interpretation of cosmological four-dimensional matter–energy components. We describe the basic concepts of the model, the complete field equations formalism and the 5-D to 4-D reduction procedure. Five-dimensional  $f(R)$  field equations turn out to be equivalent, on the four-dimensional hypersurfaces orthogonal to the extra coordinate, to an Einstein-like cosmological model with three matter–energy tensors related with higher derivative and higher-dimensional counter-terms. By considering the gravity model with  $f(R) = f_0 R^n$  the possibility is investigated to obtain five-dimensional power law solutions. The effective four-dimensional picture and the behaviour of the geometrically induced sources are finally outlined in correspondence to simple cases of such higher-dimensional solutions.

## 1 Introduction

Type Ia supernovae (SNeIa) observations depicted a late-time speeding up universe [1–7], driven by an unknown component, whose properties can be ascribed to some sort of exotic fluid. This elusive component has been addressed in terms of a form of dark energy that exhibits an anti-gravitational negative equation of state (EoS) [8,9]. The combination

of standard matter and dark energy provides an Einstein-like universe where gravitational attraction is counterbalanced by such an unusual gravitational source, thus reproducing cosmological observations. Soon after this discovery a plethora of theoretical proposals has been suggested in order to explain the dark-energy origin. However, as of today, a well endowed and self-consistent physical interpretation is so far unknown [10–13]. The puzzling quest of a satisfactory explanation as regards these phenomenological results led cosmologists to explore several research lines. In particular, two main directions were followed. On the one side, standard Einstein gravity has been reviewed by introducing a new cosmological component: the cosmological constant or any well-behaved fluid with a negative EoS. Differently, adopting an unconventional point of view, there have been considered modified gravity models that generalize Einstein theory: i.e. scalar-tensor gravity [14–16],  $f(R)$  theories [17–20], DGP gravity [21], braneworld scenarios [22–25], induced-matter theory [26–28], and so on.

Among others,  $f(R)$  theories of gravity received considerable attention. Such theories, which represent a natural generalization of Einstein gravity, are obtained by relaxing the hypothesis of linearity contained in the Hilbert–Einstein Lagrangian. Several results, sometimes controversial, have been obtained in this framework [29,30]. These models have been satisfactorily checked with cosmological observations [31–35] and suggest intriguing peculiarities in the low energy and small velocity limit [36] since the gravitational potential displays a Yukawa like correction [37,38]. In order to be viable,  $f(R)$  gravity theories have to satisfy some minimal prescriptions. In particular, they have to match cosmological observations avoiding instabilities and ghost-like solutions and they have to evade solar system tests [29,39]. Constraints from energy conditions represent a further argument to settle a suitable gravity Lagrangian [40]. However, at this stage, a fully satisfactory fourth-order gravity model is far from having been achieved. As a matter of fact, such kinds of

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models have been considered also from other points of view. For example, recently, models with a non-minimal coupling between gravity and the matter sector have been taken into account [41–43].

On the other side, historically, attempts for a unification of gravity with other interactions stimulated the search for theoretical schemes based on higher dimensions, i.e. beyond our conventional four-dimensional (4-D) spacetime. Nordström [44], who was the first to formulate a unified theory based on extra dimensions, and Kaluza [45] and Klein [46] developed a five-dimensional (5-D) version of general relativity (GR) in which electrodynamics is derived as a counter-effect of the extra dimension. Successively, a lot of work has been dedicated to such a theoretical proposal both considering fifth-coordinate compactification and large extra dimensions, allowing for non-compact mechanisms [27]. The last approach led to the so called space–time–matter (STM) or induced-matter theories (IMT) [26,28] and to braneworld theories [22]. The basic characteristic of IMT is that 5-D vacuum field equations can be recast after the reduction procedure as 4-D Einsteinian field equations with a source of geometrical origin. In such an approach, the matter–energy source of 4-D spacetime represents a manifestation of extra dimensions. Generalizations of Kaluza–Klein theory present themselves as a suitable scheme to frame modern cosmological observations. In fact, after SNIe observations, several attempts have been made to match induced-matter theories with the dark-energy puzzle [47–52].

In this paper we try to merge the two approaches. Since vacuum fourth-order gravity theories, as well as IMT, can be cast as an Einstein model with a matter–energy source of geometrical origin [53,54], it seems a significant proposal to confront the two theoretical schemes. The genuine idea underlying this work is to investigate 5-D  $f(R)$ -gravity from the point of view of a completely geometric self-consistent approach. In this scheme, all matter–energy sources will represent a byproduct of the dimensional reduction related to higher-order terms and higher-dimensional quantities. Such an approach determines a completely different conceptual framework with respect to standard five-dimensional fourth-order gravity models. The main purpose is to recover along this scheme both dark-matter and dark-energy dynamical effects. In the standard realm, in fact, extra dimensions and curvature counter-terms will only play the role of mending dark-energy phenomenology in the presence of ordinary matter. Along this orthodox paradigm, 5-D models of fourth-order gravity have been studied in presence of perfect fluid sources [55–57] under peculiar assumptions on the metric potentials. In a similar fashion, accelerating 4-D cosmologies, induced by generalized 5-D  $f(R)$  gravity, have also been studied considering a curvature–matter coupling [58]. Furthermore, a vacuum fourth-order Kaluza–Klein theory, defined in terms of the Gauss–Bonnet invariant [59], has

been studied from the point of view of the predictions of field equations. In particular, in the cylinder approximation, the propagation of its electromagnetic degrees of freedom [60] and the particle spectrum in the linear regime [61] have been investigated.

Attempts to meet  $f(R)$ -gravity and IMT have also been developed in time. For example the possibility has been investigated to obtain information on the space–time–matter tensor starting from the energy-momentum tensor induced by higher-order curvature counter-terms [62]. On the other side a new effective coupled  $F^{(4)}R, \varphi$  gravity theory has been proposed as a consequence of a five-dimensional  $f(R)$  model [63]. It has also been demonstrated that the Dolgov–Kawasaki stability criterion of the 5-D  $f(R)$  theory remains the same as in usual  $f(R)$  theories:  $f''(R) > 0$ , with the prime indicating the derivative with respect to the Ricci scalar.

In our work we try to develop a more general framework; 5-D  $f(R)$ -gravity is discussed in a complete analytical form. We present general field equations in the  $f(R)$ -IMT approach and, by exploiting the reduction procedure from 5-D to the 4-D ordinary spacetime (considering a suitable extension of the Friedmann–Robertson–Walker metric), we obtain a complete set of new Einstein-like field equations with matter–energy sources induced by higher-order derivative terms and higher dimensions (top–down reduction). In other words, we develop a fully geometrical cosmology where ordinary matter and dark components could be addressed, in principle, as the outgrowth of the GR formalism adopted within the top–down reduction mechanism.

The paper is organized as follows. In Sect. 2 we provide a brief review of 4-D  $f(R)$ -gravity. Section 3 is devoted to summarizing induced-matter theory. The 5-D  $f(R)$ -gravity formalism is outlined in Sect. 4, and, thereafter, we discuss the 5-D to 4-D reduction procedure of the field equations. In Sect. 5 a routine is described to find 5-D power law solutions. Section 6 is dedicated to the analysis of the results and, in particular, to their interpretation from the point of view of the 4-D induced cosmologies. Finally, Sect. 7 presents our conclusions.

## 2 The cosmological equations for $f(R)$ gravity

Dark-energy models are based on the underlying assumption that Einstein’s general relativity is indeed the correct theory of gravity. However, adopting a different point of view, both cosmic speed up and dark matter can be viewed as a breakdown of GR. As a matter of fact, one should consider the possibility to generalize the Hilbert–Einstein (H–E) Lagrangian. With these premises in mind, the choice of the gravity Lagrangian can be settled by means of data with the only prescription of adopting an “economic” strategy, i.e.

only a minimal generalization of the H–E action is taken into account. The so-called  $f(R)$ -gravity [17, 18, 29, 30, 64, 65], which considers an analytic function in terms of the Ricci scalar and leads to fourth-order field equations, is based on this conceptual scheme. It has to be recalled that higher-order gravity theories represent the natural effective result of several theoretical schemes [66, 67], i.e. quantum field theories on curved spacetimes and M-theory. In addition, they have been widely studied as inflationary models in the early universe [68, 69].

Fourth-order gravity is favoured by the Ostrogradski theorem. It has in fact been demonstrated [70, 71] that  $f(R)$ -Lagrangians are the only metric-based, local and potentially stable modifications of gravity among several that can be constructed by means of the curvature tensor and possibly by means of its covariant derivatives.

Let us consider the generic  $f(R)$  action; we have

$$S = \frac{1}{2} \int d^4x \sqrt{-g} [f(R) + S_M(g_{\mu\nu}, \psi)], \tag{1}$$

where we have used natural units  $8\pi G = c = \hbar = 1$ ,  $g$  is the determinant of the metric,  $R$  is the Ricci scalar and  $\psi$  characterizes the matter fields. Varying with respect to the metric we get the field equations,

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu}\square] f'(R) = T^M_{\mu\nu}, \tag{2}$$

where the prime, as said in the introduction, denotes a derivative with respect to  $R$  and, as usual,

$$T^M_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{3}$$

Taking the trace of (2) we obtain

$$f'(R)R - 2f(R) + 3\square f'(R) = T^M, \tag{4}$$

where  $T^M = g^{\mu\nu}T^M_{\mu\nu}$ , so that the relation between the Ricci scalar and  $T^M$  is obtained by means of a differential equation, different from GR where  $R = -T^M$ . This result suggest that  $f(R)$ -gravity field equations admit a larger variety of solutions than Einstein’s theory. In particular,  $T^M = 0$  solutions will no longer imply Ricci-flat cosmologies and therefore  $R = 0$ . Actually, a suitable property of this model is that the field equations (2) can be recast in the Einstein form [17, 18, 53]:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{\text{curv}}_{\mu\nu} + T^M_{\mu\nu}/f'(R), \tag{5}$$

with

$$T^{\text{curv}}_{\mu\nu} = \frac{1}{f'(R)} \left\{ \frac{1}{2}g_{\mu\nu} [f(R) - Rf'(R)] + f'(R)^{;\alpha\beta} (g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta}) \right\}, \tag{6}$$

which represents a *curvature stress-energy tensor* induced by higher-order derivative terms (the terms  $f'(R)_{;\mu\nu}$  render the equations of fourth order). The limit  $f(R) \rightarrow R$  reduces Eq. (5) to the standard second-order Einstein field equations.

The Einstein-like form of fourth-order field equations (5) suggests that higher-order counter-terms can play the role of a source-like component within gravity field equations. In practice, it is possible to postulate that geometry can play the role of a mass–energy component when higher than second order quantities are taken into account. Indeed, the trace equation (4) propagates a scalar-like degree of freedom. As a matter of fact, in principle, one can imagine to address the universe’ dark phenomenology in terms of such a kind of effective fluid. From the cosmological point of view a significant role is played by the barotropic factor of the curvature–matter–energy fluid. In the vacuum case this quantity will discriminate accelerating solutions from standard matter ones. In the case of a Friedman spacetime, for example, one has [53, 54]

$$\rho_{\text{curv}} = \frac{1}{f'(R)} \left\{ \frac{1}{2} [f(R) - Rf'(R)] - 3H\dot{R}f''(R) \right\}, \tag{7}$$

$$w_{\text{curv}} = -1 + \frac{\ddot{R}f''(R) + \dot{R}[\dot{R}f'''(R) - Hf''(R)]}{[f(R) - Rf'(R)]/2 - 3H\dot{R}f''(R)}, \tag{8}$$

where  $H = \dot{a}/a$  stands for the Hubble parameter and a dot means a time derivative.

In the following we will resort to a similar quantity in order to check the behaviour of the cosmological fluids induced by 5-D  $f(R)$ -gravity on the 4-D hypersurfaces that slice the higher dimensional spacetime along the extra coordinate.

In our conventions we will consider Latin indices like  $A, B, C$ , etc. running from 0 to 4, Latin indices like  $i, j$ , etc., running from 1 to 3 and Greek indices taking values from 0 to 3.

### 3 Induced-matter theory

Modelling a unification theory that contains gravity and particle physics forces, typically, implies the resort to extra-dimensional models [27, 72]. Among these 5-D Kaluza–Klein theory [45, 46] and its modern revisits induced-matter and membrane theory represent significant approaches; in addition these kinds of models represent the low energy limit

of more sophisticated theories (i.e. supergravity) [73–75]. The main difference between the seminal approach of Kaluza and Klein and induced-matter theory is related with the role of the extra dimension. In its first conception the fifth dimension was “rolled up” to a very small size, answering the question of why we do not “see” the fifth dimension. Modern theories like IMT postulate that we are constrained to live in a smaller 4-D hypersurface embedded in a higher-dimensional spacetime. The key point for embedding the 4-D Einstein theory into the 5-D Kaluza–Klein induced-matter theory is represented by the Campbell–Magaard (CM) theorem. The problem of embedding a Riemannian manifold in a Ricci-flat space was studied by Campbell soon after the discovery of GR [76], and finally it was demonstrated by Magaard in 1963 [77]. Further on, Tavakol et al. [78] used these studies to establish mathematical well-entwined bases to the 4-D reinterpretation of the 5-D Kaluza–Klein theory that is dubbed induced-matter theory [79, 80]. In such a framework, the 5-D to 4-D reduction procedure determines on the 4-D hypersurface an Einstein gravity theory plus induced-matter components of geometrical origin. The CM theorem can be formalized as follows:

Any analytic Riemannian space  $V_n(x^\mu, t)$  can be locally embedded in a Ricci-flat Riemannian space  $V_{n+1}(x^A, t)$ .

Here the “smaller” space has dimensionality  $n$  with  $\mu = 0, \dots, n - 1$ , while the “host” space has dimensionality  $N = n + 1$  with  $A$  running from 0 to  $n$ ; the extra coordinate can be both space-like and time-like. We are interested in the case  $N = 5$ . It is important to remark that the CM theorem is a *local* embedding theorem. Therefore, more general issues related with global embedding, i.e. initial-value problems, stability or general induced solutions [81, 82], cannot be addressed by resorting to this achievement. However, for our purposes, the theorem guarantees the right analytic framework in order to frame 4-D matter phenomenology in relation to 5-D field equations [80].

Because of the CM theorem it is possible to write down the 5-D metric as follows:

$$dS^2 = \gamma_{AB} dx^A dx^B = g_{\mu\nu}(x, y) dx^\mu dx^\nu + \epsilon \Phi^2(x, y) dy^2, \tag{9}$$

where  $x^\mu = (x^0, x^1, x^2, x^3)$  are 4-D coordinates,  $g_{\mu\nu}$  turns out to be the spacetime metric and  $y$  is the fifth coordinate,  $\epsilon = \pm 1$  allows for a space-like or a time-like extra dimension. Throughout the paper we adopt the spacetime signature  $(+, -, -, -)$ . With these premises in mind ordinary 4D spacetime results as a hypersurface  $\Sigma_y : y = y_0 = \text{constant}$ , orthogonal to the 5D extra-coordinate basis vector

$$\hat{n}^A = \frac{\delta^A_4}{\Phi}, \quad n_{AN} = \epsilon. \tag{10}$$

If one considers vacuum 5-D field equations (Ricci flat):  $R_{AB} = 0$ , the CM theorem suggests a natural reduction process to a 4-D pseudo-Riemannian spacetime. In practice, one can obtain an Einstein-like 4-D model<sup>1</sup>  $G_{\mu\nu} = T_{\mu\nu}$  [79, 80], once the right hand side of these equations fulfills the definition

$$T_{\mu\nu}^{\text{IMT}} = \frac{\Phi_{,\mu;\nu}}{\Phi} - \frac{\epsilon}{2\Phi^2} \left\{ \frac{\Phi^* g_{\mu\nu}^*}{\Phi} - g_{\mu\nu}^{**} + g^{\alpha\beta} g_{\mu\alpha}^* g_{\nu\beta}^* + \right. \\ \left. - \frac{g^{\alpha\beta} g_{\alpha\beta}^* g_{\mu\nu}^*}{2} + \frac{g_{\mu\nu}}{4} \left[ g^{*\alpha\beta} g_{\alpha\beta}^* + \left( g^{\alpha\beta} g_{\alpha\beta}^* \right)^2 \right] \right\}. \tag{11}$$

Here, a comma is the ordinary partial derivative, a semi-colon denotes the ordinary 4D covariant and starred quantities describe terms derived with respect to the fifth coordinate. This quantity defines the so-called induced-matter tensor; the only hypothesis underlying this achievement has been relaxing the cylinder condition within the Kaluza–Klein scheme (independence on the fifth coordinate).

As observed from the previous result, fourth-order gravity theories and Kaluza–Klein IMT models provide the same conceptual scheme. Both of the approaches allow one to obtain an Einstein-like gravity model where matter–energy sources are of geometrical origin. In particular, these effective matter–energy tensors descend, respectively, from the higher-order derivative contributions and from the higher-dimensional counter-terms. Therefore, it seems that deviations from GR can be naturally recast as sources of standard Einsteinian models.

In that respect let us notice that IMT and 5-D  $f(R)$  gravity are built in the framework of a different philosophy with respect to conventional higher dimensional approaches like braneworld models. In fact, despite the same working scenario: bulk universe with non-trivial dependence on the extra-coordinate, 4-D metric obtained evaluating the background metric at specific 4-D hypersurfaces and matter fields confined in the 4-D spacetime, there are intrinsic conceptual differences. 5-D  $f(R)$  gravity and induced-matter theory are based on the hypothesis that standard matter is nothing else than a 4-D manifestation of geometrical deviations with respect to GR. Braneworlds model our universe as a four-dimensional singular hypersurface, the brane, embedded in a five-dimensional anti de Sitter spacetime. In such a case the motivation for a non-compactified extra dimension is to solve the hierarchy problem. The differences in terms of physical motivations for large extra dimensions imply also different technical approaches. Within IMT and 5-D  $f(R)$  gravity one considers a Ricci-flat vacuum bulk and develops

<sup>1</sup> We recall that natural units are adopted, therefore  $8\pi G/c^4$  is set equal to unity.

the 4-D physics as a byproduct. In braneworlds the opposite point of view is assumed. In this case, one deals with suitable 4-D solutions on the brane for some matter distributions and these solutions are matched with an appropriate 5-D bulk considering Israel junction conditions. Nevertheless, despite such conceptual differences it has been shown in time that brane theory, IMT and 5-D  $f(R)$  gravity can have a suitable matching. In particular, IMT and braneworlds have been put in strict analogy [83]. The key point is to consider the induced-matter approach in terms of the spacetime extrinsic curvature. In this case it is possible to write down braneworld-like field equations with a 4-D matter–energy source and a brane tension that are defined in terms of the extrinsic curvature. On the same line very recently it has been shown that  $Z_2$  braneworld-like solutions can be obtained as particular maximally symmetric solutions of 5-D  $f(R)$  gravity with matter [63]. In future work one can imagine to deepen the interconnections that seem to arise among these different higher-dimensional approaches.

Such considerations suggest, in principle, the intriguing possibility that GR experimental shortcomings could be actually referred to the effective property of modified gravity models. In the following we will exploit such an approach to study a general fourth-order 5-D model, where higher-order derivative gravity is “merged” with higher dimensions.

#### 4 A 5-D $f(R)$ -gravity model and its 4-D reduction

A  $f(R)$  theory of gravity in five dimensions can be described by the action

$$\mathcal{S}^{(5)} = \frac{1}{2} \int d^5x \sqrt{g^{(5)}} f(R^{(5)}) + \frac{1}{2} \int d^5x \sqrt{g^{(5)}} \mathcal{L}_m(g_{AB}, \psi), \tag{12}$$

where  $R^{(5)}$  is the 5-D Ricci scalar,  $\mathcal{L}_m(g_{AB}, \psi)$  is a Lagrangian density for matter fields denoted, as in the 4-D case, by  $\psi$  and  $g^{(5)}$  represents the determinant of the 5-D metric tensor  $g_{AB}$ . 5-D field equations can be obtained by action (12) varying with respect to the metric:

$$f_{,R}^{(5)} R_{AB}^{(5)} - \frac{1}{2} f(R^{(5)}) g_{AB} + [\nabla_A \nabla_B - g_{AB} \square^{(5)}] f_{,R}^{(5)} = T_{AB}^{(5)M}, \tag{13}$$

here  $T_{AB}^{(5)M}$  is the energy-momentum tensor for matter sources, while  $\nabla_A$  is the 5-D covariant derivative,  $\square^{(5)} = g^{AB} \nabla_A \nabla_B$  is the 5-D d’Alembertian operator. In order to simplify the notation we have defined  $f_{,R}(R^{(5)}) \equiv f_{,R}^{(5)}$ , therefore henceforth by  $f_{,R}^{(5)}$  is intended to write the 5-D Ricci derivative of the  $f(R^{(5)})$  gravity Lagrangian.

Together with the field equations (13) one can obtain the trace

$$4 \square^{(5)} f_{,R}^{(5)} + f_{,R}^{(5)} R^{(5)} - \frac{5}{2} f(R^{(5)}) = T^{(5)M} \tag{14}$$

where  $T^{(5)M} = g^{AB} T_{AB}^{(5)M}$ .

Actually, 5-D field equations (13) can be naturally recast as a generalization of Einstein 5-D equations in the same manner of the ordinary 4-D formalism given in Sect. 2. In fact, with some algebra and isolating 5-D Einstein tensor one gets

$$R_{AB}^{(5)} - \frac{1}{2} R^{(5)} g_{AB} = \frac{1}{f_{,R}^{(5)}} \left\{ \frac{1}{2} \left( f(R^{(5)}) - f_{,R}^{(5)} R^{(5)} \right) g_{AB} + [\nabla_A \nabla_B - g_{AB} \square^{(5)}] f_{,R}^{(5)} \right\} + \frac{1}{f_{,R}^{(5)}} T_{AB}^{(5)M}, \tag{15}$$

by fact determining a 5-D new source of geometrical origin in the right member of field equations

$$T_{AB}^{(5)Curv} = \frac{1}{f_{,R}^{(5)}} \left\{ \frac{1}{2} \left( f(R^{(5)}) - f_{,R}^{(5)} R^{(5)} \right) g_{AB} + [\nabla_A \nabla_B - g_{AB} \square^{(5)}] f_{,R}^{(5)} \right\}. \tag{16}$$

This fact, which can resemble only the byproduct of a mathematical trick, determines significant physical consequences at 5-D and, as a consequence, also in the ordinary space-time. In fact, as we will see in the following, the 4-D dimensional reduction of Eqs. (15) and (16) implies a “new” set of Einstein-like equations with three matter–energy components all of geometrical origin. One of these quantities descends from the 4-D reduction of the Einstein tensor and the two terms derivate from the relative 4-D reduction applied to  $T_{AB}^{(5)Curv}$ . Since we want to explore how geometric counter-terms can effectively mimic cosmological sources we neglect ordinary matter, i.e. from now on we will assume  $T_{AB}^{(5)M} = 0$ , considering gravity equations in vacuum.

Let us now develop the 5-D to 4-D reduction procedure of our higher dimensional framework. At first, by assuming the metric (9), it is possible to draw the reduction rules for the differential operators [49]:

$$\nabla_\mu \nabla_\nu f_{,R}^{(5)} = D_\mu D_\nu f_{,R}^{(5)} + \frac{\epsilon}{2\Phi^2} g_{\mu\nu}^* f_{,R}^{(5)*}, \tag{17}$$

$$\nabla_4 \nabla_4 f_{,R}^{(5)} = \epsilon \Phi (D_\alpha \Phi) (D^\alpha f_{,R}^{(5)}) + f_{,R}^{(5)**} - \frac{\Phi}{\Phi} f_{,R}^{(5)*}, \tag{18}$$

$$\square^{(5)} f_{,R}^{(5)} = \square f_{,R}^{(5)} + \frac{(D_\alpha \Phi) (D^\alpha f_{,R}^{(5)})}{\Phi} + \frac{\epsilon}{\Phi^2} \left[ f_{,R}^{(5)**} + f_{,R}^{(5)*} \left( \frac{g^{\alpha\beta} g_{\alpha\beta}^*}{2} - \frac{\Phi}{\Phi} \right) \right]; \tag{19}$$

here, again, the asterisk denotes the partial derivative with respect to the extra coordinate (i.e.,  $\partial/\partial y = *$ );  $D_\alpha$  is the four-dimensional covariant derivative defined on the hypersurface  $\Sigma_{y=y_0}$ , calculated with  $g_{\mu\nu}$ , and the usual d'Alembertian  $\square$  is referred to 4-D quantities. In the same way, all the quantities that are not labelled in terms of the fifth coordinate will be intended to refer to ordinary spacetime. On this basis, it is possible to rewrite the 5-D field equations (15) by separating the spacetime part ( $\mu = 0, \dots, 3$ ;  $\nu = 0, \dots, 3$ ) from the extra-coordinate ( $y$ ) one:

$$G_{\mu\nu}^{(5)} = \frac{1}{f_{,R}^{(5)}} \left\{ \frac{1}{2} \left( f(R^{(5)}) - f_{,R}^{(5)} R^{(5)} \right) g_{\mu\nu} + \left[ \square f_{,R}^{(5)} + \frac{(D_\alpha \Phi) (D^\alpha f_{,R}^{(5)})}{\Phi} + \frac{\epsilon}{\Phi^2} \left( f_{,R}^{(5)**} + f_{,R}^{(5)*} \left( \frac{g^{\alpha\beta} g_{\alpha\beta}^*}{2} - \frac{\Phi}{\Phi} \right) \right) \right] g_{\mu\nu} + \nabla_\mu \nabla_\nu f_{,R}^{(5)} + \frac{\epsilon}{2\Phi^2} g_{\mu\nu}^* f_{,R}^{(5)*} \right\}; \tag{20}$$

$$G_{44}^{(5)} = \frac{1}{f_{,R}^{(5)}} \left\{ \frac{1}{2} \left( f(R^{(5)}) - f_{,R}^{(5)} R^{(5)} \right) g_{44} + \left[ \square f_{,R}^{(5)} + \frac{(D_\alpha \Phi) (D^\alpha f_{,R}^{(5)})}{\Phi} + \frac{\epsilon}{\Phi^2} \left( f_{,R}^{(5)**} + f_{,R}^{(5)*} \left( \frac{g^{\alpha\beta} g_{\alpha\beta}^*}{2} - \frac{\Phi}{\Phi} \right) \right) \right] g_{44} + \epsilon \Phi (D_\alpha \Phi) (D^\alpha f_{,R}^{(5)}) + f_{,R}^{(5)**} - \frac{\Phi}{\Phi} f_{,R}^{(5)*} \right\}. \tag{21}$$

Our purpose is now to disentangle the extra-coordinate dependence from the Einstein tensor. In this way it will be possible to isolate the 4-D part of Einstein tensor on the left member and move on the r.h.s. all quantities that depend on higher derivative terms and on the extra coordinate. All the higher-order and higher-dimensional counter-terms will play the role of effective source terms on the 4-D hypersurface. It is evident that since the fourth-order Lagrangian  $f(R^{(5)})$  depends on the 5-D Ricci scalar, this dependence cannot be

completely untwined until the Lagrangian dependence is not specified.

The reduction rules for the Einstein tensor [80] give

$$R_{\mu\nu}^{(5)} = R_{\mu\nu} - \frac{D_\mu D_\nu \Phi}{\Phi} + \frac{\epsilon}{2\Phi^2} \left( \frac{\Phi g_{\mu\nu}^{**}}{\Phi} - g_{\mu\nu}^{**} + g^{\lambda\alpha} g_{\mu\lambda}^* g_{\nu\alpha}^* - \frac{g^{\alpha\beta} g_{\alpha\beta}^* g_{\mu\nu}^*}{2} \right), \tag{22}$$

$$R_{44}^{(5)} = -\epsilon \Phi \square \Phi - \frac{g^{\alpha\beta} g_{\alpha\beta}^*}{4} - \frac{g^{\alpha\beta} g_{\alpha\beta}^{**}}{2} + \frac{\Phi g^{\alpha\beta} g_{\alpha\beta}^*}{2\Phi}. \tag{23}$$

The next step is to calculate the 5-D Ricci scalar  $R^{(5)} = g^{\mu\nu} R_{\mu\nu}^{(5)} + g^{44} R_{44}^{(5)}$ . After the substitution of Eq. (23) into Eq. (21) and some tedious algebra one gets the relation

$$\frac{\square \Phi}{\Phi} = \frac{\epsilon}{\Phi^2} \left( \frac{\Phi g^{\alpha\beta} g_{\alpha\beta}^*}{\Phi} - \frac{g^{\alpha\beta} g_{\alpha\beta}^*}{4} - \frac{g^{\alpha\beta} g_{\alpha\beta}^{**}}{2} \right) + \frac{1}{2} \frac{f(R^{(5)})}{f_{,R}^{(5)}} + \frac{\square^{(5)} f_{,R}^{(5)}}{f_{,R}^{(5)}} - \frac{(D_\alpha \Phi) (D^\alpha f_{,R}^{(5)})}{\Phi f_{,R}^{(5)}} + \frac{\epsilon}{\Phi^2} \left( \frac{f_{,R}^{(5)**}}{f_{,R}^{(5)}} - \frac{\Phi f_{,R}^{(5)*}}{\Phi f_{,R}^{(5)}} \right), \tag{24}$$

which reminds one of an analogous expression given in [49] for the 4-D reduction of 5-D Brans–Dicke theory. Further manipulations of (24) allow one to obtain a relatively simple expression:

$$\frac{\square \Phi}{\Phi} = \frac{\epsilon}{\Phi^2} \left( \frac{\Phi g^{\alpha\beta} g_{\alpha\beta}^*}{\Phi} - \frac{g^{\alpha\beta} g_{\alpha\beta}^*}{4} - \frac{g^{\alpha\beta} g_{\alpha\beta}^{**}}{2} \right) + \frac{1}{2} \frac{f(R^{(5)})}{f_{,R}^{(5)}} + \frac{\square f_{,R}^{(5)}}{f_{,R}^{(5)}} - \frac{\epsilon}{\Phi^2} \frac{g^{\alpha\beta} g_{\alpha\beta}^* f_{,R}^{(5)*}}{2 f_{,R}^{(5)}}, \tag{25}$$

which finally can be used in the reduction of Ricci scalar to collect some terms. Therefore, by using also the trace equation, one has

$$R^{(5)} = R - 2 \frac{\square \Phi}{\Phi} + \left( 2 \frac{\Phi}{\Phi} g^{\alpha\beta} g_{\alpha\beta}^* + 2 g^{\alpha\beta} g_{\alpha\beta}^{**} - \frac{3}{2} g^{\alpha\beta} g_{\alpha\beta}^* - \frac{(g^{\alpha\beta} g_{\alpha\beta}^*)^2}{2} \right). \tag{26}$$

It is easy to check that this result reproduces the analogous relation of 5-D GR [27] when  $f(R^{(5)}) \rightarrow R^{(5)}$ . We remember that in such a case the vacuum Einstein field equations reduce to the system  $R_{AB}^{(5)} = 0$  and  $R^{(5)}$  vanishes.

Once having developed the several aspects of the 5-D to 4-D reduction procedure it is possible to obtain the induced 4-D field equations. By inserting (22), (23) and (26) into Eqs. (20), (21) the 4-D spacetime equations get the form<sup>2</sup>:

$$G_{\mu\nu} = T_{\mu\nu}^{IMT} + T_{\mu\nu}^{curv} + T_{\mu\nu}^{Mix}. \tag{27}$$

Here

$$T_{\mu\nu}^{IMT} = \frac{\Phi_{,\mu;\nu}}{\Phi} - \frac{\epsilon}{2\Phi^2} \left\{ \frac{\Phi^* g_{\mu\nu}^*}{\Phi} - g_{\mu\nu}^{**} + g^{\alpha\beta} g_{\mu\alpha}^* g_{\nu\beta}^* + \frac{g^{\alpha\beta} g_{\alpha\beta}^* g_{\mu\nu}^*}{2} + \frac{g_{\mu\nu}}{4} \left[ g^{*\alpha\beta} g_{\alpha\beta}^* + \left( g^{\alpha\beta} g_{\alpha\beta}^* \right)^2 \right] \right\}$$

is the usual IMT tensor, derived from the 5-D to 4-D reduction of the Einstein tensor, and

$$T_{\mu\nu}^{Curv} = \frac{1}{f_{,R}^{(5)}} \left\{ \frac{1}{2} \left( f(R^{(5)}) - f_{,R}^{(5)} R \right) g_{\mu\nu} + [g_{\mu\nu} \square - \nabla_\mu \nabla_\nu] f_{,R}^{(5)} \right\} \tag{28}$$

is the curvature tensor written preserving the 4-D form. In such a case extra-dimension contributions are still hidden in the scalar curvature nested within the definition of the gravity Lagrangian and of its derivatives. Finally,

$$T_{\mu\nu}^{Mix} = \frac{\epsilon}{2\Phi^2} g_{\mu\nu}^* \frac{f_{,R}^{(5)*}}{f_{,R}^{(5)}} - \frac{(D_\alpha \Phi)(D^\alpha f_{,R}^{(5)})}{\Phi f_{,R}^{(5)}} g_{\mu\nu} + \frac{\epsilon}{\Phi^2} \left[ \frac{f_{,R}^{(5)*}}{f_{,R}^{(5)}} + \frac{f_{,R}^{(5)*}}{f_{,R}^{(5)}} \left( \frac{g^{\alpha\beta} g_{\alpha\beta}^*}{2} - \frac{\Phi^*}{\Phi} \right) \right] g_{\mu\nu} + \frac{\epsilon}{2\Phi^2} \left[ \frac{g^{\alpha\beta} g_{\alpha\beta}^*}{4} + \frac{(g^{\alpha\beta} g_{\alpha\beta}^*)^2}{4} \right] g_{\mu\nu} \tag{29}$$

represents a mixed tensor containing terms that depend explicitly on the fifth coordinate and on the derivative of  $f_{,R}^{(5)}$  with respect to  $y$ .

<sup>2</sup> The equations presented throughout this section partially reproduce previous results obtained in [63]. In fact, besides the different conceptual scheme, there are also some analytic differences that make the two schemes non-completely indistinguishable.

Therefore, we have obtained a framework where 4-D gravity is fully geometrized. In fact, within such a scheme matter-energy sources are related only with geometrical counter-effects deriving either from higher-dimensional metric quantities or from higher-order derivative ones.

In order to complete our discussion one should take into account also the off-diagonal equation. By considering that  $R_{4\alpha}^{(5)} = \Phi P_{\alpha;\beta}^\beta$  with  $P_{\alpha\beta} = \frac{1}{2\Phi} (g_{\alpha\beta}^* - g_{\sigma\rho} g^{*\sigma\rho} g_{\alpha\beta})$  [80], one obtains

$$\Phi P_{\alpha;\beta}^\beta = \frac{f_{,R,\alpha}^{(5)*}}{f_{,R}^{(5)}} - g_{\alpha\sigma}^* g^{\beta\sigma} \frac{f_{,R,\beta}^{(5)}}{f_{,R}^{(5)}} - \frac{\Phi_{,\alpha}}{\Phi} \frac{f_{,R}^{(5)*}}{f_{,R}^{(5)}}, \tag{30}$$

which coincides with similar expressions obtained elsewhere [49,63] and in the limit  $f(R^{(5)}) \rightarrow R^{(5)}$  gives back, in the absence of ordinary matter, the conservation law  $P_{\alpha;\beta}^\beta = 0$  [27]. Equation (30) resembles a more general conservation equation which relates the spacetime derivatives of  $P_\alpha^\beta$  and  $f_{,R}^{(5)}$ . It has been conjectured [84] that the spacetime components of the field equations relate geometry with the macroscopic properties of matter, while the extra-coordinate part ( $\binom{\alpha}{4}$  and  $\binom{4}{\alpha}$ ) might describe their microscopic ones. With these hypotheses, within fourth-order gravity Kaluza–Klein models, one would find that microscopic properties of matter are influenced by spacetime derivatives of the scalar degree of freedom intrinsic in  $f(R)$  gravity. Furthermore, in our case, the  $P_\alpha^\beta$  dynamics depends also on  $\Phi$ , which, in turn (see Eq. (25)), shows an evolution driven by the higher-order gravity counter-terms. It seems that there is a strict interconnection between the extra-dimension properties and the intrinsic  $f(R)$  scalar degree of freedom. For example, observing Eq. (25), it seems that  $f'(R)$  guarantees a sort of Machian effect for this kind of models. However, all these considerations represent, at this stage, nothing more than speculations and, therefore, we do not discuss this issue further here.

## 5 A solving algorithm for 5-D f(R) gravity

### 5.1 5-D f(R) gravity cosmological solutions

Let us now verify what kind of solutions can be derived for our 5-D  $f(R)$ -gravity model. Once general cosmological solutions have been obtained, it is possible to conclude to 5-D effects on  $\Sigma_{y=y_0}$  hypersurfaces, deriving the effective 4-D picture. In particular, one can determine the spacetime effective matter–energy behaviour of the different geometrical components described in the previous section. In order to search for field equation solutions we assume the 5-D metric:

$$dS^2 = n^2(t, y)dt^2 - a^2(t, y) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] + \epsilon \Phi^2(t, y)dy^2, \quad (31)$$

where  $k = 0, +1, -1$  is referred to the 3-D spacetime curvature and  $(t, r, \theta, \phi)$  are the usual coordinates for spherically symmetric spatial sections. It is important to notice that, because of the metric choice, our  $f(R^{(5)})$  Lagrangian and its derivatives do not depend on the 4-D spatial coordinates. In fact, the 5-D Ricci scalar calculated over the metric (31) is a function of the  $(t, y)$  coordinates alone. If the metric (31) is introduced in Eqs. (20)–(21), we obtain the system of equations, respectively, for the components:  ${}^0_i, {}^i_i$  ( $i = 1, 2, 3, 4$ ) and  ${}^0_4$ :

$$-\frac{f(R^{(5)})}{2f_{,R}^{(5)}} + \frac{3\epsilon a^* f_{,R}^{(5)*}}{\Phi^2 a f_{,R}^{(5)}} - \frac{3\epsilon a^{**}}{\Phi^2 a n} - \frac{\epsilon f_{,R}^{(5)*} \Phi}{f_{,R}^{(5)} \Phi^3} + \frac{\epsilon n^* \Phi}{n \Phi^3} + \frac{\epsilon f_{,R}^{(5)**}}{\Phi^2 f_{,R}^{(5)}} - \frac{\epsilon n^{**}}{\Phi^2 n} + \frac{3\dot{a} f_{,R}^{(5)}}{a f_{,R}^{(5)} n^2} + \frac{3\dot{a} \dot{n}}{a n^3} + \frac{f_{,R}^{(5)} \dot{\Phi}}{n^2 f_{,R}^{(5)} \Phi} + \frac{\dot{n} \Phi}{n^3 \Phi} - \frac{3\ddot{a}}{n^2 a} - \frac{\ddot{\Phi}}{n^2 \Phi} = 0, \quad (32)$$

$$-\frac{f(R^{(5)})}{2f_{,R}^{(5)}} - \frac{2\epsilon a^{*2}}{\Phi^2 a^2} + \frac{2\epsilon a^* f_{,R}^{(5)*}}{\Phi^2 a f_{,R}^{(5)}} - \frac{\epsilon a^{**}}{\Phi^2 a n} + \frac{\epsilon f_{,R}^{(5)*} n^*}{\Phi^2 f_{,R}^{(5)} n} + \frac{\epsilon a^* \Phi}{a \Phi^3} - \frac{\epsilon f_{,R}^{(5)*} \Phi}{\Phi^3 f_{,R}^{(5)}} - \frac{\epsilon a^{**}}{\Phi^2 a} + \frac{\epsilon f_{,R}^{(5)**}}{\Phi^2 f_{,R}^{(5)}} + \frac{2\dot{a}^2}{a^2 n^2} + \frac{2\dot{a} f_{,R}^{(5)}}{a f_{,R}^{(5)} n^2} + \frac{\dot{a} \dot{n}}{a n^3} - \frac{f_{,R}^{(5)} \dot{n}}{f_{,R}^{(5)} n^3} + \frac{\dot{a} \dot{\Phi}}{a n^2 \Phi} + \frac{f_{,R}^{(5)} \dot{\Phi}}{f_{,R}^{(5)} n^2 \Phi} - \frac{\ddot{a}}{a n^2} + \frac{f_{,R}^{(5)} \ddot{n}}{f_{,R}^{(5)} n^2} = 0, \quad (33)$$

$$-\frac{f(R^{(5)})}{2f_{,R}^{(5)}} + \frac{3\epsilon a^* f_{,R}^{(5)*}}{\Phi^2 a f_{,R}^{(5)}} + \frac{\epsilon f_{,R}^{(5)*} n^*}{\Phi^2 f_{,R}^{(5)} n} + \frac{3\epsilon a^{**}}{a \Phi^3} + \frac{\epsilon n^* \Phi}{n \Phi^3} + \frac{3\epsilon a^{**}}{\Phi^2 a} - \frac{\epsilon n^*}{\Phi^2 n} + \frac{3\dot{a} f_{,R}^{(5)}}{a f_{,R}^{(5)} n^2} + \frac{f_{,R}^{(5)} \dot{n}}{f_{,R}^{(5)} n^3} - \frac{3\dot{a} \dot{\Phi}}{a n^2 \Phi} + \frac{\dot{n} \Phi}{n^3 \Phi} + \frac{f_{,R}^{(5)} \ddot{n}}{f_{,R}^{(5)} n^2} - \frac{\ddot{\Phi}}{n^2 \Phi} = 0, \quad (34)$$

$$-\frac{3\dot{a} n^*}{a n} + \frac{n^* f_{,R}^{(5)}}{n f_{,R}^{(5)}} + \frac{3\dot{a} \dot{\Phi}}{a \Phi} - \frac{f_{,R}^{(5)} \dot{\Phi}}{f_{,R}^{(5)} \Phi} - \frac{3\dot{a}}{a} - \frac{f_{,R}^{(5)}}{f_{,R}^{(5)}} = 0. \quad (35)$$

Equation (35) reminds one, again, of the result obtained in [49] for 5-D Brans–Dicke vacuum solutions. Therefore it seems a suitable choice to pursue the same approach performed in this work to get some particular solutions of our system of equations (32)–(35). If one considers the metric coefficients as separable functions of their arguments,

$$n(t, y) = N(y), \quad a(t, y) = P(y)Q(t), \quad \Phi(t, y) = F(t), \quad f_{,R}^{(5)}(t, y) = U(y)W(t), \quad (36)$$

from Eq. (35) one gets

$$\frac{3\dot{F} P^*}{FP} + \frac{3\dot{N} Q^*}{NQ} - \frac{3\dot{P} Q^*}{PQ} + \frac{\dot{F} U^*}{FU} + \frac{\dot{N} W^*}{NW} - \frac{\dot{U} W^*}{UW} = 0. \quad (37)$$

This equation suggests the possibility to look for power law cosmological solutions. In the following section we will discuss some examples in this sense.

### 5.2 Preliminary power law solutions

In order to look for 5-D  $f(R)$  gravity power law solutions we consider a set of metric potentials defined as follows [49]:

$$n(t, y) = N_0 y^\delta, \quad a(t, y) = A_0 t^\alpha y^\beta, \quad \Phi(t, y) = \Phi_0 t^\gamma y^\sigma, \quad (38)$$

with  $N_0, A_0, \Phi_0$  some constants with appropriate units and  $\alpha, \beta, \gamma, \delta, \sigma$  that represent the unknown parameters required to satisfy 5-D field equations.

Up to now we have developed a completely general scheme, without any assumption on the gravity Lagrangian. However, to completely define the solving algorithm one has to make a choice about the Ricci scalar function entering the gravity action. To remain conservative with the solution procedure, we consider a power law function of the Ricci scalar  $f(R) = f_0 R^n$ . Such a model has been extensively studied in the literature at four dimensions both in cosmology (curvature quintessence) [17, 18, 53, 54] and in the low energy limit [36, 85]. In addition, the 5-D phenomenology of a power law fourth-order gravity has been investigated in the standard approach in the presence of a perfect fluid matter source [56, 58]. In particular, in such a case, a compact fifth dimension is assumed and it is supposed that the homogeneously distributed fluid does not travel along the fifth dimension. As a matter of fact the matter–energy density and the pressure experienced by a four-dimensional observer have to be integrated throughout the compact extra-dimensional ring. In our approach we have overcome this point of view, we relax the hypotheses on the extra coordinate and, above all, we discard ordinary matter in favour of a completely geometric self-consistent approach. In accordance with cosmological



observations [86–90], we will study field equations considering a flat spacetime geometry ( $k = 0$ ).

A suitable recipe to study Eqs. (32)–(35) is to plug the metric functions (38) into the last of these relations and to search for its solutions. This approach allows one to obtain, with some effort, a number of constraints on the model parameters. Then one can try to satisfy also the other more complicated field equations (32)–(34). By inserting the metric potentials into Eq. 4, one obtains a rather cumbersome expression:

$$\frac{\Xi(\alpha, \beta, \gamma, \delta, \sigma, n, t, y)}{\Upsilon(\alpha, \beta, \gamma, \delta, \sigma, n, t, y)} = 0 \tag{39}$$

where

$$\begin{aligned} \Xi(\alpha, \beta, \gamma, \delta, \sigma, n, t, y) = & y^{-1-2\delta} \left( 3(\beta\gamma + \alpha(-\beta + \delta)) \right. \\ & \times (N_0^2 t^2 y^{2\delta} (6\beta^2 + 3\beta(-1 + \delta - \sigma) + \delta(-1 + \delta - \sigma)) + \\ & + t^{2\gamma} y^{2+2\sigma} (6\alpha^2 + 3\alpha(-1 + \gamma) + (-1 + \gamma)\gamma) \epsilon \Phi_0^2)^2 - \\ & + 2(-1 + n) (N_0^4 t^4 y^{4\delta} \gamma (\delta + (-1 + 2n)(1 + \sigma)) \\ & \times (-6\beta^2 + 3\beta(1 - \delta + \sigma) + \delta(1 - \delta + \sigma))^2 + \\ & + N_0^2 t^{2+2\gamma} y^{2(1+\delta+\sigma)} (6\alpha^2 + 3\alpha(-1 + \gamma) + (-1 + \gamma)\gamma) \\ & \times \epsilon (6\beta^2 + 3\beta(-1 + \delta - \sigma) + \\ & + \delta(-1 + \delta - \sigma)) (3\delta + 2(-2 + n)(1 + \sigma) + \\ & + \gamma(3 - 2\delta + 2\delta n + 3\sigma)) \Phi_0^2 + \\ & \left. + t^{4\gamma} y^{4+4\sigma} (6\alpha^2 + 3\alpha(-1 + \gamma) + (-1 + \gamma)\gamma)^2 \right. \\ & \left. \times \delta \epsilon^2 (-1 + \gamma + 2n) \Phi_0^4 \right) \end{aligned}$$

and

$$\begin{aligned} \Upsilon(\alpha, \beta, \gamma, \delta, \sigma, n, t, y) = & N_0^2 t \left( N_0^2 t^2 y^{2\delta} \right. \\ & \times \left( 6\beta^2 + 3\beta(-1 + \delta - \sigma) + \delta(-1 + \delta - \sigma) \right) + \\ & \left. + t^{2\gamma} y^{2+2\sigma} (6\alpha^2 + 3\alpha(-1 + \gamma) + (-1 + \gamma)\gamma) \epsilon \Phi_0^2 \right)^2. \end{aligned}$$

This result, evidently, suggests the possibility to obtain, in principle, much more solutions than the Brans–Dicke case studied in [49]. To simplify our search, as a preliminary step, we make some trivial hypotheses about the model parameters, leaving a complete analysis of the field equations solutions to a future dedicated work.

### 5.2.1 Kaluza–Klein GR limit

As a first step we want to verify that the Kaluza–Klein GR limit is recovered. To perform the standard KK limit we have to settle  $n = 1$  and, in addition, following customary approaches to the model, we assume  $\sigma = 0$ .

In such a case Eq. (39) becomes very simple and can be verified if  $\beta\gamma + \alpha(-\beta + \delta) = 0$ . In particular, if  $\gamma \neq \alpha$  one obtains  $\beta \rightarrow \frac{\alpha\delta}{\alpha - \gamma}$  (a result that is in agreement with

[49]). Looking to the other equations, together some trivial non-evolving solutions ( $\alpha = 0^3$ ), one obtains the solving sets of parameters:

$$\begin{aligned} \delta = 1, \gamma = 1, \alpha = -1, \beta = 1 \\ \text{and } N_0^2 = 4\Phi_0^2 \text{ with } \epsilon = -1, \end{aligned} \tag{40}$$

$$\delta = 0, \gamma = -1/2, \alpha = 1/2, \beta = 0, \tag{41}$$

with  $\epsilon = \pm 1$  (no restrictions on the  $y$  coordinate), and

$$\begin{aligned} \delta = 1, \gamma = 1, \alpha = -1, \beta = \frac{\alpha}{\alpha - 1} \\ \text{and } \Phi_0^2 = \frac{N_0^2}{(\alpha - 1)^2} \text{ with } \epsilon = -1. \end{aligned} \tag{42}$$

All these solutions are well known in the literature [49,91], therefore, our model completely reproduces standard GR Kaluza–Klein models in the  $f(R^{(5)}) \rightarrow R^{(5)}$  limit.

### 5.2.2 Standard 4-D $f(R)$ gravity limit

What about standard 4-D fourth-order gravity? One should expect that this framework has to represent a natural subcase of 5-D  $f(R)$  gravity. In order to get this limit one has to assign a solutions set with  $\delta = 0, \beta = 0, \sigma = 0$  and  $\gamma = 0$ , which means there is no dependence on  $y$  and no dynamics on the fifth coordinate. Starting with this assumptions and by considering equations (32), (33) and (35), one obtains

$$\alpha = \frac{-1 + 3n - 2n^2}{-2 + n}, \tag{43}$$

which exactly matches the well-known power law  $f(R)$  gravity solution in the case of the vacuum field equations [17, 18]. However, in the 5-D case, to fully satisfy the theory we have to fulfill one more equation, Eq. (34). This requirement, in fact, settles the power law index  $n$ . Therefore the only allowed combination is  $\delta = 0, \beta = 0, \sigma = 0, \gamma = 0, n = 5/4$ , and  $\alpha = 1/2$ .

### 5.2.3 Cylinder-type solutions

We can now relax some hypotheses in order to look for generalizations of 5-D GR. To search for such a kind of solutions we start, as a first simple case, from the assumption of no dependence on the extra coordinate. This means to study our theory, looking for solutions of Eqs. (32)–(35), in the limit of the cylinder condition. Different from the standard case, in

<sup>3</sup> It is possible to find some trivial solutions with  $\alpha = 0, \beta = 0, \delta = (0, 1), \gamma = 1$ .

the  $f(R)$  generalization of induced-matter theory this condition, as we will see, does not imply radiation as the only possible kind of induced matter.

At first, let us observe Eq. (25). In the case of 5-D  $f(R)$  gravity, independence on the fifth coordinate does not mean that one has a massless Klein–Gordon equation for the extra-coordinate potential  $\Phi$ . Higher-order gravity counter-terms will play the role of a mass term and  $\Phi$  will play a completely new role in the dynamics:

$$\square\Phi = - \left[ \frac{1}{2} \frac{f(R^{(5)})}{f_{,R}^{(5)}} + \frac{\square f_{,R}^{(5)}}{f_{,R}^{(5)}} \right] \Phi. \tag{44}$$

In addition, as a consequence,  $T_{\mu\nu}^{\text{IMT}}$  will no more have zero trace [27], therefore the induced matter tensor can span more general kinds of matter in relation to the underlying cosmological solution.

From the field equations point of view, power law solutions of cylinder-type are obtained when the model parameters in (38) are settled as  $\delta = 0, \beta = 0, \sigma = 0$  with free  $n$ . Neglecting again static solutions<sup>4</sup> it is possible to find a set of implicit solutions that can be expressed in terms of the  $f(R)$  Lagrangian power index  $n$ :

$$\alpha_1 = \frac{1}{12} (-3 + 6n - f(n)),$$

$$\gamma_1 = - \frac{17 + 28n^2 + 3f(n) - 2n(24 + f(n))}{2(-5 + 2n + f(n))}; \tag{45a}$$

$$\alpha_2 = \frac{1}{12} (-3 + 6n + f(n)),$$

$$\gamma_2 = \frac{17 + 28n^2 - 3f(n) + 2n(-24 + f(n))}{2(5 - 2n + f(n))}; \tag{45b}$$

$$\alpha_3 = \gamma_3 = \frac{2 - 6n + 4n^2}{5 - 2n}, \tag{45c}$$

with  $f(n) = \sqrt{-39 + 108n - 60n^2}$ . Equations (45a)–(45c) represent an interesting achievement. In the following we will show that these solutions provide, after the 5-D to 4-D reduction, cosmological significant behaviours of the 4-D matter–energy tensors induced by 5-D geometry.

### 6 4-D induced $f(R)$ gravity: a top–down geometrization of matter

The effective 4-D picture induced by higher dimensions can be obtained once cosmological solutions obtained in Sect. 5 are plugged into the 5-D to 4-D reduction framework previously outlined. In particular the 4-D setting is obtained, as

<sup>4</sup> There are also in this case trivial solutions i.e.  $n = 1, \alpha = 0, \gamma = (0, 1)$ .

already said, by considering hypersurfaces  $\Sigma_{y=y_0}$  that slice the 5-D universe along the fifth coordinate. In this scheme, the extra-coordinate effects on 4-D physical quantities are evaluated taking  $y = y_0 = \text{const}$ . The resulting cosmological model is a 4-D induced-matter  $f(R)$  theory, where field equations are given in the form of Einstein equations equipped with three different matter–energy sources of geometrical origin. Deviations from standard GR provide, in the effective 4-D spacetime description, matter–energy sources and one could wonder if these contributions are related with the elusive nature of dark energy and dark matter. Actually, we study the 4-D induced cosmologies in correspondence with the solutions given in Sect. 5. In particular, we are interested in investigating the behaviour of the three different induced cosmological fluids (11), (28), (29) on the  $y = y_0 = \text{const}$  hypersurfaces. One can notice that for all of these quantities we have  $T_1^{1K} = T_2^{2K} = T_3^{3K}$  (with  $K = \text{IMT, Curv, Mix}$ ); therefore it is possible to describe such matter–energy tensors as perfect fluid sources with  $\rho_K = T_0^{0K}$  and the isotropic pressure defined as  $p_K = -T_1^{1K}$ . The cosmological nature of each component can be evaluated by means of the respective equation of state (EoS)  $\omega_K = -T_1^{1K}/T_0^{0K}$ .

#### 6.1 4-D GR limit

If one considers solutions obtained along the standard KK GR limit, the related 4-D effective dynamics is described by well-known cosmological models. In particular, the solution (41) represents the standard radiation universe with  $p = 1/3\rho$ . In such a case the three matter–energy tensors collapse into one matter–energy source and, in particular, the curvature quintessence tensor and the mixed tensor, respectively, vanish. On the other side, the solution (42) represents a spatially flat FRW metric once the extra coordinate is fixed:

$$ds_{|\Sigma_y}^2 = N_0^2 y_0^2 dt^2 - A_0^2 y_0^{\frac{2\alpha}{\alpha-1}} t^{2\alpha} \left[ dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right]. \tag{46}$$

The induced-matter tensor plays, again, the role of the only one effective cosmological fluid. In fact, the other two quantities combine to give a cancelling result. The cosmological energy density scales as  $\rho = \rho_0 t^{-2}$ , with  $\rho_0 = \frac{3\alpha^2}{N_0^2 y_0^2}$ , while the barotropic factor is  $\omega = \frac{2 - 3\alpha}{3\alpha}$ . This result is in complete agreement with previous achievements on non-compactified KK gravity [49].

### 6.2 4-D cosmology from standard $f(R)$ gravity limit

We have seen that the attempt to get the standard 4-D  $f(R)$  gravity limit is frustrated by the higher equations number of our model. In fact, it is possible to get the right 4-D power law solution only if the extra-coordinate equation is neglected. When this equation enters the game the power law index  $n$  gets fixed. The behaviour of the three matter–energy tensors describing the top–down effect of higher dimensions and higher derivative terms on the 4-D gravity hypersurface confirm such a result. In fact, if we evaluate  $\omega_{\text{curv}}$  on the 4-D  $f(R)$  solution (43) we obtain

$$\omega_{\text{curv}} = \frac{1 + 7n + 6n^2}{3 - 9n + 6n^2}, \tag{47}$$

which, again, matches the already known result for this kind of model [17,18]. However, since the fifth equation selects the allowed values for  $n$ , we know that the admissible solution is indeed  $\delta = 0$ ,  $\beta = 0$ ,  $\sigma = 0$ ,  $\gamma = 0$ ,  $n = 5/4$  and  $\alpha = 1/2$ . The effective 4-D consequent cosmology shows  $\rho_{\text{IMT}} = \rho_{\text{Mix}} = 0$ ,  $\rho_{\text{curv}} = \frac{3}{4N_0^2 t^2}$  and  $\omega_{\text{curv}} = 1/3$ . As a matter of fact, we find a new radiation like universe with  $n = 5/4$ . The radiation component derives from the higher derivatives counter-terms, while the induced-matter tensor vanishes.

### 6.3 4-D Cylinder models

Cosmologies induced by cylinder solutions are quite interesting and allow different dynamics. In particular, the matter–energy sector, characterized by means of the three matter–energy sources of geometrical origin, assumes a variety of significant behaviours. A relevant aspect of these solutions is that the mixed tensor  $T_{\mu\nu}^{\text{Mix}}$  provides a cosmological constant-like source. In fact, as is possible to observe from the definition (29), in such a case (no extra-coordinate dependence) it is  $T_0^{0\text{Mix}} = T_i^{i\text{Mix}}$ , therefore  $\omega_{\text{Mix}} = -1$ . Since for each solution,  $\rho_{\text{Mix}} = T_0^{0\text{Mix}} \sim X_i(n)N_0 t^{-2}$ , with  $X_i(n)$  depending on the solution we consider, we obtain an inverse square law cosmological constant. This behaviour is favoured in string cosmologies [92] and in time-varying  $\Lambda$  theories that accommodate a large  $\Lambda$  for early times and a negligible  $\Lambda$  for late times [93]. On the other side, the effective barotropic factors of the other two matter–energy sources depend on the parameter  $n$  in relation to the solutions (45a)–(45c). In order to have a schematic portrait of these results we have plotted the behaviour of each  $\omega_K$  with respect to  $n$  in Fig. 1. It is possible to observe that the value of the gravity Lagrangian power index  $n$  determines the possibility to have different cosmological components. Both the standard matter sector and the dark energy one can be mimicked by the geometrically induced fluids according with previous results in this

sense obtained in the framework of IMT [79,94–96] and fourth-order gravity [31]. Very interesting is the coexistence of different regimes with the same value of  $n$ .

Together the behaviour of  $\omega_K$  we display, for all the components (see Fig. 2), the energy density value  $\rho_K$  and the overall sum of these quantities.<sup>5</sup> It is evident that energy conditions are violated for certain values of  $n$ . For example it is easy to observe that the weak energy condition,  $\rho_i \geq 0$ , for the first solution, is not satisfied by  $\rho_{\text{IMT}}$  and alternatively by the other two energy densities  $\rho_{\text{Curv}}$  and  $\rho_{\text{Mix}}$ . For the second solution we have again no  $n$  intervals where all components satisfy this energy prescription, since  $\rho_{\text{IMT}}$  is always quite well behaved but  $\rho_{\text{Curv}}$  and  $\rho_{\text{Mix}}$  are alternatively negative. Finally, for the third solution we have a small region around  $n \approx 1.5$  where all energy densities are positive, while the WEC is violated elsewhere. Although single matter–energy sources can violate the energy conditions, the total geometrized matter (dotted line in each panel of Fig. 2) satisfies the WEC for all the ranges of the parameter space in which cylinder solutions are defined. In addition, it has been recently showed that cosmological models with negative energy components can be suitably matched with observations if these “disturbing” quantities are coupled with greater magnitude of positive energy forms [97]. Therefore, the total top–down geometrized matter is physical and preserves conceptions about energy conditions in 4-D. Such a result is quite customary when dealing with induced matter theory models [27].

Finally, let us discuss the cosmological evolution of the four-dimensional sections determined by each of the solutions (45a)–(45c) and the behaviour of the extra dimension. Figure 3 allows one to acquire significant information as regards this behaviour describing the dependence of  $a(t, y)$  and  $\Phi(t, y)$  with respect to  $n$  in the different cases.

In order to display accelerating dynamics, a power law scale factor requires that the power index has to be negative or greater than 1. In particular, universes expanding at an accelerating rate need  $\alpha > 1$  if we consider the definition given in Eq. (38).

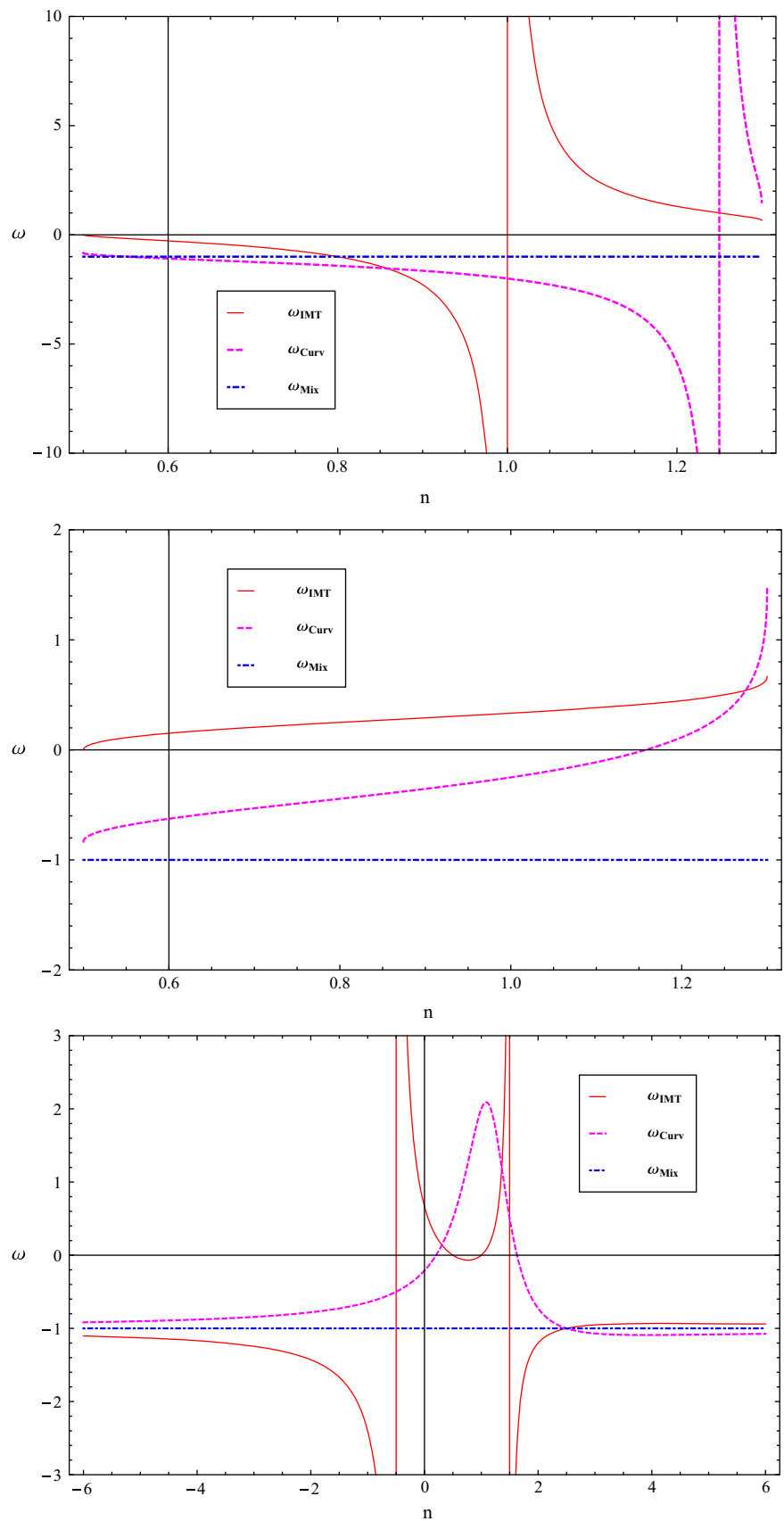
Actually, since our approach in the 4-D limit determines a multi-fluid model, the acceleration condition for the cosmological dynamics turns out to be characterized by the general relation for the scale factor,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_{\text{tot}} + 3p_{\text{tot}}).$$

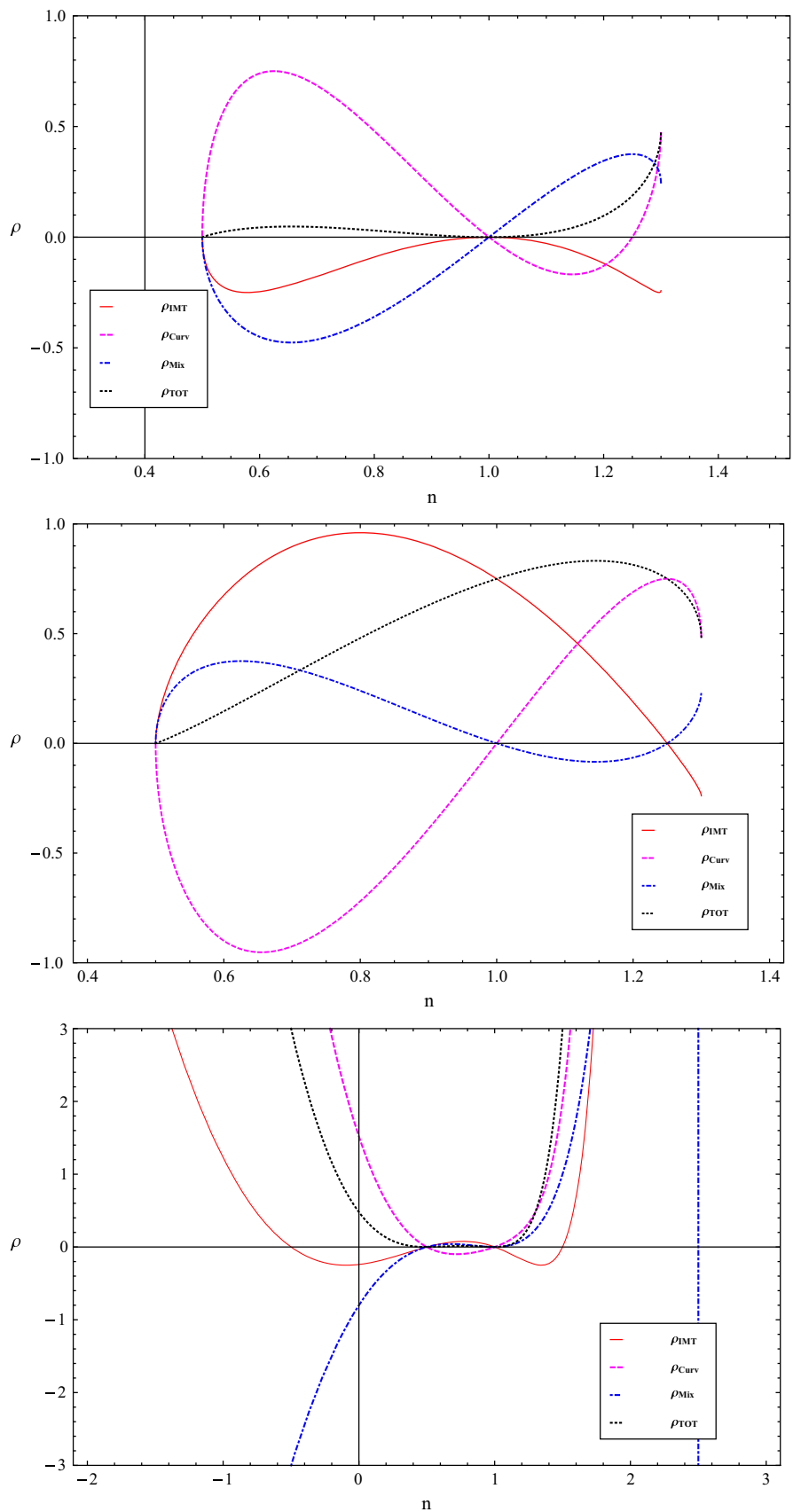
Therefore, barotropic fluids like ones outlined by our model, with  $\rho_{\text{tot}} = \sum_K \rho_K$  and  $p_{\text{tot}} = \sum_K p_K$ , imply that such a relation can be written as

<sup>5</sup> In order to achieve these plots we settle the quantity  $N_0^2 t^2 = 1$ .

**Fig. 1** The behaviour of the barotropic factors related to three matter–energy tensors that arise after the 5-D to 4-D reduction. The three *panels* refer to the EoS compartment with respect to  $n$  along each of the cylinder solutions (45a)–(45c) (from *top to down*)



**Fig. 2** Energy densities for the different species of matter–energy obtained in a 5D f(R)-gravity model when reduced to the standard 4-D spacetime. Curves labelled with  $\rho_{\text{tot}}$  describe the behaviour of the whole matter–energy cosmological budget; natural units are adopted. The upper panel refers to solution (45a), the middle one to (45b), finally the last one refers to the solution (45c)



$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_K (1 + 3\omega_K) \rho_K, \tag{48}$$

and the condition for accelerating cosmologies becomes  $\Upsilon = \sum_K (1 + 3\omega_K) \rho_K < 0$ . As one can see in Fig. 4 for our model the regions of the parameters space that fulfill this requirement exactly determine accelerating expansions (the deceleration parameter  $q = -\frac{\ddot{a}a}{\dot{a}^2}$  is negative). Let us remember that the observations suggest  $q \approx -0.5$  [98,99], albeit models considering power law solutions can also allow for higher limits, i.e.  $q \approx -0.3$  [100].

A peculiar aspect of our model is represented by the possibility that single energy densities can get negative defined values. Of course, as said above, the overall evolution is still physical since the effective energy density budget is always positive definite. This behaviour, however, can affect the effective cosmological evolution. For example, it can prevent the possibility to obtain models with speeding up evolution that accord with cosmological observations. In fact, it is possible to observe (cf. Figs. 1, 4) that there are regions of the parameter space where the induced cosmological fluids all assume negative barotropic factors, while the cosmological dynamics is still decelerating ( $q \geq 0$ ). On the other side, this peculiarity can also determine suitable conditions to frame models that experience a cosmic speeding up.

It is evident that, at this stage, our results can only be considered as a toy model achievement. Power law solutions can only describe limited stages of universe evolution. More general analyses are required in order to understand the real predictions of the model. In particular, the possibility to have a coexistence of negative defined equations of state with both decelerating dynamics and accelerating ones can represent an intriguing perspective to deepen in the framework of more general cosmological solutions.

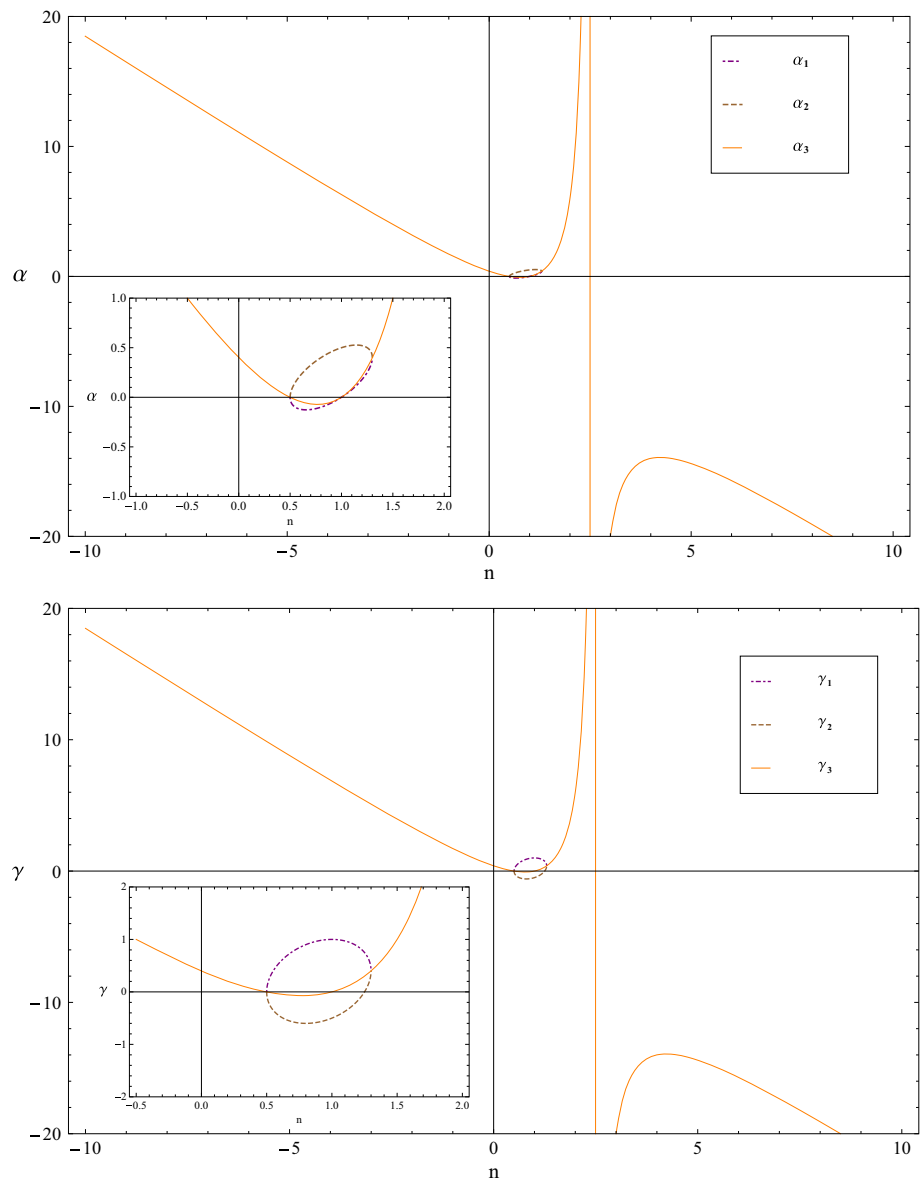
Let us discuss in some detail the results contained in Figs. 1, 2, 3, 4.

- Solution  $(\alpha_1, \gamma_1)$ -(45a) This solution is allowed in the interval  $\frac{1}{2} < n < \frac{13}{10}$ . Values of  $n < 1$  give three negative EoS fluids within the cosmic pie. The GR limit ( $n = 1$ ) implies a trivial behaviour with vanishing  $\rho_K$ . For  $n \sim 1.3$  we have two standard matter components and the cosmological constant-like source driven by  $T_{\mu\nu}^{Mix}$ . In such a case  $\rho_{IMT}, \rho_{Curv} > 0$  while  $\rho_{Mix} < 0$ , the whole cosmic mass-energy budget is positively defined as already said above. Looking to the cosmological dynamics one can notice that this solution implies a negative power law index for the scale factor within the interval  $\frac{1}{2} < n < 1$  while  $0 \leq \alpha < 1$  for  $1 < n < \frac{13}{10}$ . In accordance with Figs. 3 and 4 this behaviour indicates accelerated contracting universes in the first interval and standard matter-like dominated cosmologies in the second one. The extra

dimension is always decreasing, since  $0 < \gamma < 1$ . It is interesting to notice that the ordinary matter-like regime is achieved: in the interval  $1 < n < 1.25$  with two of the three fluids that behave as sources with a negative defined EoS (the other one is a stiffed fluid with  $\omega_{IMT} > 1$ ) and in the range  $1.25 < n < \frac{13}{10}$  with two standard matter-like sources ( $0 < \omega_{IMT} < 1$  and  $\omega_{Curv} > 1$ ) together the cosmological constant contributes. The divergence of the barotropic factor  $\omega_{Curv}$  for  $n = 1.25$  is regulated by the correspondent vanishing of  $\rho_{Curv}$ .

- Solution  $(\alpha_2, \gamma_2)$ -(45b) We have again  $\frac{1}{2} < n < \frac{13}{10}$ . For this solution, the scale factor shows a standard matter rate since always  $0 < \alpha < \frac{1}{2}$ , there is no acceleration ( $q(\alpha_2, \gamma_2) > 0$ ). The induced-matter component  $\rho_{IMT}$  is always a well-behaved thermodynamical fluid with  $0 \leq \omega_{IMT} \leq 2/3$ . The curvature induced sources can have a negative EoS or a positive one with  $-5/6 \leq \omega_{Curv} \leq 3/2$ . For values  $n \gtrsim 1.5$  we have two standard matter components and a cosmological constant-like source. Energy densities can be defined positive or negative. Around  $n \sim 1.2$  we have two standard matter-like sources with positive  $\rho_i$ , while  $\rho_{Mix}$  is negative,  $\rho_{tot}$  is always  $\geq 0$ . For  $n = 1$ , the two curvature induced sources collapse to a zero-valued energy density, while the induced-matter tensor plays the role of a radiation fluid with  $\omega_{IMT} = 1/3$  as in usual KK-GR cylinder models. The fifth dimension can experience a reduction when  $\frac{1}{2} < n < \frac{5}{4}$  and an increasing evolution for  $\frac{5}{4} < n < \frac{13}{10}$ , the case is static when  $\gamma = \frac{1}{2}, \frac{5}{4}$ .
- Solution  $(\alpha_3, \gamma_3)$ -(45c) Admissible values for  $n \in \mathbb{R} - \{5/2\}$ . This solution explicitly resembles the standard 4-D  $f(R)$ -gravity solution given in [53]. The scale factor and the extra dimension show the same dynamics since  $\alpha_3 = \gamma_3$ . There are two interesting regions around  $n \sim 0.2$  and  $n \sim 1.6$ . In the first case we have two standard matter components, dust-like and radiation-like, with  $\omega_{Curv} \approx 0, \omega_{IMT} \approx 0.26$  together a cosmological constant-like one with  $\omega_{Mix} = -1$ . The correspondent scale factor is, however, non-accelerating since  $q$  is positive. The second region shows a dust-like source with  $\omega_{Curv} \approx 0$  and two dark-energy-like components, one of them phantom-like with  $\omega_{IMT} \approx -2.6$ . In such a case the 4-D section shows an accelerating expansion since  $q < 0$  ( $q_0 = -0.5$  for  $n \approx 1.69$ ) as well as an increasing extra dimension. The zeros of  $\omega_{IMT}$  (dust-like behaviour of this source) are trivial since the corresponding value of the energy density is vanishing. Actually, this solution allows for a plethora of different behaviours. In particular, expanding solutions with no acceleration are obtained for  $-\frac{1}{2} \leq n < \frac{1}{2}$  and  $1 < n \leq \frac{3}{2}$  with two standard matter-like sources plus the effective cosmological constant contribute given by  $T_{\mu\nu}^{Mix}$ ; on the other side acceler-

**Fig. 3** The scale factor and the extra coordinate power law indices vs.  $n$  for the three cylinder solutions obtained along the 5-D to 4-D reduction of higher-order gravity power law models. Both plots contain a detail that enlightens the behaviour in the region  $n \approx 1$



ated expansions are given by models with  $n < -\frac{1}{2}$  and  $\frac{3}{2} < n < \frac{5}{2}$ . In these intervals  $T_{\mu\nu}^{\text{Curv}}$  plays the role of a baryonic component and the other two matter-energy sources act as dark-energy-like sources. The energy densities have, again, both negative and positive values with  $\rho_{\text{tot}} \geq 0$ . For  $n > 1.5$  we have all positive  $\rho_K$  with  $\omega_K \sim -1$ , in such a case the three induced components behave all as cosmological constant-like sources.

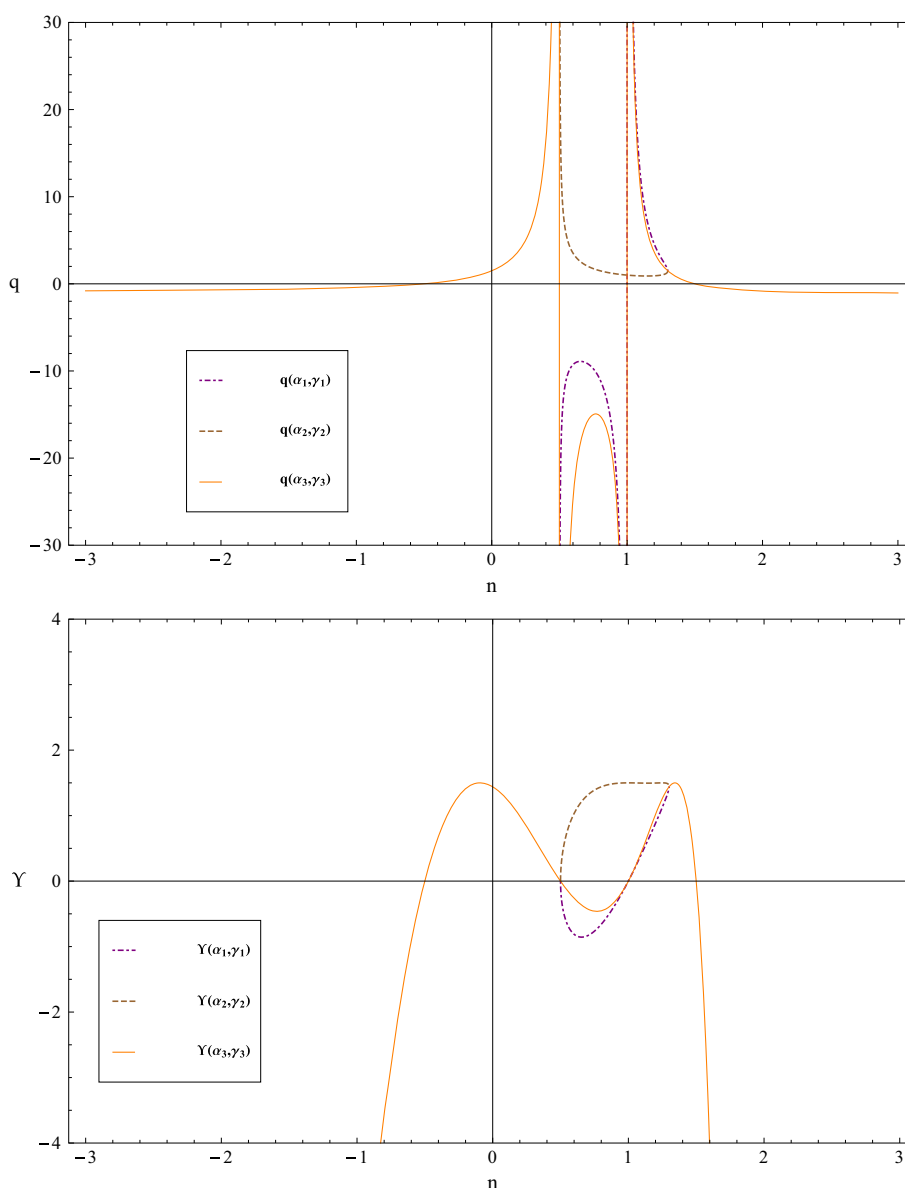
As a matter of fact, fourth-order gravity provides, in the cylindrical case, a completely different framework from the standard induced-matter theory. The standard scheme implies, in the 4-D limit, that induced matter behaves as a radiation-like fluid ( $p = \rho/3$ ) [27] according with the result obtained with an initial condition of cylindricity by Kaluza [45]. In the case of higher-order Kaluza–Klein mod-

els the energy-momentum tensors entering the 4-D induced field equations can describe more general matter sources. Therefore one can obtain dynamically significant cosmologies also in the presence of a Killing symmetry for the extra coordinate. Of course, the significance of such solutions has to be corroborated with cosmological and astrophysical data.

### 7 Conclusions

We discussed the possibility to develop a higher-order and higher dimensional gravity model. Within such an approach, after the 5-D to 4-D reduction procedure, it has been possible to suggest a new interpretive scheme for the 4-D cosmological phenomenology. Specifically, we have considered a vac-

**Fig. 4** The deceleration parameter and the  $\Upsilon$  term, which enters the scale factor acceleration equation, vs.  $n$ . Again the three *plot* are related to the cylinder solutions (45a)–(45c)



uum 5-D  $f(R)$ -gravity model. The 4-D effective reduction provides, in this case, a GR-like cosmology characterized by means of three induced-matter–energy tensors of geometrical origin. Within this matter geometrization paradigm the cosmological fluid sources can be obtained considering the standard induced matter tensor of non-compactified KK theory  $T_{\mu\nu}^{IMT}$ , the curvature quintessence tensor  $T_{\mu\nu}^{Curv}$  coming from higher-order derivative terms of  $f(R)$ -gravity and  $T_{\mu\nu}^{Mix}$ , a mixed quantity that arises from both higher-dimensional and higher-order curvature counter-terms. This last quantity embodies the 5-D to 4-D reduction of the two combined approaches. We have obtained the complete field equations formalism and in order to check the prediction of the model we have worked out a routine to search for power law solutions in the case  $f(R) = f_0 R^n$ . The whole

scheme provides the right KK-GR limit once  $f(R) \rightarrow R$ . In such a case, on shell, radiation dominated cosmologies ( $p = 1/3\rho$ ) are obtained if no dependence on the extra coordinate is taken into account. More generally, the KK-GR limit furnishes solutions that agree with similar results already present in the literature. In particular, the curvature induced-matter–energy tensors vanish, while the induced-matter tensor, which collects all metric terms depending on extra coordinate, turns out to be the only source of the 4-D field equation.

Relaxing the hypotheses on  $n$ , i.e. by assuming a generic power law fourth-order gravity model, it is possible to search for more general solutions. As a first preliminary step, cylindricity has been taken into account. A more general study has been left to a dedicated forthcoming publication. Dif-



ferent from the standard KK-GR theory, the combination of fourth-order gravity with higher dimensions implies, in such a case, non-trivial solutions that can give rise to interesting cosmologies. We have obtained a set of solutions, parameterized by the  $f(R)$  power index  $n$ , that determine a variety of possible barotropic fluids in the 4-D mass–energy sector. In relation to the value of  $n$  one can have both standard matter components and dark-energy-like ones. The cylinder condition implies that  $T_{\mu\nu}^{\text{Mix}}$ , according to its definition, behaves as an effective cosmological constant-like term with  $\omega_{\text{Mix}} = -1$ . Actually, in such a case, the energy density  $\rho_{\text{Mix}}$  depends on time with an inverse square law. This fact suggests the intriguing possibility to have a time varying cosmological constant. The cosmological behaviour of the 4-D sections shows that it is possible to have both accelerating and non-accelerating expansions. In particular, non-accelerated expansion can be obtained also in the presence of subdominant dark-energy-like components. On the other side, a dark-energy-dominated speeding up universe can easily be framed generalizing results obtained in the standard 4-D  $f(R)$ -gravity. The interesting aspect of this last solution is that geometry can provide both standard matter-like sources and dark-energy-like ones, and one of these can mimic a phantom fluid. The extra dimension follows an independent evolution, that, in the third solution, is equivalent to one of the scale factors. Therefore, in such a case, expanding accelerated evolutions are characterized also by a growing fifth coordinate.

The price to pay for geometrizing matter–energy sources is that cosmological energy densities can assume negative values violating the WEC. We have shown that although this is true for some values of  $n$ , also in interesting regions of the parameters space, the whole mass–energy budget  $\rho_{\text{tot}}$  is always positive definite and, therefore, total top–down geometrized matter is physical.

The fact that geometrically induced-matter–energy tensors could, alternatively, play the role of standard matter components and of dark-energy-like sources can represent an intriguing perspective in order to interpret cosmological observations. In particular, this achievement seems to deserve more studies in the direction of dark-energy and dark-matter interpretation. Of course, more insights are necessary in order to evaluate such a theoretical scheme. In particular, the matter sector (i.e. matter-like components), which has been satisfactory checked within standard IMT [94,95,101], requires a careful investigation since matter properties should be completely induced by gravity. In addition, model predictions also have to be checked against cosmological and astrophysical data.

Furthermore, if  $f(R)$  Kaluza–Klein gravity determines, in the 5-D to 4-D reduction of cylinder solutions, cosmological models with a suitable top–down matter geometrization, in principle, more general solutions could allow one to develop

cosmological models with a wider phenomenology. On the other side, the fifth dimension role requires a careful analysis when the extra-coordinate Killing symmetry is discarded. In fact, if the cylinder condition is relaxed one naturally gets into the game of a length scale related with the extra coordinate. In this sense, it has been shown that 5-D  $f(R)$  gravity suggests that, eventually, there is a strict interconnection between the fifth dimension dynamics and the higher derivative counter-terms.

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