

Robinson–Trautman solution with nonlinear electrodynamics

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Abstract Explicit Robinson–Trautman solutions with an electromagnetic field satisfying nonlinear field equations are derived and analyzed. The solutions are generated from the spherically symmetric ones. In all studied cases the electromagnetic field singularity is removed while the gravitational one persists. The models resolving the curvature singularity in spherically symmetric spacetimes could not be generalized to the Robinson–Trautman geometry using the generating method developed in this paper, which indicates that the removal of a singularity in the associated spherically symmetric case might be just a consequence of high symmetry. We show that the obtained solutions are generally of algebraic type II and reduce to type D in spherical symmetry. Asymptotically they tend to the spherically symmetric case as well.

1 Introduction

Robinson–Trautman spacetimes represent an important class of geometries defined by the requirement of possessing an expanding, nontwisting, and nonshearing null geodesic congruence [1–4]. This class contains non-spherical generalizations of black holes without rotation (Schwarzschild solution is a special case in this class, unlike the Kerr black hole). In general, Robinson–Trautman spacetimes do not possess any Killing vectors, thus providing solutions devoid of symmetry. Another important aspect of this family is the presence of gravitational radiation connected with the dynamical nature of general solutions within this class. Many global properties of this class in four dimensions have been studied analytically, especially in the last 25 years. In particular, based on the only nontrivial Einstein equation (the so-called Robinson–Trautman equation) the asymptotic evolution and global structure of vacuum Robinson–Trautman

spacetimes of type II with spherical topology were investigated by Chruściel and Singleton [5–7]. They showed that the characteristic initial value problem for generic, arbitrarily strong smooth initial data converges asymptotically in retarded time to the corresponding Schwarzschild metric near its future horizon. Extensions across such a “Schwarzschild-like” future event horizon are only of a finite order of smoothness. These results were later extended to cover the presence of a cosmological constant which naturally modifies the asymptotic behavior and the solutions tend to Schwarzschild–(anti-)de Sitter cases [8,9]. Finally, the Chruściel–Singleton analysis was used for Robinson–Trautman spacetimes admitting additionally pure radiation [10,11], where the asymptotic state is described by spherically symmetric Vaidya–(anti-)de Sitter metric. The location of the past quasilocal horizon (which cannot be determined as an event horizon due to the impossibility to extend the solutions to past null infinity) together with the proof of its existence and uniqueness for the vacuum Robinson–Trautman solutions has been studied by Tod [12]. Later, Chow and Lun [13] further analyzed the properties of this horizon and made a numerical study of both the horizon equation and the Robinson–Trautman equation. The analytic results were generalized to a nonvanishing cosmological constant [14]. The relation between the asymptotic momentum and the local horizon curvature in the Robinson–Trautman class was used in the analytic explanation of an “antikick” appearing in numerical studies of an asymmetric binary black hole merger [15]. Recently, the solution with minimally coupled free scalar field was derived in [16] and shown to possess a singularity which is initially naked and only later gets covered by a horizon.

Type D vacuum solutions of the Robinson–Trautman family contain, apart from the Schwarzschild solution, also the C-metric [17] representing a uniformly accelerated pair of black holes. The C-metric is as well a natural future asymptotic in the case of non-smooth initial data for a certain subclass of Robinson–Trautman spacetimes [18]. By leaving out

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the usual spherical topology assumption one can obtain a special case of the Kasner metric [3]. Including a null radiation source into type D leads to, e.g., a Vaidya solution or Kinnerley's rocket [19], which is interpreted as an object propelled by emitting directional null radiation (these "rocket" solutions can be generalized to type II). All type D null radiation metrics are known [20].

Vacuum type N solutions which correspond to spacetimes containing just gravitational radiation have a singularity at each wave surface which combines into singular lines [3,4]. The general solution was given by Foster and Newman [21] and they are frequently used to form various sandwich-type waves [22]. There are no pure radiation solutions of type N.

There is also a higher-dimensional generalization of Robinson–Trautman spacetimes (containing aligned pure radiation and a cosmological constant) which, however, lacks the rich dynamics present in four dimensions [23]. The existence of horizons (which generally have a richer topology than in four dimensions) in this case was subsequently analyzed in [24]. These higher-dimensional solutions were as well generalized to admit a source in the form of p-form fields [25].

Robinson–Trautman spacetimes with Maxwell field were already derived in the founding paper of this family of solutions [2]. Later they were studied more extensively [26–28]. Among the special cases of these solutions belong the Reissner–Nordström black hole and the charged C-metric, but it was also shown that this subfamily suffers from non-well-posedness [29]. The higher-dimensional generalization of Robinson–Trautman spacetimes coupled to Maxwell electrodynamics was derived in [30].

Nonlinear electrodynamics (NE) was founded almost a century ago and used mainly as a solution to the problem of the divergent field of a point charge in the vicinity of its position (see e.g. [31]), also giving a satisfactory self-energy of the charged particle. The best-known and most frequently used form of the theory was introduced already in 1934 by Born and Infeld [32]. A nice overview with a lot of useful information was given in the book by Plebański [33]. More recently, the attention to nonlinear electrodynamics was increased thanks to the discovery that string-generated corrections to the Maxwell field have the form of original Born–Infeld theory [42]. However, it was also noted that the electric displacement vector in Born–Infeld model has two possible values for a single value of the electric field. This non-uniqueness was soon solved by adding the so-called Hoffmann term [34] in the Born–Infeld Lagrangian. Additionally, this new model was also used to resolve the spacetime singularity in the spherically symmetric case [35].

Later, other NE models were considered for both solving the point charge singularity and resolving the spacetime singularity [36,37]. Note, however, that these results were obtained in the Hamiltonian framework and their Legendre

transform is far from trivial [38]. One important example is the so-called Bardeen black hole [39], which was originally discovered as a regular solution with a horizon generated by a certain stress energy tensor. Only later this source was interpreted as being created by a specific model of NE [40], which, however, does not have a Maxwell limit (in the weak field regime).

Many spherically symmetric solutions of Einstein equations with NE were studied, mostly with the Born–Infeld Lagrangian [41–43], logarithmic Lagrangians [44,45], square root lagrangian [46–48], power Maxwell models [49–52], and other forms [54–56]. These solutions are mostly thought of as a model of a charged particle. However, since General Relativity is a nonlinear theory, one should study the stability of such models nonperturbatively.

Our aim here is to derive Robinson–Trautman solutions coupled to several forms of NE Lagrangians in order to investigate the influence of nonsphericity in this family on the results obtained previously in highly symmetric situations. Since spherically symmetric solutions are a special subcase of Robinson–Trautman spacetimes it is a natural family to consider for nonlinear stability investigations.

2 Vacuum Robinson–Trautman solutions and field equations

The general form of a vacuum Robinson–Trautman spacetime can be represented by the following line element [1–4]:

$$ds_{RT}^2 = -2H du^2 - 2 du dr + \frac{r^2}{P^2} (dx^2 + dy^2), \quad (1)$$

$$\text{with } 2H = \Delta(\ln P) - 2r(\ln P)_{,u} - 2m/r - (\Lambda/3)r^2,$$

$$\Delta \equiv P^2(\partial_{xx} + \partial_{yy}), \quad (2)$$

and where Λ is the cosmological constant. The metric depends on two functions, $P(u, x, y)$ and $m(u)$, which satisfy the nonlinear fourth-order PDE (the so-called Robinson–Trautman equation)

$$\Delta\Delta(\ln P) + 12m(\ln P)_{,u} - 4m_{,u} = 0. \quad (3)$$

The function $m(u)$ might be set to a constant by a suitable coordinate transformation for the vacuum solution. However, for a null radiation field source which is aligned with the principal null direction the solution represents a generalization of a Vaidya spacetime with time-dependent $m(u)$.

The spacetime is defined by a geodesic, shear-free, twist-free, and expanding null congruence generated by $\mathbf{k} = \partial_r$. The coordinate $r \in (0, \infty)$ is an affine parameter along this congruence, $u \in (u_0, \infty)$ is a retarded time coordinate (the initial data for this class of spacetimes are specified on the

null hypersurface $u = u_0$), and x, y are spatial coordinates spanning a transversal 2-space, which has a Gaussian curvature (for $r = 1$),

$$K(u, x, y) \equiv \Delta(\ln P). \tag{4}$$

If we select the transversal 2-spaces to be topological spheres (a standard assumption in this class) then x, y are real stereographic-type coordinates on a deformed spheres $r = \text{const}, u = \text{const}$. For general fixed values of r and u , the Gaussian curvature is K/r^2 so that, as $r \rightarrow \infty$, they become locally flat.

3 Robinson–Trautman and static spherically symmetric solutions

In this section we will develop a generating method to obtain a Robinson–Trautman solution coupled to NE based on a static spherically symmetric spacetime solution with NE. In order to have a straightforward generalization, we use the Kerr–Schild-type modification of the vacuum Robinson–Trautman metric. First, we find the Einstein equations for the Robinson–Trautman spacetime coupled to NE and then compare them to SSS ones. Based on the similarities we formulate a theorem summarizing the generating method.

We consider the following action, describing nonlinear electrodynamics coupled to gravity:

$$S = \int d^4x \sqrt{-g} [\mathcal{R} - 2\Lambda + \mathcal{L}(F)] \tag{5}$$

where \mathcal{R} is the Ricci scalar for the metric $g_{\mu\nu}$ and the Lagrangian of the nonlinear electromagnetic field $\mathcal{L}(F)$ is an arbitrary function of the electromagnetic field invariant $F = F_{\mu\nu}F^{\mu\nu}$ constructed from a closed Maxwell 2-form $F_{\mu\nu}$. We use units in which $c = \hbar = 8\pi G = 1$. By applying the variation with respect to the metric for the action (5), we get the Einstein equations,

$$G^\mu{}_\nu = T^\mu{}_\nu - \Lambda \delta^\mu{}_\nu. \tag{6}$$

The energy momentum tensor generated by the NE Lagrangian is given by

$$T^\mu{}_\nu = \frac{1}{2} \{ \delta^\mu{}_\nu \mathcal{L} - 4 (F_{\nu\lambda} F^{\mu\lambda}) \mathcal{L}_F \} \tag{7}$$

and the modified Maxwell (nonlinear electrodynamics) field equations are then given in the following form:

$$\partial_\mu (\sqrt{-g} \mathcal{L}_F F^{\mu\nu}) = 0 \tag{8}$$

in which $\mathcal{L}_F = \frac{d\mathcal{L}(F)}{dF}$.

Our metric ansatz is

$$ds^2 = ds_{RT}^2 - Q(u, r) du^2, \tag{9}$$

where we modify the vacuum Robinson–Trautman metric, i.e. (1), by adding the function $Q(u, r)$ to the g_{uu} component. This corresponds to the general Kerr–Schild metric form given by the background vacuum Robinson–Trautman geometry $g_{\mu\nu}^{RT}$ and the null, shear-free, twist-free, and geodesic vector k

$$g_{\mu\nu} = g_{\mu\nu}^{RT} - Q(u, r) k_\mu k_\nu. \tag{10}$$

The Einstein tensor has now more nontrivial components compared to the original Robinson–Trautman metric (1),

$$G^u{}_u = G^r{}_r = -\Lambda + \frac{Q_{,r}}{r} + \frac{Q}{r^2}, \tag{11}$$

$$G^x{}_x = G^y{}_y = -\Lambda + \frac{1}{2} Q_{,rr} + \frac{Q_{,r}}{r}, \tag{12}$$

$$G^r{}_u = {}^{RT}G^r{}_u - \frac{1}{r} [(\ln P)_{,u} \{ r Q_{,r} - 2Q \} + Q_{,u}], \tag{13}$$

where

$${}^{RT}G^r{}_u = -\frac{1}{2r^2} \{ \Delta \Delta(\ln P) + 12m(\ln P)_{,u} - 4m_{,u} \}.$$

We assume the following specific Maxwell 2-form in the coordinates of (9):

$$\mathbf{F} = F_{\mu\nu} dx^\mu dx^\nu = E(u, r) du \wedge dr \tag{14}$$

where the electromagnetic invariant $F = F_{\mu\nu}F^{\mu\nu}$ for the above Maxwell 2-form simplifies to $-2E^2$. Then from (8) and the metric (9) one can find the dynamical equation for the electromagnetic field

$$\frac{r^2}{P(u, x, y)^2} \mathcal{L}_F F_{ur} = F_0(x, y), \tag{15}$$

which can be solved by

$$\mathcal{L}_F F_{ur} = \frac{q(u)^2}{r^2}, \tag{16}$$

in which $P(u, x, y)^2 F_0(x, y) = q(u)^2$ in order to satisfy the assumed form of \mathbf{F} (14). The energy momentum tensor given by (7) can then be expressed in diagonal form,

$$T^\mu{}_\nu = \text{diag} \left\{ \frac{\mathcal{L}}{2} - F \mathcal{L}_F, \frac{\mathcal{L}}{2} - F \mathcal{L}_F, \frac{\mathcal{L}}{2}, \frac{\mathcal{L}}{2} \right\} \tag{17}$$

for our form of Maxwell field (14) and with respect to the coordinates (u, r, x, y) of the metric (9).

Let us note that the assumption (14) necessarily leads to $F_{\mu\nu}F^{\mu\nu} \neq 0$ and $F_{\mu\nu}^*F^{\mu\nu} = 0$ (where $*$ means the Hodge

dual) as in the static spherically symmetric cases considered below. This also means that our field is non-null and purely electric. At the same time we do not need to consider a generalization of the Lagrangian that would additionally contain terms dependent on the second invariant.

Now, we will turn our attention to the general form of the previously derived static spherically symmetric (SSS) solutions with NE. The metric has the following form encompassing all the models that we wish to generalize:

$$ds^2 = - \left[1 - \frac{\Lambda}{3}r^2 + f(r) \right] dt^2 + \frac{dr^2}{\left[1 - \frac{\Lambda}{3}r^2 + f(r) \right]} + r^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{18}$$

Here, we prefer not to introduce the null coordinate in order to keep the correspondence with the published spherically symmetric models. In this case one can of course as well use the Kerr–Schild form to introduce the function f into the background Minkowski/(anti-)de Sitter metric.

We obtain these nonzero components of the Einstein tensor for the line element (18):

$$G^t_t = G^r_r = -\Lambda + \frac{f,r}{r} + \frac{f}{r^2}, \tag{19}$$

$$G^\theta_\theta = G^\phi_\phi = -\Lambda + \frac{1}{2}f,rr + \frac{f,r}{r}. \tag{20}$$

The Maxwell 2-form has the following form, resembling (14):

$$\mathbf{F} = \tilde{E}(r)dt \wedge dr, \tag{21}$$

although the electromagnetic field is static now (like before, the field invariant simplifies considerably, $F = F_{\mu\nu}F^{\mu\nu} = -2\tilde{E}^2$). From the modified Maxwell equation (8) and the metric (18), we find

$$\mathcal{L}_F F_{tr} = \frac{q_0^2}{r^2}, \tag{22}$$

where q_0 is a constant. One can easily calculate the energy momentum tensor in this case and recovers the diagonal form (17) but now with respect to the coordinates (t, r, θ, ϕ) of the metric (18).

We can see the structural similarity between the set of equations (11), (12) and (19), (20). This suggests trying to generalize the SSS solutions with NE source to the Robinson–Trautman geometry via this similarity. Note that this similarity is not analyzed based on some coordinate transformation but purely from the perspective of a coincidence between the forms of the differential equations. Of course, this similarity has an underlying reason in the fact that Robinson–Trautman metrics are generalizations of spherically symmetric models but this is not important for the following observations. Namely, the solution $f(r)$ of Eqs. (19) and (20) can evidently be transformed into a particular solution $Q(u, r)$ of equations (11) and (12) by promoting the

integration constants in f into functions of the coordinate u . However, one has to ensure that the newly constructed function $Q(u, r)$ satisfies the additional equation $G^r_u = 0$ coming from combining corresponding components of the Einstein tensor (13) and energy momentum tensor (17) into the field equations (6). This equation is the basic constraint on generalizing SSS solutions with NE into the Robinson–Trautman case.

We summarize the method in the following theorem.

Theorem 1 Consider the SSS solution of the Einstein equations coupled to an arbitrary NE model which is given by a line element in the form of (18) and a Maxwell field (21) with the specific function $f(r)$. This SSS solution can be generalized to a Robinson–Trautman solution coupled to the same NE model with metric (9) and Maxwell field (14). Here the function $Q(u, r)$ is obtained from $f(r)$ by promoting the integration constants (appearing in f) into functions of u , provided the additional constraint equation

$$(\ln P)_{,u} \{rQ_{,r} - 2Q\} + Q_{,u} = 0 \tag{23}$$

is satisfied for such $Q(u, r)$.

4 Explicit examples

Now, we will present several important forms of NE Lagrangians and the associated solutions for both an electromagnetic field and a Robinson–Trautman metric (giving the form of the metric function $Q(u, r)$). For each model we first solved the spherically symmetric equations (19) and (20) to obtain the most general function $f(r)$, then we used Theorem 1 to generate $Q(u, r)$ with which we try to solve the constraint (23). We do not present the original function $f(r)$ since it can easily be read off from $Q(u, r)$ by putting all functions of u to constants. Finally, we specify in which references one can find the spherically symmetric case.

We plotted the profile of Lagrangian (Fig. 1) for all the models considered below and the electromagnetic field invariant F (Fig. 2) in the case of static spherically symmetric solutions (alternatively it can be viewed as a profile of invariant on $u = \text{const}$ hypersurface). The evolution of the invariant for the Robinson–Trautman solutions is presented in three-dimensional plots in the appendix (Figs. 3, 4, 5, 6, and 7).

4.1 Maxwell

As a starting point we briefly show the linear Maxwell case which generalizes the standard Reissner–Nordström solution,

$$\mathcal{L}(F) = -F. \tag{24}$$

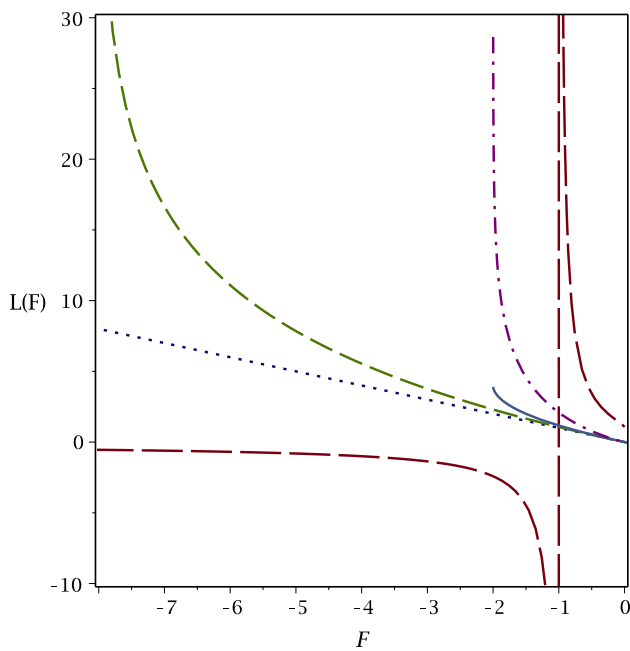


Fig. 1 The plot of all considered Lagrangians for NE: Maxwell (*dotted*), Born–Infeld (*solid line*—it is defined only for restricted values of F), logarithmic model (*dashed*), New Lagrangian 1 (*dash-dotted*), New Lagrangian 2 (*long-dashed*). The free parameter was fixed to following value: $\beta = 1$. Evidently, all the nonlinear Lagrangians have steeper growth with increasing absolute value of F than the Maxwell model and as well either diverge for certain finite values of F or fail to be defined (Born–Infeld). Except for the New Lagrangian 2 model the value at $F = 0$ is the same for all models

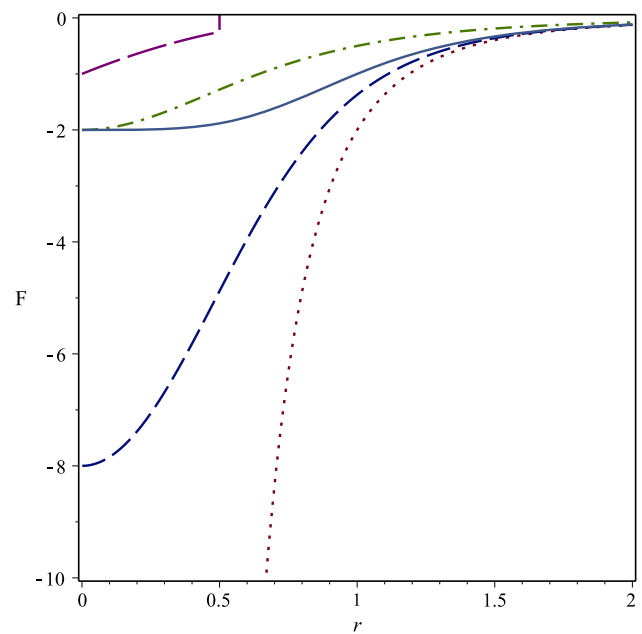


Fig. 2 The plot of the electromagnetic field invariant F for the Maxwell (*dotted*), Born–Infeld (*solid line*), logarithmic (*dashed*), New Lagrangian 1 (*dash-dotted*), New Lagrangian 2 (*long-dashed*) models. The free parameter was fixed at the value $\beta = 1$. We have also fixed the function q at a constant $q(u) = 1$, which can be considered as a $u = \text{const}$ slice of the complete three-dimensional plots given in the appendix. The graph corresponding to New Lagrangian 2 is limited only to the coordinate range where the model satisfies energy conditions. Only in the Maxwell case the invariant F diverges at the origin

Applying the method described in Sect. 3 and summarized in Theorem 1 we obtain this solution on a Robinson–Trautman background,

$$Q(u, r) = \frac{q(u)^4}{r^2} + \frac{C_1(u)}{r}. \tag{25}$$

If we put this form into the field equation constraint (23), we find

$$\begin{aligned} \Delta\Delta(\ln P) + 12 M(\ln P)_{,u} - 4 M_{,u} &= 0, \\ q P_{,u} - q_{,u} P &= 0, \end{aligned} \tag{26}$$

where

$$M = m(u) - \frac{C_1(u)}{2}. \tag{27}$$

This result is a special case of algebraic type II Einstein–Maxwell spacetimes analyzed in [3] where a theorem (Theorem 28.3 in the reference) for generating these solutions from the vacuum Robinson–Trautman spacetime is presented.

4.2 Born–Infeld

As was mentioned before, NE started with this model of dynamics and it has been studied widely in three, four, and

higher dimensions; also their physical properties were analyzed in detail. The famous Born–Infeld Lagrangian has the following form [32]:

$$\mathcal{L}_{BI} = 4\beta^2 \left(1 - \sqrt{1 + \frac{F}{2\beta^2}} \right), \tag{28}$$

β is the critical field length. When $\beta \rightarrow \infty$ this Lagrangian goes to a Maxwell form, which is as well recovered in the weak field regime.

Applying the method described in Sect. 3 we obtain this solution on a Robinson–Trautman background:

$$F = -\frac{2q(u)^4}{r^4 + q(u)^4/\beta^2}. \tag{29}$$

As expected, the electric field remains regular at $r = 0$. The Lagrangian (28) is only defined for field invariant $F \geq -2\beta^2$ (see as well Fig. 1) which corresponds to $r \geq 0$ according to (29) so it covers the whole range of coordinates. The metric function corresponding to the NE source becomes

$$Q(u, r) = \frac{C_1(u)}{r} + \frac{2\beta^2}{3}r^2 - \frac{2\beta^2}{r} \int \sqrt{r^4 + q(u)^4/\beta^2} dr. \tag{30}$$

If we put these forms in the field equations we will find the same restrictions as for Maxwell theory (26). However, in this case the constraint (23) enforces $C_1(u) = 0$ and so $M = m$.

Such a solution is a straightforward generalization of the SSS solutions given for example in [53,54]. This shows that there is indeed a solution in the Robinson–Trautman class that settles down smoothly to a known spherically symmetric one when the function P attains its special form describing the geometry on a sphere (this is elaborated in Sect. 5). Then, from (26) one immediately concludes that $q = \text{const}$.

4.3 Logarithmic form of nonlinear electrodynamics theory

The logarithmic form of the Lagrangian can be written

$$\mathcal{L}_{LNE} = -8\beta^2 \ln \left(1 + \frac{F}{8\beta^2} \right). \tag{31}$$

In [44] it was used to present a model of a point particle without divergence in electromagnetic field. However, even with vanishing “Schwarzschild mass” and finite electromagnetic field the curvature singularity at the origin is present. For special value of β the horizon radius shrinks to zero and a so-called black point is created (see, e.g., [57,58] for their previous occurrences).

Applying the method described in Sect. 3 we obtain

$$F = -\frac{8\beta^2}{q(u)^4} \left(\beta r^2 - \sqrt{\beta^2 r^4 + q(u)^4} \right)^2. \tag{32}$$

The presence of a curvature singularity can be understood from Lagrangian (31), since it has a singularity for the field invariant value $F = -8\beta^2$, which corresponds to $r = 0$ according to (32) and so this Lagrangian provides a singular source for the Einstein equations. The Robinson–Trautman geometry is specified by the following metric function:

$$\begin{aligned} Q(u, r) = & \frac{C_1(u)}{r} + \frac{20}{9}\beta^2 r^2 - 4\beta\sqrt{r^4\beta^2 + q(u)^4} \\ & - \frac{4}{3}\beta^2 r^2 \ln \left(\frac{-2r^4\beta^2 + 2\beta r^2\sqrt{r^4\beta^2 + q(u)^4}}{q(u)^4} \right) \\ & + \frac{16\beta^3}{r} \int \frac{r^4}{\sqrt{r^4\beta^2 + q(u)^4}} dr. \end{aligned} \tag{33}$$

If we put these Q and F into the constraint (23) we recover conditions (26) and (27).

This solution is again a straightforward generalization of the SSS solutions given in [44]. The results concerning the spherically symmetric limit are the same as in the previous analysis of the Born–Infeld example.

4.4 New Lagrangian 1

Now we will consider another form of dynamics for NE given by the following Lagrangian [59]:

$$\mathcal{L}_{New1} = -\frac{2}{\alpha^4} \ln \left(1 - \alpha^2 \sqrt{-F} \right) - \frac{2\sqrt{-F}}{\alpha^2}, \tag{34}$$

which has the correct Maxwell limit in both the weak field regime and for $\alpha \rightarrow 0$. The generating method from Sect. 3 yields the solution

$$F = -\frac{2q(u)^4}{(r^2 + q(u)^2\beta^2)^2} \tag{35}$$

in which $\beta^2 = \sqrt{2}\alpha^2$. Again, the field is regular; however, the Lagrangian has a singularity for $F = -\alpha^{-4}$ (see Fig. 1), which is attained at $r = 0$. Note that in this case one needs to select a negative root when solving for $E(r, u)$ in order to satisfy the field equations. The metric function becomes

$$\begin{aligned} Q(u, r) = & \frac{C_1(u)}{r} - \frac{2}{3} \frac{q(u)^2}{\beta^2} - \frac{4}{3} \frac{q(u)^3}{r\beta} \arctan \left(\frac{r}{q(u)\beta} \right) \\ & - \frac{2r^2}{3\beta^4} \ln \left(\frac{r^2}{r^2 + q(u)^2\beta^2} \right). \end{aligned} \tag{36}$$

If we put these forms in the field equations we will find the already known set of restrictions (26) and (27). This solution generalizes the SSS solutions given in [59] where this type of Lagrangian is used for the first time.

4.5 New Lagrangian 2

Finally, we consider the following NE Lagrangian [60]:

$$\mathcal{L}_{New2} = \frac{1}{\beta - \sqrt{-F}}, \tag{37}$$

which does not have a Maxwell limit in the weak field regime. However, models containing a square root (or arbitrary powers) and generally devoid of a Maxwell limit were extensively discussed before [46–52]. These examples can serve as an approximation of the true dynamics in the strong field regime. Additional motivation to consider a model without Maxwell limit is to test whether the generating method used here is not limited (via the constraint (23)) only to those NE models with correct Maxwell limit (which are all those considered so far).

The generating method from Sect. 3 yields the solution

$$F = -\frac{(\beta q(u) - r)^2}{q(u)^2} \tag{38}$$

and we need $q(u) > 0$, $\beta q(u) - 2r > 0$ to fulfill the weak energy condition so this solution cannot cover the whole coordinate range but can serve as an inner solution for small

r . Evidently, without introducing the parameter β we cannot satisfy the energy conditions at all (for more discussion and interpretation see [60]).

The geometry is in this case defined by

$$Q(u, r) = \frac{C_1(u)}{r} - \frac{\beta q(u)^2}{2} + \frac{q(u)}{2}r. \tag{39}$$

If we put these forms in the field equations we will find the already known set of restrictions (26) and (27). This solution is a straightforward generalization of the SSS solutions given for example in [60].

The Lagrangian (37) has a singularity at $\sqrt{-F} = \beta$ (see as well Fig. 1), which corresponds to $r = 0$ according to (38). This means that although the field invariant F is regular the curvature singularity is created via singular behavior of the Lagrangian.

5 Asymptotic behavior

The metric (1) admits coordinate freedom, as already noted by Robinson and Trautman,

$$u' = U(u), \quad r' = \frac{r}{U_{,u}}, \quad m' = \frac{m}{U_{,u}^3}, \quad P' = \frac{P}{U_{,u}}, \tag{40}$$

which can be used to set the mass M' to a positive constant by a proper function $U(u)$. In the case of nonlinear electrodynamics, when considering the metric (9) and modified Maxwell equations (8), one has to supplement (40) (now with $M = m - \frac{C_1}{2}$ instead of m) by transforming $q(u)$ as well,

$$q' = \frac{q}{U_{,u}}. \tag{41}$$

Note that one cannot set $q(u)$ and $M(u)$ to constants simultaneously since one has only a single function at hand. This means that $q(u)$, although looking like a physical charge based on (16), shares the interpretation problems with $m(u)$. Only now these difficulties are combined together.

Now we are ready to investigate the asymptotic behavior for our solutions, separately for the retarded time $u \rightarrow \infty$ and $r \rightarrow \infty$.

5.1 Asymptotic $u \rightarrow \infty$

One can use (40) to put M' to a constant in (27) for all models; exactly the form considered in Chruściel and Singleton’s [5–7] analysis of asymptotic behavior to recover the spherically symmetric final state and also exponentially fast decay of the dependence of the function P' on the new coordinate u' . Due to the last equation in (26), the asymptotic behavior of the function q' is the same as P' . Namely, it tends to a constant, which is completely consistent with the final state approaching the corresponding spherically symmetric solution.

5.2 Asymptotic $r \rightarrow \infty$

In the limit $r \rightarrow \infty$ the electromagnetic field vanishes for all cases and for the Born–Infeld, Logarithmic and New Lagrangian 1 models the asymptotic behavior is identical to the Maxwellian case. Analyzing the behavior of the function $Q(u, r)$ (after moving the C_1 term into a redefinition of the “mass” $m \rightarrow M$) we immediately see that in all cases the metric is locally asymptotically flat (or (anti-)de Sitter) as for the vacuum Robinson–Trautman solution. We have ignored the New Lagrangian 2 model due to its restricted coordinate range.

6 Horizons

Since in all cases the Robinson–Trautman spacetimes with arbitrary form of the electrodynamics still possess curvature singularities at $r = 0$, as one can confirm by computing the Kretschmann scalar (the singularity is milder for NE models with a regular EM field at the origin), we would like to know if it is covered by a horizon. Due to the dynamical nature of our spacetimes we will look for a quasilocal horizon. So we need to find a marginally trapped surface and one can select any of the most popular horizon definitions—the apparent [61], trapping [62] or dynamical horizon [63]. We will be looking for a horizon hypersurface given by the equation $r = \mathcal{N}(u, x, y)$ with the $u = u_0 = \text{const}$ slices

$$r = \mathcal{N}(u_0, x, y) = N(x, y), \tag{42}$$

being marginally trapped surfaces. We shall investigate the expansions of both null normals to this surface

$$\begin{aligned} \mathbf{k} &= \partial_r, \\ \mathbf{l} &= \partial_u + \left[\frac{P^2}{r^2}(N_{,x}^2 + N_{,y}^2) - H - \frac{Q}{2} \right] \partial_r \\ &\quad + \frac{P^2}{r^2}(N_{,x}\partial_x + N_{,y}\partial_y), \end{aligned} \tag{43}$$

which are normalized using $g(\mathbf{l}, \mathbf{k}) = -1$. The congruence generated by \mathbf{k} is the one defining Robinson–Trautman family and thus apart from being shear-free and twist-free it has a positive expansion everywhere, $\Theta_{\mathbf{k}} > 0$. So by requiring the other expansion to vanish, $\Theta_{\mathbf{l}} = 0$, we are looking for the past horizon according to the definition by Hayward [62]. One can express this expansion in the following form:

$$\Theta_{\mathbf{l}} = \frac{-1}{r} \left[K - \frac{2m}{r} - \frac{\Lambda}{3}r^2 + Q - \frac{r\Delta N - P^2(N_{,x}^2 + N_{,y}^2)}{r^2} \right], \tag{44}$$

which leads upon evaluation on the horizon surface to the equation

$$K - \frac{2m}{N} - \frac{\Lambda}{3}N^2 - \Delta \ln N + Q(u_0, N) = 0 \tag{45}$$

where only the last term represents a generalization of the horizon equation derived in [14]. We can check using an expansion at the origin and for infinity that $Q(u, r)$ (after moving the C_1 term into a redefinition of the “mass”) is regular everywhere for all NE models considered above, while it naturally diverges at the origin for Maxwell theory. For the last NE model this regularity is in fact caused by the restricted range of coordinate r .

First, let us use the redefinition of the function describing the horizon, $N(x, y) = 2mce^{-\Phi(x,y)}$, given in [14] to obtain Eq. (45) in better form,

$$\Delta\Phi = \frac{1}{c}e^\Phi + \frac{4}{3}\Lambda m^2 c^2 e^{-2\Phi} - K - Q(u_0, 2mce^{-\Phi}), \tag{46}$$

where we assume $\Phi > 0$.

In the case of NE models the regularity of Q at the origin and for infinity (or the maximum allowed value of r) means that it has finite supremum and infimum,

$$Q_{\text{sup}} = \sup_{r \in (0, \infty)} Q(u_0, r); \quad Q_{\text{inf}} = \inf_{r \in (0, \infty)} Q(u_0, r). \tag{47}$$

These, together with the minimum K_{min} and maximum K_{max} of Gaussian curvature of the compact surface spanned by x and y , can be used to straightforwardly generalize the results given in [14] which use a theorem (Theorem 1 therein) relying on the existence of sub- and super-solutions¹ Φ_{\pm} for (46) satisfying $0 < \Phi_- \leq \Phi_+$. In our case the constant sub- and super-solutions can be given depending on the value of the cosmological constant:

- $\Lambda \leq 0$

$$\begin{aligned} \Phi_- &= \ln(c[K_{\text{min}} + Q_{\text{inf}}]), \\ \Phi_+ &= \ln\left(c[K_{\text{max}} + Q_{\text{sup}}] - \frac{4}{3}\Lambda m^2 c^3\right), \end{aligned} \tag{48}$$

provided $c[K_{\text{min}} + Q_{\text{inf}}] > 1$,

- $\Lambda > 0$

$$\begin{aligned} \Phi_- &= \ln\left(c[K_{\text{min}} + Q_{\text{inf}}] - \frac{4}{3}\Lambda m^2 c^3\right), \\ \Phi_+ &= \ln(c[K_{\text{max}} + Q_{\text{sup}}]), \end{aligned} \tag{49}$$

if $c[K_{\text{min}} + Q_{\text{inf}}] - \frac{4}{3}\Lambda m^2 c^3 > 1$. By using the optimal choice of constant c this constraint reduces to a condition on the “physical” quantities,

$$9\Lambda m^2 < (K_{\text{min}} + Q_{\text{inf}})^3. \tag{50}$$

So the restrictions on the existence of a horizon are stronger for positive cosmological constant, which is natural since as a special case for $Q = 0$ we have asymptotically a (there we have $K_{\text{min}} = 1$) Schwarzschild–de Sitter solution which can have a naked singularity (see [14] for extended discussion).

For the Maxwell theory one has to be more careful when constructing sub- and super-solutions. The sub-solutions for both cases of Λ can be used here straightforwardly by setting $Q_{\text{inf}} = 0$. For a super-solution one uses the explicit form $Q = q(u_0)^4/r^2$ to derive a quadratic equation for $z = e^\Phi$,

$$\frac{1}{c}z + \frac{4}{3}\Lambda m^2 c^2 - K_{\text{max}} - \frac{q(u_0)^4}{4m^2 c^2} z^2 = 0, \tag{51}$$

based on the right-hand side of (46) after using the upper bound for the second and third term. Upon finding a positive solution of (51) for an optimal choice of the free constant c one can give the following super-solutions (notice the slightly different division of the cases according to Λ):

- $\Lambda < 0$

$$\Phi_+ = \ln\left(\frac{2m^2 c_1}{q(u_0)^4}\right) \tag{52}$$

where $c_1 = -\frac{\sqrt{3\Lambda(K_{\text{max}}q(u_0)^4 - m^2)}}{2\Lambda q(u_0)^2 m}$, so the necessary condition is

$$m^2 > K_{\text{max}}q(u_0)^4. \tag{53}$$

- $\Lambda \geq 0$ In this case one can neglect the cosmological constant term since it has the preferable sign anyway and one directly obtains

$$\Phi_+ = \ln\left(\frac{2mc}{q(u_0)^4}(m + \sqrt{m^2 - K_{\text{max}}q(u_0)^4})\right) \tag{54}$$

and the condition is the same as in the previous case (53).

Asymptotically $K_{\text{max}} \rightarrow 1$ since the solution tends to the spherically symmetric case and $q(u)$ approaches the constant q_0 . If one would choose the traditional notation $q_0^4 = Q^2$ (here and only here Q denotes the charge of a Reissner–Nordström solution) one recovers the natural condition $m^2 > Q^2$.

¹ E.g. a super-solution satisfies $\Delta u^+ \leq F(x, u^+, \nabla u^+)$.

7 Algebraic type of the solution

Now, we would like to see if the geometry of our spacetime is sufficiently general. Since the vacuum Robinson–Trautman spacetime is generally of algebraic type II we would like our solution to be at least of the same type and not more special. Our preferred tetrad for determining the Weyl scalars of our solution is given by different null vectors compared to (43),

$$\begin{aligned} \tilde{\mathbf{k}} &= \partial_r, \\ \tilde{\mathbf{l}} &= \partial_u - (H + Q/2)\partial_r, \\ \tilde{\mathbf{m}} &= \frac{P}{\sqrt{2}r}(\partial_x + i\partial_y), \end{aligned} \tag{55}$$

where i is the complex unit. The Weyl spinor computed from this tetrad has only the following nonzero components:

$$\begin{aligned} \Psi_2 &= \frac{1}{12} \left[Q_{,rr} - \frac{2}{r} Q_{,r} + \frac{2}{r^2} Q \right] - \frac{m}{r^3}, \\ \Psi_3 &= -\frac{\sqrt{2}P}{4r^2} (K_{,x} - iK_{,y}), \\ \Psi_4 &= \frac{1}{4r^2} [\{P^2(K_{,x} - iK_{,y})\}_{,x} - i\{P^2(K_{,x} - iK_{,y})\}_{,y}]. \end{aligned} \tag{56}$$

Now, we can easily determine the type irrespective of possible non-optimal choice of tetrad by using the review of explicit methods for determining the algebraic type in [64] that are based on [65]. Namely, when we use the invariants

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2, \quad J = \det \begin{pmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{pmatrix},$$

we can immediately confirm that $I^3 = 27J^2$ is satisfied so that we are dealing with type II or a more special case. At the same time generally $IJ \neq 0$, so it cannot be just type III. Moreover, the spinor covariant R_{ABCDEF} has nonzero components

$$R_{011111} = \frac{1}{2}\Psi_2(2\Psi_3^2 - 3\Psi_2\Psi_4), \tag{57}$$

$$R_{111111} = \Psi_3(2\Psi_3^2 - 3\Psi_2\Psi_4), \tag{58}$$

which means that generally the spacetime cannot be of type D. So indeed our NE solution is of the most general type possible for the Robinson–Trautman vacuum class. This does not mean that there cannot be a NE solution of type I when one considers a completely general Maxwell tensor \mathbf{F} . Moreover, inspecting the components of the Weyl spinor (56) one concludes that in the special case of $K(x, y) = \text{const} > 0$ (constant positive Gaussian curvature of compact two-space spanned by x, y) the algebraic type becomes D, consistent with spherical symmetry. Finally, since $\Psi_3 = 0$ implies $\Psi_4 = 0$, we cannot have all components of the spinor covariant Q_{ABCD} (see [64,65]) vanishing while having a nonvan-

ishing Weyl spinor. This means that our family of solutions does not contain type N geometries.

8 Conclusion and final remarks

We have derived Robinson–Trautman solutions with a source given by nonlinear electrodynamics for several specific models of the NE Lagrangian (both with Maxwell limit and without). The solutions were derived based on known spherically symmetric ones by the method described in Sect. 3. The Maxwell case was included for comparison as well. In all cases of NE the singularity of the electromagnetic field is resolved as in the static spherically symmetric cases. However, it was not possible to satisfy an additional constraint for having a Robinson–Trautman solution with the Hoffmann–Born–Infeld model or with the NE model, which provides a source of the Bardeen black hole. Both these models can be used to construct spherically symmetric solutions without a curvature singularity. The impossibility to generalize these models in the absence of this symmetry suggests that this kind of resolution of the curvature singularity might not be stable under nonlinear perturbations (at least within the Robinson–Trautman class). However, the Robinson–Trautman class does not contain rotating black holes (due to the twist-free condition) and therefore our results do not need to be universally valid. Unfortunately, the twisting class of solutions does not permit analysis on the level presented here (there are no asymptotic behavior studies in the dynamical regime).

Since in all models the curvature singularity is present we analyzed the existence of horizons using quasilocal concepts. All solutions are generally of algebraic type II and asymptotically in retarded time approach their spherically symmetric versions. All models with unrestricted coordinate ranges also remain locally asymptotically flat (or (anti-)de Sitter) as their vacuum counterparts.

The interpretation of the “charge” $q(u)$ suffers from the same difficulties as that of the “mass” in vacuum Robinson–Trautman solutions. The asymptotic behavior of q in all our models is identical to that of the function P , which describes the geometry of two spaces of constant u and r ; namely, in the preferred coordinate u it settles exponentially fast to a constant.

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Appendix

Here we plot the graphs of the electromagnetic field invariant F for all the considered models. We have put $C_1(u) = 1$, $\beta = 1$, $\alpha = 2^{-1/4}$, and we have selected an exponential decay behavior for the function $q(u) = 1 + e^{-u}$, which corresponds to the asymptotic behavior derived in Sect. 5. The plots show that, except for the Maxwell case, F is finite at $r = 0$ for all models. The difference is only in the rate of approach to this finite value. All the models give evidently asymptotically ($r \rightarrow \infty$) vanishing F , and also the fast exponential decay to the SSS form of F is clear.

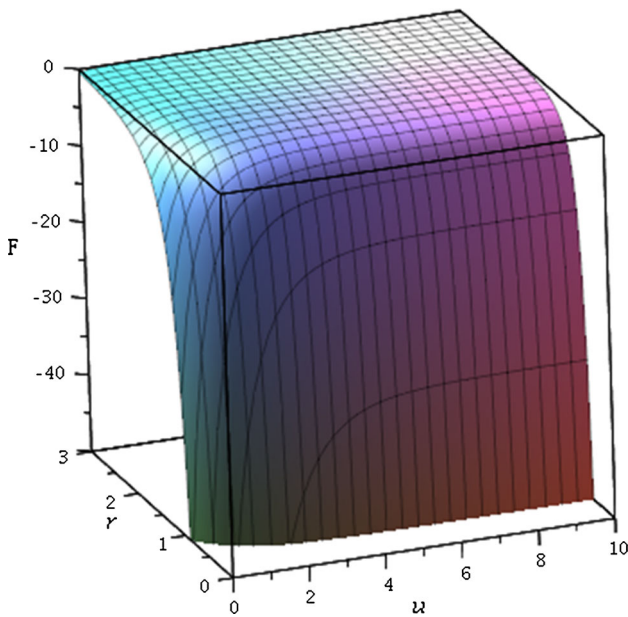


Fig. 3 The plot for Maxwell model

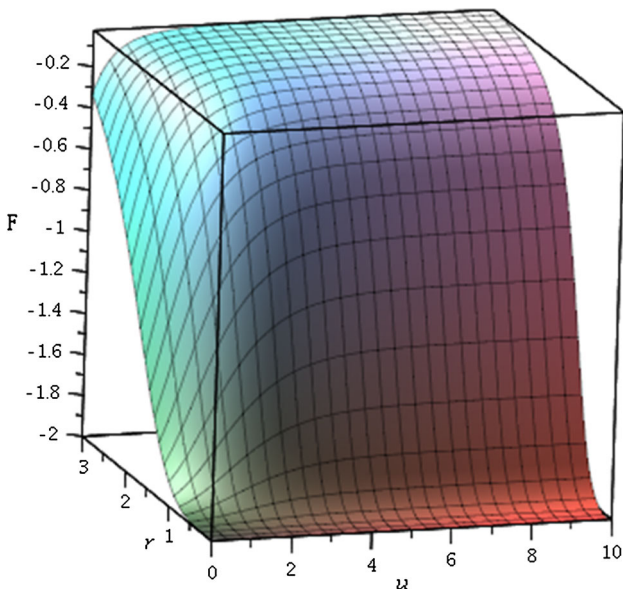


Fig. 4 The plot for Born-Infeld model

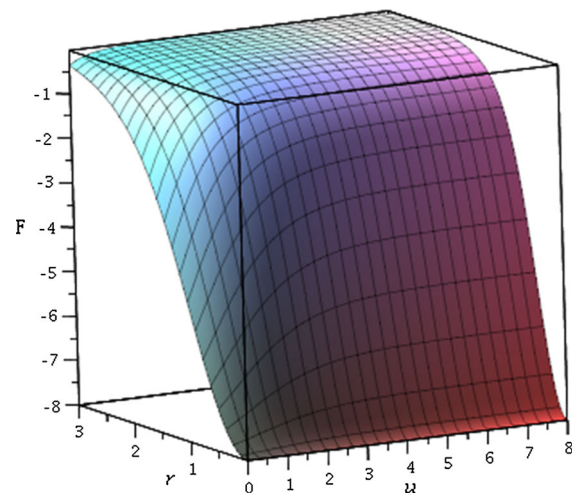


Fig. 5 The plot for logarithmic model

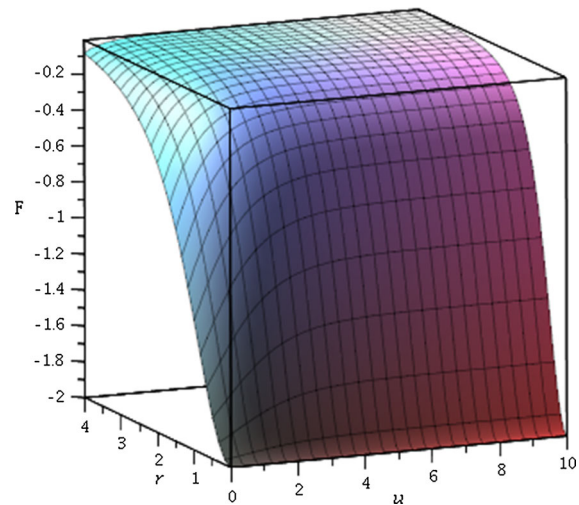


Fig. 6 The plot for New Lagrangian 1 model

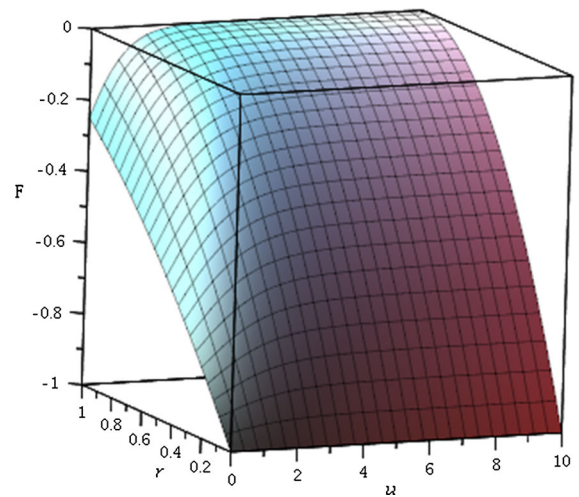


Fig. 7 The plot for New Lagrangian 2 model

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