On some aspects of isospin breaking in the decay

$$K^\pm \rightarrow \pi^0 \pi^0 e^\pm \bar{\nu}_e^{(-)}$$

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Abstract Two aspects of isospin breaking in the decay $K^\pm \rightarrow \pi^0 \pi^0 e^\pm \bar{\nu}_e^{(-)}$ are studied and discussed. The first addresses the possible influence of the phenomenological description of the unitarity cusp on the extraction of the normalization of the form factor from data. Using the scalar form factor of the pion as a theoretical laboratory, we find that this determination is robust under variations of the phenomenological parameterizations of the form factor. The second aspect concerns the issue of radiative corrections. We compute the radiative corrections to the total decay rate for $K^\pm \rightarrow \pi^0 \pi^0 e^\pm \bar{\nu}_e^{(-)}$ in a setting that allows comparison with the way radiative corrections were handled in the channel $K^\pm \rightarrow \pi^+ \pi^- e^\pm \bar{\nu}_e^{(-)}$. We find that, once radiative corrections are included, the normalizations of the form factor as determined experimentally from data in the two decay channels come to a better agreement. The remaining discrepancy can easily be accounted for by other isospin-breaking corrections, mainly those due to the difference between the masses of the *up* and *down* quarks.

1 Introduction

The program of analyzing $K_{\ell 4}$ decays of the charged kaon conducted by the NA48/2 Collaboration at the CERN SPS has so far been very successful. In the $\pi^+ \pi^-$ channel of the electron mode, $\ell = e$ (the decay $K^\pm \rightarrow \pi^+ \pi^- e^\pm \bar{\nu}_e^{(-)}$ will henceforth be referred to as K_{e4}^{+-}), it has led, besides a more precise determination of the corresponding branch-

ing ratio and hadronic form factors [1], to a very accurate determination of the $\pi\pi$ *S*-wave scattering lengths a_0^0 and a_0^2 [2,3], which constitutes a stringent test of the QCD prediction obtained within the framework of chiral perturbation theory [4–7]. These data have also contributed to improvement of the accuracy of the determination of some of the low-energy constants of chiral perturbation theory [8–10].

More recently, the results concerning an analysis of the data obtained in the $\pi^0 \pi^0$ channel of the electron mode (the decay $K^\pm \rightarrow \pi^0 \pi^0 e^\pm \bar{\nu}_e^{(-)}$ will henceforth be referred to as K_{e4}^{00}) have also become available [11]. Although the number of events is lower ($\sim 6.5 \cdot 10^4$ events in the K_{e4}^{00} mode vs. $\sim 10^6$ events for K_{e4}^{+-}), this allows for some cross checks at the level of the structure of one of the form factors, that is identical for the two channels in the isospin limit. The normalization of this common form factor, as measured in the two channels, reads [1,11]

$$\begin{aligned} |V_{us}| f_s [K_{e4}^{+-}] &= 1.285 \pm 0.001_{\text{stat}} \pm 0.004_{\text{syst}} \\ &\quad \pm 0.005_{\text{ext}}, \\ (1 + \delta_{EM}) |V_{us}| f_s [K_{e4}^{00}] &= 1.369 \pm 0.003_{\text{stat}} \pm 0.006_{\text{syst}} \\ &\quad \pm 0.009_{\text{ext}}. \end{aligned} \quad (1.1)$$

Ignoring for the time being the correction factor δ_{EM} (we will discuss radiative corrections below), the difference of the two values, as compared to the value measured in the K_{e4}^{+-} channel, amounts to 6.5 % in relative terms. This might be considered as a small difference, but given the uncertainties, it is, in statistical terms, quite significant. Adding all errors in quadrature¹ this gives

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¹ Obtaining the values in Eq. (1.1) from the measurements [1,11] of the corresponding branching ratios involves the lifetime $\tau_{K^\pm}$ of the charged kaon, whose uncertainty contributes to the “external” error bars.

$$(1 + \delta_{EM}) \frac{f_s[K_{e4}^{00}]}{f_s[K_{e4}^{+-}]} = 1.065 \pm 0.010. \quad (1.2)$$

It seems difficult to ascribe a variation of 6.5 % to the radiative correction factor δ_{EM} alone. While in some regions of phase space radiative corrections can reach the ± 10 % level, they usually sum up to ± 1 % in the decay rate. The radiative corrections to the K_{e4}^{+-} decay mode have been discussed in several places [12–14] at one-loop precision in the low-energy expansion. But no comparable study has been done for the K_{e4}^{00} decay mode. There exists an older, less systematic, analysis [15] that covers the corrections due to virtual photon exchanges and real photon emission, which could provide the relevant contributions at a first stage, but its practical use is somewhat limited, since the expressions given there are not always very explicit, and, moreover, need to be checked. Furthermore, not all radiative corrections occurring in the charged K_{e4}^{+-} channel [12–14] have been taken into account in the analysis of the experimental data. These additional radiative corrections could affect $f_s[K_{e4}^{00}]$ and $f_s[K_{e4}^{+-}]$ in different ways, and make up for another part of the discrepancy.

If δ_{EM} alone does not explain the discrepancy (1.2), one has to look for other sources of isospin-breaking effects. These can be due to the difference between the *up* and *down* quark masses m_u and m_d , conveniently described by the parameter R , with $1/R = (m_d - m_u)/(m_s - \hat{m})$, where m_s is the mass of the *strange* quark, whereas $\hat{m}$ denotes the average mass of the *up* and *down* quarks, $\hat{m} = (m_u + m_d)/2$. For instance, at lowest order in the chiral expansion, one has [12, 16]

$$\frac{f_s[K_{e4}^{00}]}{f_s[K_{e4}^{+-}]} = \left(1 + \frac{3}{2R}\right). \quad (1.3)$$

Barring contributions of higher-order corrections, values of R as small as [17] $R = 35.8$ (1.9) (1.8) can account for about two thirds of the effect in Eq. (1.2).

Finally, there are also isospin-breaking effects induced by the mass difference between charged and neutral pions. Most notable from this point of view is the presence of a unitarity cusp [11] in the form factor describing the amplitude of the K_{e4}^{00} mode. The interpretation of this cusp is by now well understood, and as in the case of the $K^\pm \rightarrow \pi^0 \pi^0 \pi^\pm$ decay [18–22], it arises from the contribution of a $\pi^+ \pi^-$ intermediate state in the unitarity sum [for a general discussion of the properties of the K_{e4} form factors from the point of

view of analyticity and unitarity, see Ref. [23] and references therein].

This cusp contains information on the combination $a_0^0 - a_0^2$ that describes the amplitude for the process $\pi^+ \pi^- \rightarrow \pi^0 \pi^0$ at threshold. Although this information probably cannot be extracted from the K_{e4}^{00} data in a way as statistically significant as the determination from the cusp in $K^\pm \rightarrow \pi^0 \pi^0 \pi^\pm$ [24], it is nevertheless important to include a correct description of this cusp in the parameterization of the form factor used to analyze the data. This necessity has been demonstrated in full detail in the case of the $K^\pm \rightarrow \pi^0 \pi^0 \pi^\pm$ decay, and it is to be expected that the same attention to these matters should be paid also in the analysis of the K_{e4}^{00} data. Failure to do so may introduce a systematic bias which would make the comparison with the information on the form factor extracted from the K_{e4}^{+-} data spurious to some extent.

It is the purpose of the present note to address some of these issues. In a first step, we investigate the possible influence that various parameterizations of the form factors could have on the outcome of the analysis. In order to control inputs and outputs fully, we choose to work with a simplified model, where the exact form factors are known from a theoretical point of view, and where one can assess the effects of various choices of parameterizations for the form factors used in order to analyze the numerically generated data (which we will henceforth refer to as pseudo-data). This framework is provided by the scalar form factors of the pions, defined as

$$\begin{aligned} &\langle \pi^0(p_1) \pi^0(p_2) | \hat{m}(\bar{u}u + \bar{d}d)(0) | \Omega \rangle \\ &= +F_S^{\pi^0}(s) [s \equiv (p_1 + p_2)^2], \\ &\langle \pi^+(p_+) \pi^-(p_-) | \hat{m}(\bar{u}u + \bar{d}d)(0) | \Omega \rangle \\ &= -F_S^\pi(s) [s \equiv (p_+ + p_-)^2]. \end{aligned} \quad (1.4)$$

Expressions of these form factors, with isospin-breaking contributions due to the difference of masses between charged and neutral pions included, have been recently obtained in [25] up to and including two loops in the low-energy expansion. We will use these expressions in order to generate pseudo-data, which we can then submit to analysis, using various parameterizations for the form factors, inspired by those in use for the analyses of the K_{e4}^{+-} and K_{e4}^{00} experimental data. The reason for working with the scalar form factors is at least twofold. First, the form factors, with isospin-breaking effects included, are known at two loops in both channels, whereas in the K_{e4} case, only the form factors in the channel with two charged pions have been studied at the same level of accuracy as far as isospin-breaking corrections are concerned [23] (see Ref. [26] for a systematic study at one loop). Second, the K_{e4} form factors depend on two more kinematical variables, besides the di-pion invariant mass. The scalar form factors depend

Footnote 1 continued

The ratio in Eq. (1.2), however, does not depend on $\tau_{K^\pm}$ anymore, which lowers the contribution of the “external” uncertainties to Eq. (1.2). At the level of precision shown, this does not impinge on the uncertainty in Eq. (1.2). We are indebted to B. Bloch-Devaux for drawing our attention to this point.

only on the latter, and offer therefore a simple kinematical environment, so that the issues we wish to focus on can be addressed without unnecessary additional complications.

In a second step, we address the issue of radiative corrections to the total decay rate of the decay $K^\mp \rightarrow \pi^0 \pi^0 e^\mp \bar{\nu}_e$. Our intent here is not to develop a full one-loop calculation, at the same level of precision as those that exist for the decay channel into two charged pions [12–14]. We rather aim at providing a simple estimate for the radiative corrections to the total decay rate, much in the spirit of Refs. [15] and [27] or, on a more general level, of Ref. [28]. This will allow us to assess how much of the discrepancy (1.2) has to be ascribed to other isospin-breaking effects in the form factors, such as discussed above.

The remainder of this study is then organized in the following way. First, we give (Sect. 2) a theoretical discussion of the structure of the scalar form factors of the pions using the explicit expressions obtained in Ref. [25]. We will thus adapt the discussion of Ref. [11] to the case at hand. Working on this analogy will allow us to give an assessment of some additional assumptions regarding the structure of the form factors implicitly made in Ref. [11]. Next, we generate pseudo-data (Sect. 3) using the known two-loop expressions of the form factors, which we then analyze using various phenomenological parameterizations, which do not necessarily comply with the outcome of Sect. 2. The purpose here is to discuss in a quantitative way the possible systematic biases that can be induced by these different choices. The last part of Sect. 3 addresses the determination of the combination $a_0^0 - a_0^+$ of S -wave scattering lengths. Radiative corrections, aiming at an estimate of the correction factor δ_{EM} in Eq. (1.1), are discussed in Sect. 4. We first compute radiative corrections to the K_{e4}^{00} decay rate in a similar set-up to the one used for the treatment of the data in the K_{e4}^{+-} channel, in order to obtain a meaningful comparison between the two channels. Then we compute the effects of additional photonic corrections, not included in this treatment. Finally, we end our study with a summary and conclusions. Two appendices contain technical details relevant for the discussions in Sects. 2 and 4, respectively.

2 Describing the cusp: theory

According to the general analysis of Ref. [20], the occurrence of both $\pi^0 \pi^0$ and $\pi^+ \pi^-$ intermediate states at different thresholds leads to the following structure for the scalar form factor of the neutral pion $F_S^{\pi^0}(s)$ (M_π stands for the charged-pion mass, whereas M_{π^0} is the mass of the neutral pion):

$$e^{-i\delta(s)} F_S^{\pi^0}(s) = \begin{cases} \mathcal{F}_0^{\pi^0}(s) - i\mathcal{F}_1^{\pi^0}(s) & [s \geq 4M_\pi^2] \\ \mathcal{F}_0^{\pi^0}(s) + \mathcal{F}_1^{\pi^0}(s) & [4M_{\pi^0}^2 \leq s \leq 4M_\pi^2] \end{cases}, \quad (2.1)$$

where $\mathcal{F}_0^{\pi^0}(s)$ is a function of s that is smooth as long as no other threshold, corresponding to higher intermediate states, is reached. Here $\delta(s)$ represents a phase. It can be chosen arbitrarily, as long as it is also a smooth function of s . The cusp at $s = 4M_\pi^2$ observed in the differential decay rate corresponding to this simplified (as compared to K_{e4}) situation then results from this decomposition, since

$$|F_S^{\pi^0}(s)|^2 = \begin{cases} |\mathcal{F}_0^{\pi^0}(s)|^2 + |\mathcal{F}_1^{\pi^0}(s)|^2 \\ -2\text{Im} [\mathcal{F}_0^{\pi^0}(s)\mathcal{F}_1^{\pi^0*}(s)] & [s \geq 4M_\pi^2] \\ |\mathcal{F}_0^{\pi^0}(s)|^2 + |\mathcal{F}_1^{\pi^0}(s)|^2 \\ +2\text{Re} [\mathcal{F}_0^{\pi^0}(s)\mathcal{F}_1^{\pi^0*}(s)] & [4M_{\pi^0}^2 \leq s \leq 4M_\pi^2]. \end{cases} \quad (2.2)$$

Apart from the dependence with respect to the second kinematical variable s_e , the empirical parameterization used for the fit to the K_{e4}^{00} data, Eq. (9.1) in Ref. [11], complies with this general representation provided (the variable S_π used in this reference corresponds to the variable s used here):

- (1) $\mathcal{F}_1^{\pi^0}(s)$ is parameterized as a *real* constant times $\hat{\sigma}(s)$, with

$$\hat{\sigma}(s) = \sqrt{\left|1 - \frac{4M_\pi^2}{s}\right|} = \sqrt{\left|\frac{q^2}{1+q^2}\right|}, \quad (2.3)$$

for $4M_{\pi^0}^2 \leq s \leq 4M_\pi^2$, or $q^2 \leq 0$, with $s = 4M_\pi^2(1+q^2)$.

- (2) $\mathcal{F}_1^{\pi^0}(s)$ is set to zero (its value for $s = 4M_\pi^2$) for $s \geq 4M_\pi^2$ ($q^2 \geq 0$).
- (3) For $4M_{\pi^0}^2 \leq s \leq 4M_\pi^2$ ($q^2 \leq 0$), $\mathcal{F}_0^{\pi^0}(s)$ is replaced by a constant, equal to $\mathcal{F}_0^{\pi^0}(4M_\pi^2)$.

A more theoretically based parameterization, adapted from the simple discussion of the cusp in $K^\pm \rightarrow \pi^0 \pi^0 \pi^\pm$ given in Ref. [19], is considered in Sect. 9.4 of Ref. [11], though not used for the data analysis. As compared to Eq. (2.2), its validity also rests on additional assumptions, which, once transposed to the present situation, read:

- (1') The phase $\delta(s)$ can be chosen such as to make the two functions $\mathcal{F}_0^{\pi^0}(s)$ and $\mathcal{F}_1^{\pi^0}(s)$ simultaneously real, so that Eq. (2.2) takes the simpler form

$$|F_S^{\pi^0}(s)|^2 = \begin{cases} |\mathcal{F}_0^{\pi^0}(s)|^2 + |\mathcal{F}_1^{\pi^0}(s)|^2 & [s \geq 4M_\pi^2] \\ [\mathcal{F}_0^{\pi^0}(s) + \mathcal{F}_1^{\pi^0}(s)]^2 & [4M_{\pi^0}^2 \leq s \leq 4M_\pi^2] \end{cases}. \quad (2.4)$$

(2') $\mathcal{F}_1^{\pi^0}(s)$ is related to the scalar form factor $F_S^\pi(s)$ of the charged pion, multiplied by a combination of the two S -wave $\pi\pi$ scattering lengths a_0^0 and a_0^2 in the isospin limit,

$$\mathcal{F}_1^{\pi^0}(s) = -\frac{2}{3} (a_0^0 - a_0^2) \mathcal{F}_S^\pi(s) \hat{\sigma}(s). \quad (2.5)$$

In view of the discussion in Ref. [11], $\mathcal{F}_S^\pi(s)$ should be identified with the phase-removed scalar form factor of the charged pion. The latter is given by $e^{-i\delta_0^\pi(s)} F_S^\pi(s)$, where the phase $\delta_0^\pi(s)$ is defined as $F_S^\pi(s + i0) = e^{2i\delta_0^\pi(s)} \times F_S^\pi(s - i0)$.

Our purpose in this section is twofold. First, we will rewrite the two-loop representation of the form factor $F_S^{\pi^0}(s)$ obtained in Ref. [25] in the form (2.2) that makes the cusp structure explicit. Second, we will assess to which extent the additional features mentioned above and assumed in Ref. [11] are actually reproduced by the structure of the form factors at two loops in the low-energy expansion. In particular, we will establish the precise relation between $\mathcal{F}_S^\pi(s)$ and $F_S^\pi(s)$ in Eq. (2.5) at this order. In what follows, and unless otherwise stated, it will always be understood that $s \geq 4M_{\pi^0}^2$. Furthermore, in practice $s \geq 4M_\pi^2$ will actually mean $4M_\pi^2 \leq s \leq M_K^2$, where M_K is the mass of the charged kaon, so that we need not worry about thresholds other than those produced by two-pion intermediate states.

2.1 The cusp in the one-loop form factors

We start with the study of the cusp using the one-loop expression of the form factor $F_S^{\pi^0}(s)$,

$$F_S^{\pi^0}(s) = F_S^{\pi^0}(0) \left[1 + a_S^{\pi^0} s + 16\pi \frac{\varphi_0^{00}(s)}{2} \bar{J}_0(s) \right] - 16\pi F_S^\pi(0) \varphi_0^x(s) \bar{J}(s). \quad (2.6)$$

In this expression, $a_S^{\pi^0}$ denotes a subtraction constant, that we need not specify further for the time being. The loop functions $\bar{J}_0(s)$ and $\bar{J}(s)$ are given by

$$\begin{aligned} \bar{J}_0(s) &= \frac{s}{16\pi^2} \int_{4M_{\pi^0}^2}^{\infty} \frac{dx}{x} \frac{1}{x-s-i0} \sigma_0(x), \\ \bar{J}(s) &= \frac{s}{16\pi^2} \int_{4M_\pi^2}^{\infty} \frac{dx}{x} \frac{1}{x-s-i0} \sigma(x), \end{aligned} \quad (2.7)$$

with

$$\sigma_0(s) = \sqrt{1 - \frac{4M_{\pi^0}^2}{s}}, \quad \sigma(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}. \quad (2.8)$$

The functions $\varphi_0^{00}(s)$ and $\varphi_0^x(s)$ denote the lowest-order real parts of the S -wave projections of the amplitudes of the processes $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ and $\pi^0\pi^0 \rightarrow \pi^+\pi^-$, respectively. Their expressions read

$$\varphi_0^{00}(s) = a_{00}, \quad \varphi_0^x(s) = a_x + b_x \frac{s - 4M_\pi^2}{F_\pi^2}, \quad (2.9)$$

with [23, 25, 29]

$$\begin{aligned} a_{00} &= \frac{2}{3} (a_0^0 + 2a_0^2) \left(1 - \frac{\Delta_\pi}{M_\pi^2} \right) \\ a_x &= -\frac{2}{3} (a_0^0 - a_0^2) + a_0^2 \frac{\Delta_\pi}{M_\pi^2} \\ b_x &= -\frac{1}{12} (2a_0^0 - 5a_0^2) \frac{F_\pi^2}{M_\pi^2}, \end{aligned} \quad (2.10)$$

and $\Delta_\pi \equiv M_\pi^2 - M_{\pi^0}^2$.

In the range of s under consideration, the function $\bar{J}_0(s)$ is complex, but both its real and imaginary parts are smooth,

$$\begin{aligned} \bar{J}_0(s) &= \frac{1}{16\pi^2} [2 + \sigma_0(s)L_0(s) + i\pi\sigma_0(s)], \\ L_0(s) &\equiv \ln \left(\frac{1 - \sigma_0(s)}{1 + \sigma_0(s)} \right) \quad [s \geq 4M_{\pi^0}^2], \end{aligned} \quad (2.11)$$

whereas $\bar{J}(s)$ may be rewritten as

$$\bar{J}(s) = \bar{J}^{[0]}(s) + \bar{J}^{[1]}(s) \times \begin{cases} -i\hat{\sigma}(s) & [s \geq 4M_\pi^2] \\ +\hat{\sigma}(s) & [4M_{\pi^0}^2 \leq s \leq 4M_\pi^2] \end{cases}. \quad (2.12)$$

The two functions $\bar{J}^{[0]}(s)$ and $\bar{J}^{[1]}(s)$ are smooth, and read

$$\bar{J}^{[0]}(s) = \frac{1}{16\pi^2} [2 + \sigma(s)\hat{L}(s)], \quad \bar{J}^{[1]}(s) = -\frac{1}{16\pi}, \quad (2.13)$$

where

$$\hat{L}(s) = \ln \left(\frac{1 - \sigma(s)}{1 + \sigma(s)} \right) \quad [s \geq 4M_{\pi^0}^2]. \quad (2.14)$$

In these expressions, the definition of $\sigma(s)$ has been extended below $s = 4M_\pi^2$ by²

$$\sigma(s) = \begin{cases} \sqrt{1 - \frac{4M_\pi^2}{s}} = \hat{\sigma}(s) & [s \geq 4M_\pi^2] \\ i\sqrt{\frac{4M_\pi^2}{s} - 1} = i\hat{\sigma}(s) & [4M_{\pi^0}^2 \leq s \leq 4M_\pi^2] \end{cases}. \quad (2.15)$$

According to Eqs. (2.1) and (2.2), $\bar{J}(s)$ exhibits a cusp structure at $s = 4M_\pi^2$. One thus obtains, at this order, the decomposition of the form (2.1) for $F_S^{\pi^0}(s)$, with

² This extension follows from the usual analytical continuation resulting from the replacement $s \rightarrow s + i0$.

$$\begin{aligned}\mathcal{F}_0^{\pi^0}(s) &= F_S^{\pi^0}(0) \left\{ 1 + a_S^{\pi^0} s + \frac{\varphi_0^{00}(s)}{2\pi} [2 + \sigma_0(s)L_0(s)] \right\} \\ &\quad - F_S^{\pi}(0) \frac{\varphi_0^x(s)}{\pi} [2 + \sigma(s)\hat{L}(s)] + \mathcal{O}(E^6), \\ \mathcal{F}_1^{\pi^0}(s) &= F_S^{\pi}(0) \varphi_0^x(s) \hat{\sigma}(s) + \mathcal{O}(E^6),\end{aligned}\quad (2.16)$$

provided one factorizes the global phase

$$\delta(s) = \frac{1}{2} \sigma_0(s) \varphi_0^{00}(s) + \mathcal{O}(E^4). \quad (2.17)$$

Therefore, up to so far unspecified higher-order corrections, the one-loop expression of the form factor can be brought into the form (2.1). Both functions $\mathcal{F}_0^{\pi^0}(s)$ and $\mathcal{F}_1^{\pi^0}(s)/\hat{\sigma}(s)$ are real and smooth for $s \geq 4M_{\pi^0}^2$ at this stage. However, the expression for $\mathcal{F}_1^{\pi^0}(s)$ in (2.16) does not quite comply with Eq. (2.5). Whereas at this stage $\mathcal{F}_S^{\pi}(s)$ is equal to the constant $F_S^{\pi}(0)$, which, at this order, can be identified with the phase-removed form factor, the combination of scattering lengths that occurs in Eq. (2.5) corresponds to $\hat{\varphi}_0^x(4M_{\pi}^2)$, where $\hat{\varphi}_0^x(s)$ is the expression of $\varphi_0^x(s)$ in the isospin limit. Thus, at this order, the expression (2.5) misses both the dependence with respect to s in $\varphi_0^x(s)$, and the isospin-breaking corrections in the scattering lengths.

For later reference, we briefly extend the discussion to the scalar form factor of the charged pion. At one loop, it is given by

$$\begin{aligned}F_S^{\pi}(s) &= F_S^{\pi}(0) [1 + a_S^{\pi} s + 16\pi \varphi_0^{+-}(s) \bar{J}(s)] \\ &\quad - 16\pi F_S^{\pi^0}(0) \frac{1}{2} \varphi_0^x(s) \bar{J}_0(s).\end{aligned}\quad (2.18)$$

Besides the subtraction constant a_S^{π} , which differs from $a_S^{\pi^0}$ (they become identical in the isospin limit), this expression involves the lowest-order real part of the S -wave projection of the amplitude for the scattering process $\pi^+\pi^- \rightarrow \pi^+\pi^-$,

$$\varphi_0^{+-}(s) = a_{+-} + b_{+-} \frac{s - 4M_{\pi}^2}{F_{\pi}^2}, \quad (2.19)$$

where [23, 25, 29]

$$\begin{aligned}a_{+-} &= \frac{1}{3} (2a_0^0 + a_0^2) - 2a_0^2 \frac{\Delta_{\pi}}{M_{\pi}^2} \\ b_{+-} &= \frac{1}{24} (2a_0^0 - 5a_0^2) \frac{F_{\pi}^2}{M_{\pi}^2}.\end{aligned}\quad (2.20)$$

After having factorized the global phase

$$\tilde{\delta}(s) = -\frac{1}{2} \sigma_0(s) \varphi_0^x(s) \frac{F_S^{\pi^0}(0)}{F_S^{\pi}(0)} + \mathcal{O}(E^4), \quad (2.21)$$

one can decompose $F_S^{\pi}(s)$ according to Eq. (2.1), with

$$\begin{aligned}\mathcal{F}_0^{\pi}(s) &= F_S^{\pi}(0) \left\{ 1 + a_S^{\pi} s + \frac{\varphi_0^{+-}(s)}{\pi} [2 + \sigma(s)\hat{L}(s)] \right\} \\ &\quad - F_S^{\pi^0}(0) \frac{\varphi_0^x(s)}{2\pi} [2 + \sigma_0(s)L_0(s)] + \mathcal{O}(E^6), \\ \mathcal{F}_1^{\pi}(s) &= -F_S^{\pi}(0) \varphi_0^{+-}(s) \hat{\sigma}(s) + \mathcal{O}(E^6).\end{aligned}\quad (2.22)$$

At this stage, both functions $\mathcal{F}_0^{\pi}(s)$ and $\mathcal{F}_1^{\pi}(s)/\hat{\sigma}(s)$ are real and smooth for $s \geq 4M_{\pi^0}^2$.

2.2 The cusp in the two-loop form factor $F_S^{\pi^0}(s)$ of the neutral pion

Let us now go through the same analysis, but with the two-loop expression of the form factor. The expressions of the pion scalar form factors at two loops and in presence of isospin breaking have been worked out in Ref. [25] using a recursive construction based on general properties like relativistic invariance, unitarity, analyticity, and chiral counting. The scalar form factor of the neutral pion can be written as³

$$\begin{aligned}F_S^{\pi^0}(s) &= F_S^{\pi^0}(0) \left(1 + a_S^{\pi^0} s + b_S^{\pi^0} s^2 \right) \\ &\quad + 8\pi F_S^{\pi^0}(0) \varphi_0^{00}(s) \left[1 + a_S^{\pi^0} s + \frac{1}{\pi} \varphi_0^{00}(s) \right] \bar{J}_0(s) \\ &\quad - 16\pi F_S^{\pi}(0) \varphi_0^x(s) \left[1 + a_S^{\pi} s + \frac{2}{\pi} \varphi_0^{+-}(s) \right] \bar{J}(s) \\ &\quad + \frac{M_{\pi}^4}{F_{\pi}^4} F_S^{\pi^0}(0) \left[\xi_{00}^{(0)}(s) \bar{J}_0(s) + \xi_{00}^{(1;0)}(s) \bar{K}_1^0(s) \right. \\ &\quad + 2\xi_{00}^{(2;0)}(s) \bar{K}_2^0(s) + \xi_{00}^{(3;0)}(s) \bar{K}_3^0(s) + \xi_{00}^{(1;\nabla)}(s) \bar{K}_1^{\nabla}(s) \\ &\quad + \xi_{00}^{(3;\nabla)}(s) \bar{K}_3^{\nabla}(s) + 2\xi_{00}^{(2;\pm)}(s) [16\pi^2 \bar{J}(s) - 2] \bar{J}_0(s) \Big] \\ &\quad - 2 \frac{M_{\pi}^4}{F_{\pi}^4} F_S^{\pi}(0) \left[\xi_x^{(0)}(s) \bar{J}(s) + 2\xi_x^{(2;\pm)}(s) \bar{K}_2(s) \right. \\ &\quad + \xi_x^{(1)}(s) K_1^x(s) + \xi_x^{(3)}(s) K_3^x(s) \\ &\quad + 2\xi_x^{(2;0)}(s) [16\pi^2 \bar{J}_0(s) - 2] \bar{J}(s) + \Delta_1 \xi_x(s) \bar{K}^x(s) \Big] \\ &\quad + \mathcal{O}(E^8).\end{aligned}\quad (2.23)$$

In this formula, the functions $\xi_{00}^{(0)}(s), \dots, \xi_x^{(0)}(s), \dots$ are polynomials of at most second order in the variable s . Their expressions can be found in Ref. [25], except for $\Delta_1 \xi_x(s)$, which reads

$$\begin{aligned}\Delta_1 \xi_x(s) &= 8 \frac{\Delta_{\pi}}{M_{\pi}^2} b_{+0} \frac{s}{M_{\pi}^2} \left[\frac{s}{9M_{\pi}^2} b_{+0} - a_{+0} \frac{F_{\pi}^2}{M_{\pi}^2} \right. \\ &\quad \left. + 2b_{+0} \left(1 + \frac{M_{\pi^0}^2}{M_{\pi}^2} \right) \right].\end{aligned}\quad (2.24)$$

³ We neglect here a tiny contribution of second order in isospin breaking.

It is also useful to be aware of the relations

$$\begin{aligned}\xi_{00}^{(2;0)}(s) &= 2 \frac{F_\pi^4}{M_\pi^4} [\varphi_0^{00}(s)]^2, \\ \xi_x^{(2;0)}(s) &= 2 \frac{F_\pi^4}{M_\pi^4} \varphi_0^{00}(s) \varphi_0^x(s), \\ \xi_{00}^{(2;\pm)}(s) &= 4 \frac{F_\pi^4}{M_\pi^4} [\varphi_0^x(s)]^2, \\ \xi_x^{(2;\pm)}(s) &= 4 \frac{F_\pi^4}{M_\pi^4} \varphi_0^x(s) \varphi_0^{+-}(s).\end{aligned}\quad (2.25)$$

In order to achieve the decomposition (2.1), we need to extend the decomposition of the function $\bar{J}(s)$ in Eqs. (2.12) and (2.13) to the other functions, denoted generically by $\bar{K}_n^\alpha(s)$, which appear in the expression (2.23). This may be done as follows. First, we may observe that, like $\bar{J}_0(s)$ or $\bar{J}(s)$, these functions can also be defined by a dispersive representation of the form [for $\bar{J}_0(s) \equiv \bar{K}_0^0(s)$, one has $k_0^0(s) = \sigma_0(s)/16\pi$, whereas for $\bar{J}(s) \equiv \bar{K}_0(s)$, $k_0(s) = \sigma(s)/16\pi$; see Eq. (2.7)]

$$\bar{K}_n^\alpha(s) = \frac{s}{\pi} \int_{s_{\text{thr}}}^{\infty} \frac{dx}{x} \frac{1}{x-s-i0} k_n^\alpha(x). \quad (2.26)$$

Explicit expressions for the functions $k_n^\alpha(s)$ are given in A. For the set of functions $K_n^0(s)$ and $\bar{K}_n^\nabla(s)$, one has $s_{\text{thr}} = 4M_{\pi^0}^2$. These functions will therefore each develop an imaginary part for $s \geq s_{\text{thr}} = 4M_{\pi^0}^2$, $\text{Im} \bar{K}_n^\alpha(s) = k_n^\alpha(s) \theta(s - 4M_{\pi^0}^2)$, while the real part displays a cusp at $s = 4M_{\pi^0}^2$, but is smooth for $s \geq 4M_{\pi^0}^2$. The situation is different for the remaining functions, $\bar{K}_n(s)$, $\bar{K}_n^x(s)$, and $\bar{K}^x(s)$, for which $s_{\text{thr}} = 4M_\pi^2$, so that, in a generic way, they have the following structure:

$$\begin{aligned}\bar{K}_n^\alpha(s) &= \text{Re } \bar{K}_n^\alpha(s) + \begin{cases} ik_n^\alpha(s) & [s \geq 4M_\pi^2] \\ 0 & [4M_{\pi^0}^2 \leq s \leq 4M_\pi^2] \end{cases} \\ &= \text{Re } \bar{K}_n^\alpha(s) + \frac{k_n^\alpha(s)}{\sigma(s)} \times \begin{cases} i\hat{\sigma}(s) & [s \geq 4M_\pi^2] \\ 0 & [4M_{\pi^0}^2 \leq s \leq 4M_\pi^2] \end{cases}.\end{aligned}\quad (2.27)$$

In general, the function $k_n^\alpha(s)/\sigma(s)$, although real, is not smooth for the whole range $s \geq 4M_{\pi^0}^2$, but only for $s \geq 4M_\pi^2$. Suppose one can find a function $\hat{k}_n^\alpha(s)$ such that it coincides with $k_n^\alpha(s)$ for $s \geq 4M_\pi^2$, and such that $\hat{k}_n^\alpha(s)/\sigma(s)$ is real and smooth for all $s \geq 4M_{\pi^0}^2$. Then one can perform the decomposition

$$\begin{aligned}\bar{K}(s) &= \bar{K}^{[0]}(s) \\ &+ \bar{K}^{[1]}(s) \times \begin{cases} -i\hat{\sigma}(s) & [s \geq 4M_\pi^2] \\ +\hat{\sigma}(s) & [4M_{\pi^0}^2 \leq s \leq 4M_\pi^2] \end{cases},\end{aligned}\quad (2.28)$$

in terms of two real and smooth functions $\bar{K}^{[0]}(s)$ and $\bar{K}^{[1]}(s)$, given by

$$\begin{aligned}\bar{K}_n^{\alpha[0]}(s) &= \begin{cases} \text{Re } \bar{K}_n^\alpha(s) & [s \geq 4M_\pi^2] \\ \text{Re } \bar{K}_n^\alpha(s) - i\hat{k}_n^\alpha(s) & [4M_{\pi^0}^2 \leq s \leq 4M_\pi^2] \end{cases}, \\ \bar{K}_n^{\alpha[1]}(s) &= -\frac{\hat{k}_n^\alpha(s)}{\sigma(s)}.\end{aligned}\quad (2.29)$$

Such a decomposition can indeed be achieved for the various functions considered here, as discussed in detail in Appendix A. The decomposition (2.1) of the form factor now follows immediately, with

$$\begin{aligned}e^{i\delta(s)} \mathcal{F}_0^{\pi^0}(s) &= F_S^{\pi^0}(0) \left(1 + a_S^{\pi^0} s + b_S^{\pi^0} s^2 \right) \\ &+ 8\pi F_S^{\pi^0}(0) \varphi_0^{00}(s) \left[1 + a_S^{\pi^0} s + \frac{1}{\pi} \varphi_0^{00}(s) \right] \bar{J}_0(s) \\ &- 16\pi F_S^{\pi^0}(0) \varphi_0^x(s) \left[1 + a_S^{\pi^0} s + \frac{2}{\pi} \varphi_0^{+-}(s) \right] \bar{J}^{[0]}(s) \\ &+ \frac{M_\pi^4}{F_\pi^4} F_S^{\pi^0}(0) \left[\xi_{00}^{(0)}(s) \bar{J}_0(s) + \xi_{00}^{(1;0)}(s) \bar{K}_1^0(s) \right. \\ &+ \xi_{00}^{(1;\nabla)}(s) \bar{K}_1^\nabla(s) + 2\xi_{00}^{(2;0)}(s) \bar{K}_2^0(s) + \xi_{00}^{(3;0)}(s) \bar{K}_3^0(s) \\ &+ \xi_{00}^{(3;\nabla)}(s) \bar{K}_3^\nabla(s) + 2\xi_{00}^{(2;\pm)}(s) [16\pi^2 \bar{J}^{[0]}(s) - 2] \bar{J}_0(s) \left. \right] \\ &- 2 \frac{M_\pi^4}{F_\pi^4} F_S^{\pi^0}(0) \left[\xi_x^{(0)}(s) \bar{J}^{[0]}(s) + \xi_x^{(1)}(s) \bar{K}_1^{x[0]}(s) \right. \\ &+ 2\xi_x^{(2;\pm)}(s) \bar{K}_2^{[0]}(s) + \xi_x^{(3)}(s) \bar{K}_3^{x[0]}(s) \\ &+ 2\xi_x^{(2;0)}(s) [16\pi^2 \bar{J}_0(s) - 2] \bar{J}^{[0]}(s) \\ &+ \Delta_1 \xi_x(s) \bar{K}^{x[0]}(s) \left. \right] + \mathcal{O}(E^8),\end{aligned}\quad (2.30)$$

and

$$\begin{aligned}e^{i\delta(s)} \mathcal{F}_1^{\pi^0}(s) &= -\hat{\sigma}(s) \left\{ 16\pi F_S^{\pi^0}(0) \varphi_0^x(s) \right. \\ &\times \left[1 + a_S^{\pi^0} s + \frac{2}{\pi} \varphi_0^{+-}(s) \right] \bar{J}^{[1]}(s) \\ &- 2 \frac{M_\pi^4}{F_\pi^4} F_S^{\pi^0}(0) \xi_{00}^{(2;\pm)}(s) \times 16\pi^2 \bar{J}_0(s) \bar{J}^{[1]}(s) \\ &+ 2 \frac{M_\pi^4}{F_\pi^4} F_S^{\pi^0}(0) \left[\xi_x^{(0)}(s) \bar{J}^{[1]}(s) + \xi_x^{(1)}(s) \bar{K}_1^{x[1]}(s) \right. \\ &+ 2\xi_x^{(2;\pm)}(s) \bar{K}_2^{[1]}(s) + \xi_x^{(3)}(s) \bar{K}_3^{x[1]}(s) \\ &+ 2\xi_x^{(2;0)}(s) [16\pi^2 \bar{J}_0(s) - 2] \bar{J}^{[1]}(s) \\ &+ \Delta_1 \xi_x(s) \bar{K}^{x[1]}(s) \left. \right\} + \mathcal{O}(E^8).\end{aligned}\quad (2.31)$$

Both functions $e^{i\delta(s)} \mathcal{F}_0^{\pi^0}(s)$ and $e^{i\delta(s)} \mathcal{F}_1^{\pi^0}(s)/\hat{\sigma}(s)$ are smooth for $s \geq 4M_{\pi^0}^2$, but complex. It remains to discuss the phase $\delta(s)$. If we want to make the function $\mathcal{F}_0^{\pi^0}(s)$ real, while keeping it smooth, then its choice is unique,

$$\delta(s) \equiv \frac{1}{2} \sigma_0(s) \left[\varphi_0^{00}(s) + \hat{\psi}_0^{00}(s) \right], \quad (2.32)$$

with $\hat{\psi}_0^{00}(s)$ given by

$$\begin{aligned} \frac{1}{2} \sigma_0(s) \hat{\psi}_0^{00}(s) = & \frac{M_\pi^4}{F_\pi^4} \left[\xi_{00}^{(0)}(s) k_0^0(s) + \xi_{00}^{(1;0)}(s) k_1^0(s) \right. \\ & + \xi_{00}^{(1;\nabla)}(s) k_1^\nabla(s) + \xi_{00}^{(2;0)}(s) k_2^0(s) \\ & + \xi_{00}^{(3;0)}(s) k_3^0(s) + \xi_{00}^{(3;\nabla)}(s) k_3^\nabla(s) \\ & \left. + \xi_{00}^{(2;\pm)}(s) \frac{1}{8\pi} \sigma_0(s) \sigma(s) \hat{L}(s) \right]. \quad (2.33) \end{aligned}$$

Note that $\hat{\psi}_0^{00}(s)$ differs from the quantity $\psi_0^{00}(s)$ defined in Eq. (4.6) of Ref. [25] by the presence of the function $\hat{L}(s)$ instead of $L(s)$ in the last term between square brackets, see Eq. (A.2). This makes $\sigma_0(s) \hat{\psi}_0^{00}(s)$ a smooth function for $s \geq 4M_{\pi^0}^2$, whereas $\sigma(s)L(s)$, and hence $\psi_0^{00}(s)$, displays a cusp at $s = 4M_{\pi^0}^2$. Making use of Eq. (2.27), the removal of the phase $\delta(s)$ indeed leads to a real and smooth expression for the function $\mathcal{F}_0^{\pi^0}(s)$:

$$\begin{aligned} \mathcal{F}_0^{\pi^0}(s) = & F_S^{\pi^0}(0) \left(1 + a_S^{\pi^0} s + b_S^{\pi^0} s^2 \right) \\ & + 8\pi F_S^{\pi^0}(0) \varphi_0^{00}(s) \left[1 + a_S^{\pi^0} s \right] \text{Re } \bar{J}_0(s) \\ & - 16\pi F_S^{\pi^0}(0) \varphi_0^x(s) \left[1 + a_S^{\pi^0} s + \frac{2}{\pi} \varphi_0^{+-}(s) \right] \bar{J}^{[0]}(s) \\ & + \frac{M_\pi^4}{F_\pi^4} F_S^{\pi^0}(0) \left[\xi_{00}^{(0)}(s) \text{Re } \bar{J}_0(s) + \xi_{00}^{(1;0)}(s) \text{Re } \bar{K}_1^0(s) \right. \\ & + \xi_{00}^{(1;\nabla)}(s) \text{Re } \bar{K}_1^\nabla(s) + \xi_{00}^{(2;0)}(s) \text{Re } \bar{K}_2^0(s) \\ & + \xi_{00}^{(3;0)}(s) \text{Re } \bar{K}_3^0(s) + \xi_{00}^{(3;\nabla)}(s) \text{Re } \bar{K}_3^\nabla(s) \left. \right] \\ & - 2 \frac{M_\pi^4}{F_\pi^4} F_S^{\pi^0}(0) \left[\xi_x^{(0)}(s) \bar{J}^{[0]}(s) + \xi_x^{(1)}(s) \bar{K}_1^{x[0]}(s) \right. \\ & + 2\xi_x^{(2;\pm)}(s) \bar{K}_2^{x[0]}(s) + \xi_x^{(3)}(s) \bar{K}_3^{x[0]}(s) \\ & \left. + \Delta_1 \xi_x(s) \bar{K}^{x[0]}(s) \right] \\ & + 8F_S^{\pi^0}(0) \left[\varphi_0^x(s) \right]^2 \left(16\pi^2 \bar{J}^{[0]}(s) - 2 \right) \text{Re } \bar{J}_0(s) \\ & - 8F_S^{\pi^0}(0) \varphi_0^x(s) \varphi_0^{00}(s) \left[16\pi^2 \text{Re } \bar{J}_0(s) - 2 \right] \bar{J}^{[0]}(s) \\ & + F_S^{\pi^0}(0) \left[\varphi_0^{00}(s) \right]^2 \left[2\text{Re } \bar{K}_2^0(s) + 8\text{Re } \bar{J}_0(s) \right. \\ & \left. + \frac{1}{8} \left(1 - \frac{4M_{\pi^0}^2}{s} \right) \right] + \mathcal{O}(E^8). \quad (2.34) \end{aligned}$$

As far as $\mathcal{F}_1^{\pi^0}(s)$ is concerned, we may even proceed in a more direct way by noticing that, up to higher-order corrections, Eq. (2.31) rewrites as

$$\begin{aligned} e^{i\delta(s)} \mathcal{F}_1^{\pi^0}(s) = & e^{\frac{i}{2} \sigma_0(s) \varphi_0^{00}(s)} \hat{\sigma}(s) \left[\varphi_0^x(s) + \hat{\psi}_0^x(s) \right] \\ & \times F_S^{\pi^0}(0) \left\{ 1 + a_S^{\pi^0} s + 16\pi \varphi_0^{+-}(s) \bar{J}^{[0]}(s) \right. \\ & \left. - 8\pi \frac{F_S^{\pi^0}(0)}{F_S^{\pi^0}(0)} \varphi_0^x(s) \bar{J}_0(s) \right\} + \mathcal{O}(E^8), \quad (2.35) \end{aligned}$$

with (for the notation, see Appendix A)

$$\begin{aligned} \hat{\psi}_0^x(s) = & 2 \frac{M_\pi^4}{F_\pi^4} \frac{1}{\sigma(s)} \left\{ \xi_x^{(0)}(s) k_0(s) + \xi_x^{(2;\pm)}(s) \hat{k}_2(s) \right. \\ & + \xi_x^{(1)}(s) k_1^x(s) + \xi_x^{(3)}(s) k_3^x(s) \\ & \left. + \xi_x^{(2;0)}(s) k_2^x(s) + \Delta_1 \xi_x(s) k^x(s) \right\}. \quad (2.36) \end{aligned}$$

Now, the phase that appears factored out on the right-hand side of this equation can be identified with the phase $\delta(s)$ on the left-hand side, since the difference generates contributions of order $\mathcal{O}(E^8)$ that are neglected anyway. Taking into account Eqs. (2.21) and (2.22), one finally obtains

$$\mathcal{F}_1^{\pi^0}(s) = \hat{\sigma}(s) \left[\varphi_0^x(s) + \hat{\psi}_0^x(s) \right] \mathcal{F}_\pi(s) + \mathcal{O}(E^8), \quad (2.37)$$

with

$$\mathcal{F}_\pi(s) \equiv e^{i\delta(s)} \mathcal{F}_0^{\pi^0}(s) + \mathcal{O}(E^6). \quad (2.38)$$

It is possible to give a more precise interpretation of the combination $\varphi_0^x(s) + \hat{\psi}_0^x(s)$ that occurs in (2.37). To this end, let us recall from Ref. [25] that the $\ell = 0$ partial-wave projection $f_0^x(s)$ for the scattering amplitude of the process $\pi^0 \pi^0 \rightarrow \pi^+ \pi^-$ is given, at the order of one loop and for $s \geq 4M_{\pi^0}^2$, by

$$\begin{aligned} f_0^x(s) = & \varphi_0^x(s) + \psi_0^x(s) + i \varphi_0^x(s) \left[\frac{1}{2} \sigma_0(s) \varphi_0^{00}(s) \right. \\ & \left. + \sigma(s) \varphi_0^{+-}(s) \theta(s - 4M_\pi^2) \right] + \mathcal{O}(E^6), \quad (2.39) \end{aligned}$$

where $\psi_0^x(s)$ is defined in Eq. (4.15) of Ref. [25] [the contribution $\Delta_2 \psi_0^x(s)$, of second order in isospin breaking, is numerically quite small, and is omitted for simplicity]. It differs from $\hat{\psi}_0^x(s)$ by the replacement of $\hat{k}_2(s)$ by $k_2(s)$ in Eq. (2.36). Then applying the decomposition (2.28) to $f_0^x(s)$, one finds

$$\begin{aligned} e^{-i\delta(s)} f_0^x(s) = & f_0^{x[0]}(s) \\ & + f_0^{x[1]}(s) \times \begin{cases} -i \hat{\sigma}(s) & [s \geq 4M_\pi^2] \\ +\hat{\sigma}(s) & [4M_{\pi^0}^2 \leq s \leq 4M_\pi^2] \end{cases}, \quad (2.40) \end{aligned}$$

with $f_0^{x[0]}(s) = \varphi_0^x(s) + \hat{\psi}_0^x(s) + \mathcal{O}(E^6)$, and $f_0^{x[1]}(s) = -\varphi_0^x(s) \varphi_0^{+-}(s) + \mathcal{O}(E^6)$.

We may summarize this theoretical study of the cusp in the scalar form factor of the neutral pion with a couple of remarks:

- It is, in general, not possible to choose the phase $\delta(s)$ in Eq. (2.1) such as to make both $\mathcal{F}_0^{\pi^0}(s)$ and $\mathcal{F}_1^{\pi^0}(s)$ real simultaneously. A relative phase remains; see Eqs. (2.37) and (2.38). At lowest order, this phase is given by the S -wave projection of the inelastic rescattering of a pair of neutral pions through a pair of charged pions, cf. Eq. (2.21).
- The structure of $\mathcal{F}_1^{\pi^0}(s)$ is more complicated than just the product of the scattering length corresponding to this rescattering amplitude times the phase-removed scalar form factor of the charged pion. At the order we have been working, it involves the decomposition (2.40) of the S -wave projection of this amplitude times the part $\mathcal{F}_0^{\pi}(s)$ of the decomposition (2.1) of $F_S^{\pi}(s)$. This is different from the phase-removed form factor, as already seen at order one loop:

$$\begin{aligned} e^{-i\delta_0^{\pi}(s)} F_S^{\pi}(s) - \mathcal{F}_0^{\pi}(s) \\ = \mathcal{F}_1^{\pi}(s) \times \begin{cases} 0 & [s \geq 4M_{\pi}^2] \\ 1 & [4M_{\pi^0}^2 \leq s \leq 4M_{\pi}^2] \end{cases} + \mathcal{O}(E^6). \end{aligned} \quad (2.41)$$

Note, however, that this difference only concerns the region $4M_{\pi^0}^2 \leq s \leq 4M_{\pi}^2$, which contributes very little to the total decay rate as defined by Eqs. (3.1) and (3.2) below.

2.3 Description of the two-loop form factor $F_S^{\pi}(s)$ of the charged pion

We now briefly address the scalar form factor of the charged pion. The issue here is not to describe the cusp, which occurs below the physical threshold at $s = 4M_{\pi}^2$, but to provide the expressions that will be used in the sequel. Again, we will rely on the results obtained in Ref. [25], and rewrite the form factor $F_S^{\pi}(s)$ at two loops in a way that is adapted to our purposes. In particular, we will consider the phase-removed form factor, which reads, in the relevant domain $s \geq 4M_{\pi}^2$,

$$\begin{aligned} e^{-i\delta_0^{\pi}(s)} F_S^{\pi}(s) &= F_S^{\pi}(0) \left(1 + a_S^{\pi} s + b_S^{\pi} s^2 \right) \\ &- 8\pi F_S^{\pi}(0) \varphi_0^x(s) \left[1 + a_S^{\pi} s + \frac{1}{\pi} \varphi_0^{00}(s) \right] \text{Re } \bar{J}_0(s) \\ &+ 16\pi F_S^{\pi}(0) \varphi_0^{+-}(s) \left[1 + a_S^{\pi} s + \frac{2}{\pi} \varphi_0^{+-}(s) \right] \text{Re } \bar{J}(s) \\ &- \frac{M_{\pi}^4}{F_{\pi}^4} F_S^{\pi}(0) \left\{ \xi_x^{(0)}(s) \text{Re } \bar{J}_0(s) + \xi_x^{(1)}(s) \text{Re } \bar{K}_1^{x0}(s) \right. \\ &\left. + 2\xi_x^{(2;0)}(s) \text{Re } \bar{K}_2^0(s) + \xi_x^{(3)}(s) \text{Re } \bar{K}_3^0(s) \right. \end{aligned}$$

$$\begin{aligned} &+ \Delta_1 \xi_x(s) \text{Re } \bar{K}^{x0}(s) \\ &+ 2\xi_x^{(2;\pm)}(s) \left[16\pi^2 \text{Re } \bar{J}(s) - 2 \right] \text{Re } \bar{J}_0(s) \Big\} \\ &+ 2 \frac{M_{\pi}^4}{F_{\pi}^4} F_S^{\pi}(0) \left\{ \xi_{+-;S}^{(0)}(s) \text{Re } \bar{J}(s) + \xi_{+-;S}^{(1;\pm)}(s) \text{Re } \bar{K}_1(s) \right. \\ &+ 2\xi_{+-;S}^{(2;\pm)}(s) \left[\text{Re } \bar{K}_2(s) + \frac{1}{32} \left(1 - \frac{4M_{\pi}^2}{s} \right) \right] \\ &+ \xi_{+-;S}^{(3;\pm)}(s) \text{Re } \bar{K}_3(s) + \xi_{+-;S}^{(1;\Delta)}(s) \text{Re } \bar{K}_1^{\Delta}(s) \\ &+ \xi_{+-;S}^{(3;\Delta)}(s) \text{Re } \bar{K}_3^{\Delta}(s) + 2\xi_{+-;S}^{(2;0)}(s) \\ &\times \left[(16\pi^2 \text{Re } \bar{J}_0(s) - 2) \text{Re } \bar{J}(s) \right. \\ &\left. + \frac{1}{64} \left(1 - \frac{4M_{\pi^0}^2}{s} \right) \right] \Big\} + \mathcal{O}(E^8). \end{aligned} \quad (2.42)$$

The functions $\bar{K}_n^{\alpha}(s)$ that appear in this expression have again a dispersive representation of the form displayed in Eq. (2.26). The absorptive parts are in part given in Appendix A. For the remaining ones, we have

$$\begin{aligned} k_n^{x0}(s) &= \frac{\sigma_0(s)}{\sigma(s)} k_n^x(s), \\ k^{x0}(s) &= \frac{\sigma_0(s)}{\sigma(s)} k^x(s), \end{aligned} \quad (2.43)$$

for the functions corresponding to $s_{\text{thr}} = 4M_{\pi^0}^2$, and, for those whose dispersive integrals start at $s_{\text{thr}} = 4M_{\pi}^2$,

$$\begin{aligned} k_1^{\Delta}(s) &= \frac{1}{8\pi} \frac{\sigma(s)}{\sigma_0(s - 4\Delta_{\pi})} L_0(s - 4\Delta_{\pi}), \\ k_3^{\Delta}(s) &= \frac{3}{16\pi} \frac{M_{\pi^0}^2}{s\sigma(s)} L_0^2(s - 4\Delta_{\pi}), \end{aligned} \quad (2.44)$$

with the definitions of the functions $\sigma(s)$, $\sigma_0(s)$ and $L_0(s)$ for $s \geq 4M_{\pi^0}^2$ given in Eqs. (2.8), (2.11), and (2.15).

3 Generation and analysis of the pseudo-data

In order to study the effect that particular choices of phenomenological parameterizations of the form factors can have on the output, we will first generate numerical data sets for the scalar form factors of the neutral and charged pions. The pseudo-data in question consist of the (unnormalized) decay distribution defined by

$$\begin{aligned} \frac{d^2 \Gamma^{\pi^0}(s, s_{\ell})}{ds ds_{\ell}} &\equiv \frac{1}{2} \sqrt{1 - \frac{4M_{\pi^0}^2}{s}} |F_{S;\text{data}}^{\pi^0}(s)|^2 \\ &\times \lambda^{3/2}(M_K^2, s, s_{\ell}), \end{aligned} \quad (3.1)$$

with $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. The total decay rate is obtained by integrating the distributions (3.1), convoluted with the $K_{\ell 4}$ phase space, over the whole physical range (we now consider the electron mode only, and set $m_e = 0$):

$$\Gamma^{\pi^0} = \mathcal{N} \int_{4M_{\pi^0}^2}^{M_K^2} ds \int_0^{(M_K^2 - \sqrt{s})^2} ds_e \frac{d^2 \Gamma^{\pi^0}(s, s_e)}{ds ds_e}. \quad (3.2)$$

Since we consider the scalar form factor instead of $K_{\ell 4}$ form factors, the integration with respect to s_ℓ involves the phase space only. The overall normalization factor has been chosen to be the one of the K_{e4} decay,

$$\mathcal{N} = \frac{G_F^2 |V_{us}|^2}{3 \cdot 2^{12} \pi^5 M_K^5 |F_S^\pi(0)|^2}, \quad (3.3)$$

up to the factor $1/|F_S^\pi(0)|^2$, introduced for convenience.

3.1 Form factors and input parameters used for the generation of pseudo-data

The form factors involved in the preceding expressions are considered as known exactly and are constructed as follows. For $F_{S;\text{data}}^{\pi^0}(s)$, we will basically use the decomposition of Eq. (2.1), with $\mathcal{F}_0^{\pi^0}(s)$ given by Eq. (2.34), and $\mathcal{F}_1^{\pi^0}(s)$ given by Eqs. (2.37) and (2.38). Unfortunately, for some of the functions involved in Eq. (2.34), like $\text{Re } \bar{K}_n^\nabla(s)$ or $\text{Re } \bar{K}_n^x(s)$, explicit analytical expressions are not known. For a numerical approach, we could use their dispersive representation, as given by Eq. (2.26). We have, however, found it more convenient to start from expressions upon which we have full analytical control. For that purpose, one may replace the functions $\text{Re } \bar{K}_n^\nabla(s)$ by the corresponding functions $\text{Re } \bar{K}_n^0(s)$, and likewise $\text{Re } \bar{K}_n^x(s)$ by $\text{Re } \bar{K}_n(s)$. We also drop the contribution proportional to $\Delta_1 \xi_x(s)$. In the range of s we are interested in, $4M_{\pi^0}^2 \leq s \leq M_K^2$, the difference induced in $\mathcal{F}_0^{\pi^0}(s)$ by these changes is numerically very small. For the scalar form factor of the neutral pion, the resulting expression then reads

$$F_{S;\text{data}}^{\pi^0}(s) = \begin{cases} \mathcal{F}_{0;\text{data}}^{\pi^0}(s) - i \mathcal{F}_{1;\text{data}}^{\pi^0}(s) & [s \geq 4M_\pi^2] \\ \mathcal{F}_{0;\text{data}}^{\pi^0}(s) + \mathcal{F}_{1;\text{data}}^{\pi^0}(s) & [4M_{\pi^0}^2 \leq s \leq 4M_\pi^2] \end{cases}, \quad (3.4)$$

with

$$\begin{aligned} \mathcal{F}_{0;\text{data}}^{\pi^0}(s) &= F_S^{\pi^0}(0) \left(1 + a_S^{\pi^0} s + b_S^{\pi^0} s^2 \right) \\ &+ 8\pi F_S^{\pi^0}(0) \varphi_0^{00}(s) \left[1 + a_S^{\pi^0} s \right] \text{Re } \bar{J}_0(s) \\ &- 16\pi F_S^{\pi^0}(0) \varphi_0^x(s) \left[1 + a_S^{\pi^0} s + \frac{2}{\pi} \varphi_0^{+-}(s) \right] \bar{J}^{[0]}(s) \\ &+ \frac{M_\pi^4}{F_\pi^4} F_S^{\pi^0}(0) \left\{ \xi_{00}^{(0)}(s) \text{Re } \bar{J}_0(s) \right. \\ &\left. + \left[\xi_{00}^{(1;0)}(s) + \xi_{00}^{(1;\nabla)}(s) \right] \text{Re } \bar{K}_1^0(s) + \xi_{00}^{(2;0)}(s) \text{Re } \bar{K}_2^0(s) \right. \end{aligned}$$

$$\begin{aligned} &+ \left[\xi_{00}^{(3;0)}(s) + \xi_{00}^{(3;\nabla)}(s) \right] \text{Re } \bar{K}_3^0(s) \Big\} \\ &- 2 \frac{M_\pi^4}{F_\pi^4} F_S^{\pi^0}(0) \left[\xi_x^{(0)}(s) \bar{J}^{[0]}(s) \right. \\ &+ \xi_x^{(1)}(s) \bar{K}_1^{[0]}(s) + 2\xi_x^{(2;\pm)}(s) \bar{K}_2^{[0]}(s) + \xi_x^{(3)}(s) \bar{K}_3^{[0]}(s) \Big] \\ &+ 8F_S^{\pi^0}(0) [\varphi_0^x(s)]^2 \left(16\pi^2 \bar{J}^{[0]}(s) - 2 \right) \text{Re } \bar{J}_0(s) \\ &- 8F_S^{\pi^0}(0) \varphi_0^x(s) \varphi_0^{00}(s) \left[16\pi^2 \text{Re } \bar{J}_0(s) - 2 \right] \bar{J}^{[0]}(s) \\ &+ F_S^{\pi^0}(0) [\varphi_0^{00}(s)]^2 \left[2\text{Re } \bar{K}_2^0(s) + 8\text{Re } \bar{J}_0(s) \right. \\ &\left. + \frac{1}{8} \left(1 - \frac{4M_{\pi^0}^2}{s} \right) \right] \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \mathcal{F}_{1;\text{data}}^{\pi^0}(s) &= e^{-\frac{i}{2} \sigma_0(s) \varphi_0^x(s) \frac{F_S^{\pi^0}(0)}{F_S^\pi(0)}} \hat{\sigma}(s) F_S^\pi(0) \\ &\times \left\{ 1 + a_S^\pi s + \frac{\varphi_0^{+-}(s)}{\pi} \left[2 + \sigma(s) \hat{L}(s) \right] \right. \\ &- \frac{F_S^{\pi^0}(0) \varphi_0^x(s)}{F_S^\pi(0) 2\pi} [2 + \sigma_0(s) L_0(s)] \Big\} \\ &\times \left\{ \varphi_0^x(s) + \frac{M_\pi^4}{F_\pi^4} \frac{1}{8\pi} \left[\xi_x^{(0)}(s) + 2\xi_x^{(1)}(s) \frac{\hat{L}(s)}{\sigma(s)} \right. \right. \\ &\left. \left. + 2\xi_x^{(2;\pm)}(s) \sigma(s) \hat{L}(s) + 3\xi_x^{(3)}(s) \frac{M_\pi^2}{s - 4M_\pi^2} \hat{L}^2(s) \right] \right\}. \end{aligned} \quad (3.6)$$

Finally, in the case of $F_{S;\text{data}}^\pi(s)$, we use the expression (2.42) of the phase-removed form factor, replacing the functions $\bar{K}_n^{x0}(s)$ by $\bar{K}_n^0(s)$, and the functions $\bar{K}_n^\Delta(s)$ by $\bar{K}_n(s)$, respectively. This then gives

$$\begin{aligned} F_{S;\text{data}}^\pi(s) &= F_S^\pi(0) \left(1 + a_S^\pi s + b_S^\pi s^2 \right) \\ &- 8\pi F_S^\pi(0) \varphi_0^x(s) \left[1 + a_S^\pi s + \frac{1}{\pi} \varphi_0^{00}(s) \right] \text{Re } \bar{J}_0(s) \\ &+ 16\pi F_S^\pi(0) \varphi_0^{+-}(s) \left[1 + a_S^\pi s + \frac{2}{\pi} \varphi_0^{+-}(s) \right] \text{Re } \bar{J}(s) \\ &- \frac{M_\pi^4}{F_\pi^4} F_S^\pi(0) \left\{ \xi_x^{(0)}(s) \text{Re } \bar{J}_0(s) + \xi_x^{(1)}(s) \text{Re } \bar{K}_1^0(s) \right. \\ &+ 2\xi_x^{(2;0)}(s) \text{Re } \bar{K}_2^0(s) + \xi_x^{(3)}(s) \text{Re } \bar{K}_3^0(s) \\ &+ 2\xi_x^{(2;\pm)}(s) \left[16\pi^2 \text{Re } \bar{J}(s) - 2 \right] \text{Re } \bar{J}_0(s) \Big\} \\ &+ 2 \frac{M_\pi^4}{F_\pi^4} F_S^\pi(0) \left\{ \xi_{+-;S}^{(0)}(s) \text{Re } \bar{J}(s) \right. \\ &+ \left[\xi_{+-;S}^{(1;\pm)}(s) + \xi_{+-;S}^{(1;\Delta)}(s) \right] \text{Re } \bar{K}_1(s) \\ &\left. + 2\xi_{+-;S}^{(2;\pm)}(s) \left[\text{Re } \bar{K}_2(s) + \frac{1}{32} \left(1 - \frac{4M_\pi^2}{s} \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \left[\xi_{+-;S}^{(3;\pm)}(s) + \xi_{+-;S}^{(3;\Delta)}(s) \right] \text{Re } \bar{K}_3(s) \\
& + 2\xi_{+-;S}^{(2;0)}(s) \left[(16\pi^2 \text{Re } \bar{J}_0(s) - 2) \text{Re } \bar{J}(s) \right. \\
& \left. + \frac{1}{64} \left(1 - \frac{4M_\pi^2}{s} \right) \right] \Bigg\}. \quad (3.7)
\end{aligned}$$

In the sequel, we will generate pseudo-data using the expressions presented in this subsection, considered to provide exact descriptions of the form factors. In particular, it is understood that higher-order contributions are considered as vanishing. As already mentioned, in order to work within a framework where we deal with fully analytical expressions of the form factors, we have made some approximations as compared to the two-loop expressions discussed in the preceding section. Numerically, these differences are small, but most important is that the approximations we have made preserve the general features of the form factors as described after Eq. (2.40).

For the numerical generation of the pseudo-data, we need to fix the values of the various parameters that occur in the expressions of the form factors. As we want to compare different methods of analysis of the $K_{\ell 4}$ form factors, we only aim at choosing values that are representative of the expected situation in these decays, with some limited arbitrariness in this choice. In the following, we consider the case

$$a_0^0 = 0.22 \quad a_0^2 = -0.045. \quad (3.8)$$

We fix the subtraction constants of the form factors by requesting that the charged scalar form factor has the typical values $r_\pi^2 = 0.60 \text{ fm}^2$ and $c_S^\pi = 10 \text{ GeV}^{-4}$ [30], leading to

$$a_S^\pi = 2.63 \text{ GeV}^{-2} \quad b_S^\pi = 2.96 \text{ GeV}^{-4} \quad (3.9)$$

Using Ref. [25], one can compute the isospin-breaking shift between $a_S^{\pi^0}$ and a_S^π . Assuming that $c_S^\pi = c_S^{\pi^0}$, we obtain

$$a_S^{\pi^0} = 2.60 \text{ GeV}^{-2} \quad b_S^{\pi^0} = 3.24 \text{ GeV}^{-4} \quad (3.10)$$

leading to $r_{\pi^0}^2 = 0.59 \text{ fm}^2$. For the remaining parameters, we use the same input values as in Ref. [25]. We normalize the charged scalar form factor to unity at $s = 0$ and rescale the neutral one accordingly:

$$\frac{F_S^{\pi^0}(0)}{F_S^\pi(0)} = 0.99. \quad (3.11)$$

As an illustration, we quote the values obtained for the total decay rate of Eq. (3.2) from these inputs, for $V_{us} = 0.2255$:

$$\Gamma^{\pi^0} = 0.73 \cdot 10^{-22} \text{ GeV}. \quad (3.12)$$

We also show, on the left panel of Fig. 1, the various contributions to the form factor $F_{S;\text{data}}^{\pi^0}(s)$ obtained with our input values. For comparison, the right panel shows the equivalent

results in the case of the parameterization of the K_{e4}^{00} form factor $\mathcal{M}(s)$ discussed in Section 9.4 of Ref. [11]. It corresponds, for $\mathcal{M}_1(s)$, to the parameterization of Eq. (3.13) below with $f_s''' = 0$ and the remaining parameters f_s'/f_s and f_s''/f_s fixed at the central values given in Table 3 of Ref. [3]. For $\mathcal{M}_0(s)$, we take the expression of $\mathcal{F}_0^{\pi^0}(s)$ given in Eq. (3.14). The parameters, f_s , f_{s0}/f_s , f_{s0}'/f_s , and f_{s0}''/f_s it involves are determined from a fit to the phase-space distribution given by Eq. (9.1) of Ref. [11] at $s_e = 0$, and with the values of the remaining coefficients a , b , and d taken at their central values as shown in Table 1 of that same reference. The overall normalization N is fixed such that the distribution is equal to unity at $s = 4M_\pi^2$, $s_e = 0$. One can observe similar features in both plots on Fig. 1 [the absence of an imaginary part in the right-hand plot has already been discussed at the beginning of Sect. 2], suggesting that our subsequent analysis, based on the scalar form factors of the pion, has also some bearing on the K_{e4} form factor. Note that due to our choice of phase space in Eqs. (3.1) and (3.2), in practice the region of interest in Fig. 1 corresponds to $s \leq 0.15 \text{ GeV}$.

In summary, the form factors used in order to generate the pseudo-data are defined by the expressions (3.5), (3.6) and (3.7) (with vanishing higher-order corrections), together with the values (central values for all parameters, no error bars) of the parameters specified above. The form factors thus defined will be referred to as the “exact” form factors, considered to represent the “truth” to which we will fit different model parameterizations of the form factors, in order to obtain a quantitative determination of the possible biases different parameterizations can have on the output of the analysis.

3.2 Phenomenological parameterizations

In order to mimic the situation in K_{e4} decays, the pseudo-data generated with the exact scalar form factors of the pion will now be analyzed using approximate phenomenological parameterizations. For the analysis itself, we will consider a framework close to the experimental set-up for the K_{e4}^{+-} [1–3] and the K_{e4}^{00} [11] decay channels. From here on, we therefore also use $q^2 \equiv s/4M_\pi^2 - 1$ in addition to s , the square of the center-of-mass energy of the di-pion system. The region below the cusp corresponds to $q^2 \leq 0$, while positive values of q^2 describe the region above the cusp.

For the charged-pion form factor, this means that we consider a parameterization of the form

$$F_S^\pi(s) = f_s \left[1 + \frac{f_s'}{f_s} q^2 + \frac{f_s''}{f_s} q^4 + \frac{f_s'''}{f_s} q^6 \right]. \quad (3.13)$$

In the case of the neutral-pion form factor, we consider two parameterizations:

Fig. 1 Contributions, as functions of s (in GeV^2), to $F_S^{\pi^0}(s)$ normalized to its value at $s = 4M_\pi^2$: $\mathcal{F}_0^{\pi^0}(s)$ (black solid), $\text{Re } \mathcal{F}_1^{\pi^0}(s)$ (blue dashed), and $\text{Im } \mathcal{F}_1^{\pi^0}(s)$ (green dotted-dashed). The left panel corresponds to our representation (3.4), (3.5), and (3.6) of the exact form

factors with the values of the parameters discussed in the text. The right panel corresponds to the parameterization used by NA48/2 in Ref. [11] for the K_{e4}^{00} form factor, with the parameters chosen as described in the text after Eq. (3.12)

– Model 1:

$$\begin{aligned}\mathcal{F}_0^{\pi^0}(s) &= f_{s0} \left[1 + \frac{f'_{s0}}{f_{s0}} q^2 + \frac{f''_{s0}}{f_{s0}} q^4 \right] \\ \mathcal{F}_1^{\pi^0}(s) &= -2/3(a_0^0 - a_0^2) f_s \hat{\sigma}(s).\end{aligned}\quad (3.14)$$

– Model 2:

$$\begin{aligned}\mathcal{F}_0^{\pi^0}(s) &= f_{s0} \left[1 + \frac{f'_{s0}}{f_{s0}} q^2 + \frac{f''_{s0}}{f_{s0}} q^4 \right] \\ \mathcal{F}_1^{\pi^0}(s) &= \hat{\sigma}(s) \varphi_0^x(s) f_s \left[1 + \frac{f'_s}{f_s} q^2 + \frac{f''_s}{f_s} q^4 + \frac{f'''_s}{f_s} q^6 \right] \\ &\quad \times e^{-\frac{i}{2} \sigma_0(s) \varphi_0^x(s)}.\end{aligned}\quad (3.15)$$

In order to check the influence of possible higher-order terms in the q^2 expansion, as compared to the parameterization considered in Ref. [11], we have introduced a coefficient f_s''' , which can be set to zero or kept as a free variable in the fit.

The first parameterization with $f_s''' = 0$ reproduces exactly the one that was considered in Sect. 9.4 of Ref. [11]. The second parameterization incorporates more information gathered from the theoretical discussion in Sect. 2, while remaining sufficiently simple. Although we have chosen not to distinguish them, the parameters f_s , f'_s , f''_s , and f_s''' appearing in Eqs. (3.14) and (3.15) are not, a priori, identical to those occurring in the expression (3.13). One issue of the analysis we will present is precisely to determine to which extent e.g. f_s in Eq. (3.14) should be expected to agree with f_s in Eq. (3.13). Our first task is therefore to provide reference values for the various parameters. This is done by performing, in Eqs. (3.5) and (3.7) the Taylor expansion around $q^2 = 0$, thus obtaining f_{s0} , f'_{s0} , f''_{s0} from the former, and f_s , f'_s , f''_s , f_s''' from the latter. In the case of $F_S^{\pi^0}(s)$, we neglect the small half-integer powers of q arising in the expansion, which do not contribute significantly in the

vicinity of $q^2 = 0$. The resulting values are shown in the last column of Table 1. These expansions are not supposed to provide accurate descriptions of the corresponding form factors over the whole physical range.

The comparison with our various fits will illustrate how different the parameters extracted from the fit and those describing the real Taylor expansion can be, and they will thus give information on the possible bias introduced by the fitting procedure. For convenience, in the following f_{s0} , f'_{s0} , ... will be called “neutral” parameters, whereas f_s , f'_s , ... are referred to as the “charged” parameters.

3.3 Fitting procedures

In order to stay close to the NA48/2 experimental set-up, we will thus assume that we have measurements of $|F_S^{\pi^0}(s)|^2$ at the 12 points corresponding to the barycenters of the experimental bins, and we assign a statistical uncertainty derived from the number of events collected in each bin [~ 2900 events in the first two bins, and ~ 5900 events in all the other ones], without any correlations between the bins. As the parameterizations given in Eqs. (3.14) and (3.15) depend on the S -wave $\pi\pi$ scattering lengths, our χ^2 will also include an uncertainty on these quantities in order to mock up the fact that in the real analysis these scattering lengths are determined from the charged form factor. Here we will use the experimental information on these quantities, namely the latest NA48/2 combination of $K \rightarrow 3\pi$ and $K_{\ell 4}$ results [3]

$$\begin{aligned}a_0^0 &= 0.2210 \pm 0.0047 \pm 0.0040, \\ a_0^2 &= -0.0429 \pm 0.0044 \pm 0.0028, \\ \rho_{a_0^0, a_0^2} &= 0.92,\end{aligned}\quad (3.16)$$

where we combine statistical and systematic uncertainties in quadrature. One should notice that the central values are

Table 1 Results of the different models (1–2) and methods (A–B–C), compared to the reference values, obtained from the Taylor expansion of the exact form factors (3.4), (3.5), and (3.6) with the values of the parameters discussed in the text. The value of the ratio r does not result

	A1	A2	Reference
χ^2/N_{dof}	$(9.1 \pm 4.3)/9$	$(6.9 \pm 3.6)/7$	
$f_{s0}/F_S^\pi(0) = f_s/F_S^\pi(0)$	1.38 ± 0.01	1.38 ± 0.01	(1.381, 1.395)
f'_{s0}/f_{s0}	0.18 ± 0.03	0.23 ± 0.03	0.191
f''_{s0}/f_{s0}	-0.03 ± 0.03	-0.33 ± 0.44	-0.059
f'_s/f_s	0	1.91 ± 6.31	0.199
f''_s/f_s	0	-1.01 ± 6.38	-0.032
r	1.00 ± 0.01	0.98 ± 0.04	1
	B1	B2	Reference
χ^2/N_{dof}	$(8.0 \pm 4.0)/8$	$(6.0 \pm 3.2)/6$	
$f_{s0}/F_S^\pi(0)$	1.38 ± 0.01	1.38 ± 0.01	1.381
f'_{s0}/f_{s0}	0.18 ± 0.04	0.21 ± 0.12	0.191
f''_{s0}/f_{s0}	-0.03 ± 0.04	-0.27 ± 0.39	-0.059
f_s/f_{s0}	1.33 ± 0.40	0.97 ± 0.45	1.010
f'_s/f_s	0	0.43 ± 12.8	0.199
f''_s/f_s	0	0.95 ± 14.6	-0.032
r	1.00 ± 0.01	0.97 ± 0.05	1
	C1	C2	Reference
χ^2/N_{dof}	$(3.3 \pm 0.1) \cdot 10^6/18$	$(15.0 \pm 5.6)/16$	
$f_{s0}/F_S^\pi(0)$	1.38 ± 0.01	1.38 ± 0.01	1.381
f'_{s0}/f_{s0}	0.17 ± 0.03	0.19 ± 0.03	0.191
f''_{s0}/f_{s0}	-0.02 ± 0.03	-0.04 ± 0.04	-0.059
f_s/f_{s0}	1.07 ± 0.01	1.01 ± 0.01	1.010
f'_s/f_s	0	0.19 ± 0.01	0.199
f''_s/f_s	0	-0.03 ± 0.01	-0.032
f'''_s/f_s	0	0.01 ± 0.01	0.012
r	1.00 ± 0.01	1.00 ± 0.02	1

close (but not identical) to the “true” values used to generate our pseudo-data.

We consider the following methods to determine the coefficients of the above models.

- Method A: fit of $|F_S^{\pi^0}(s)|^2$ for all points to determine all the (neutral, charged) parameters (setting $f_s''' = 0$ to ensure a reasonable convergence of the fit), assuming the equality of the neutral and charged normalization ($f_s = f_{s0}$).
- Method B: fit of $|F_S^{\pi^0}(s)|^2$ for all points to determine all the (neutral, charged) parameters, setting $f_s''' = 0$ and keeping the normalizations f_s and f_{s0} distinct.
- Method C: fit of $|F_S^{\pi^0}(s)|^2$ to determine the neutral parameters, injecting information on charged parameters by adding to the χ^2 a contribution corresponding to a fit of

from the fit, but is obtained once the fit has been performed; see the last paragraph of Sect. 3.3. The column label A1 (A2) refers to the fit method A using model 1 (2), and so on

the charged form factor $F_S^\pi(s)$ to the polynomial expression (3.13), effectively identifying the charged parameters in the models for $F_S^{\pi^0}(s)$ with the parameters occurring in the charged scalar form factor.

In method C, we generate pseudo-data points for the charged-pion scalar form factor with energies corresponding to the barycenters given in Ref. [1], and we use the relative uncertainties for F_S (combined in quadrature) quoted for each bin in this reference, without correlations. In agreement with Ref. [1], we add an overall 0.62 % relative uncertainty, completely correlated between all the charged bins. The curvature of the charged form factor $F_S^\pi(s)$ being more pronounced than that of the scalar $K_{e4}^\pm$ form factor, a q^6 term must be included in the polynomial in order to obtain a good description of the form factor over the whole kinematic range.

We give the resulting $\chi^2_{\min}$ (obtained from the best-fit values of each method). Even though each model provides through its fit a value of f_{s0} , one can also determine the latter by considering the branching ratio. In this case, f_{s0} is determined by integrating the decay distribution obtained by using as inputs the slope parameters determined from the different methods of fitting, and fixing the normalization by comparison with the total decay rate Γ^{π^0} defined in Eq. (3.2), and evaluated with the exact form factor $F_{S;\text{data}}^{\pi^0}(s)$. The corresponding numerical value is given in Eq. (3.12). We denote by r the ratio between the value of f_{s0} determined this way from the branching ratio, and the true value computed from the exact form factor, i.e. the reference value $f_{s0} = 1.381$.

3.4 Discussion of the results

In order to obtain an estimate of the uncertainty attached to the coefficients of models 1–2 using methods A and B, we will perform fits of the models on a series of 10000 pseudo-experiments, generated by assuming that the data are random variables with a mean given by our theoretical model for the neutral scalar form factor and a standard deviation given by the relative uncertainty of the corresponding form factor measured in K_{e4} decays by the NA48 experiment [1, 11]. We will then determine the mean and the variance of the resulting distribution for each coefficient of the parameterization considered. The results are gathered in Table 1. The column labeled “Reference” provides a comparison with the coefficients obtained from the Taylor expansions of the form factors given in Eqs. (3.5) and (3.7), as described after Eq. (3.15).

As shown by the comparison between models 1 and 2 for methods A and B, the higher powers of q^2 are only weakly constrained. Model 1 is very rough and provides a very poor description of the charged form factor (modeling it as a simple constant), which explains the very bad $\chi^2_{\min}$ for method C1. Only f_{s0} and f'_{s0}/f_{s0} can be determined with a good accuracy, but there is no significant bias introduced by the fitting procedure with respect to the reference values. Despite its shortcomings, method A gives good results for the neutral parameters. As expected, compared to method B, method C provides a much better accuracy on the neutral parameters since the charged ones are constrained in this method. As shown by the ratio r , both methods yield accurate values of f_{s0} (at the few percent level) obtained by integrating over the phase space to consider the branching ratio, even methods that do not attempt at describing the $q^2 < 0$ region correctly. This can easily be understood: both methods are constrained to describe correctly $|F_S^{\pi^0}(s)|^2$ for small $q^2 > 0$ (as can be seen by their agreement concerning f_{s0} and f'_{s0}), but they may differ for $q^2 < 0$ (which exhibit larger uncertainties). However, this region is very narrow ($4M_{\pi^0}^2 \leq s \leq 4M_{\pi^+}^2$) and its contribution is further sup-

pressed by phase space. Therefore, the impact of this region on the estimation of the branching ratio is very small, and the latter is completely dominated by the region $s \geq 4M_{\pi^+}^2$ where all parameterizations agree (the uncertainties reflecting mainly the uncertainties of the inputs and the lack of data at large q^2).

From this discussion, one thus expects that using the fit function given by Eq. (9.1) of Ref. [11], and described at the beginning of Sect. 2, will lead to similar results for the ratio r , despite the fact that this model complies with the expected structure of the cusp only if one imposes strong assumptions, and should be considered as a mere phenomenological parameterization to reproduce data smoothly. One indeed obtains $r = 1 \pm 0.01$ and $\chi^2/N_{\text{dof}} = (8.0 \pm 4.1)/9$ for method A, and $r = 1 \pm 0.01$ and $\chi^2/N_{\text{dof}} = (8.1 \pm 4.1)/8$ for method B, illustrating once more that a smooth parameterization of the curve above the cusp in good agreement with the data is enough to obtain an accurate and unbiased value for the normalization f_{s0} .

The outcome of this discussion is that the measurement of $|F_S^{\pi^0}|^2$ allows for an accurate determination of f_{s0} (at the percent level), in the current experimental setting. As shown by the ratio r , the value of f_{s0} obtained from the computation of the branching ratio is equal (within uncertainties) to its true value for all methods and parameterizations considered here. Even though one has to keep in mind that this observation is done using the pion scalar form factors rather than the actual K_{e4} form factors, it nevertheless suggests that the fit procedure adopted in Ref. [11] does not bias the determination of f_{s0} , and thus cannot explain the surprisingly higher value of f_s extracted by the NA48/2 Collaboration from the K_{e4}^{00} channel, as compared to the value for f_s determined from the K_{e4}^{+-} channel.

3.5 Constraining the scattering lengths

The presence of a cusp similar to the one observed in the three-body $K^+ \rightarrow \pi^+\pi^0\pi^0$ decay suggests that it should, in principle, be possible to extract information on the scattering lengths from an accurate measurement of the K_{e4}^{00} differential decay rate. At leading order, the cusp is related to the difference of scattering lengths $a_0^0 - a_0^2$. Going to higher orders in $\mathcal{F}_1^{\pi^0}$ (i.e. including φ_0^x) will also add a [weaker] dependence on a_0^2 . The scattering lengths can be determined only once the relative normalization of form factors involved in $\mathcal{F}_0^{\pi^0}$ and $\mathcal{F}_1^{\pi^0}$ is fixed, which requires the determination of the charged parameters in some way. We define two methods for this purpose. Method D is exactly as method A, without including any experimental information on a_0^0 and a_0^2 in the χ^2 [i.e. removing them from the χ^2 as described in Eq. (3.16)], and similarly for method E with respect to method C. We proceed as before, but now also fitting the scattering lengths.

Table 2 Results for methods D2 and E2, where $\pi\pi$ scattering lengths are fitted in addition to data on the neutral scalar form factor (for method D) or for both neutral and charged scalar form factors (method E). The

third column corresponds to the case where the uncertainties for the neutral scalar form factor are divided by 10

χ^2/N_{dof}	D2 (4.8 ± 3.1)/5	D2 with $\sigma[K_{\ell 4}^{00}]/10$ (5.6 ± 3.3)/7	Reference (s_0, s)
$f_{s0}/F_S^\pi(0) = f_s/F_S^\pi(0)$	1.38 ± 0.01	1.38 ± 0.01	(1.381, 1.395)
f'_{s0}/f_{s0}	0.20 ± 0.12	0.20 ± 0.01	0.191
f''_{s0}/f_{s0}	0.21 ± 0.88	−0.05 ± 0.06	−0.059
f'_s/f_s	−14 ± 234	0.62 ± 1.86	0.199
f''_s/f_s	32 ± 516	0.00 ± 3.95	−0.032
$a_0^0 - a_0^2$	0.21 ± 0.16	0.25 ± 0.03	0.265
a_0^2	−0.83 ± 2.87	0.00 ± 0.43	−0.045
r	0.92 ± 0.15	0.99 ± 0.02	1
χ^2/N_{dof}	E2 (13.2 ± 5.2)/14	E2 with $\sigma[K_{\ell 4}^{00}]/10$ (12.9 ± 5.1)/14	Reference
$f_{s0}/F_S^\pi(0)$	1.38 ± 0.01	1.38 ± 0.01	1.381
f'_{s0}/f_{s0}	0.22 ± 0.05	0.20 ± 0.01	0.191
f''_{s0}/f_{s0}	−0.09 ± 0.08	−0.05 ± 0.02	−0.059
f_s/f_{s0}	1.01 ± 0.01	1.01 ± 0.01	1.010
f'_s/f_s	0.19 ± 0.01	0.19 ± 0.01	0.199
f''_s/f_s	−0.03 ± 0.01	−0.03 ± 0.01	−0.032
f'''_s/f_s	0.01 ± 0.01	0.01 ± 0.01	0.012
$a_0^0 - a_0^2$	0.25 ± 0.10	0.26 ± 0.03	0.265
a_0^2	0.05 ± 0.16	0.07 ± 0.53	−0.045
r	1.00 ± 0.01	1.00 ± 0.01	1

We consider model 2, as model 1 yielded poor results in the previous section for method C. In order to discuss the potential impact future experimental improvements could have, we consider also a situation where all statistical errors are reduced by 10 (but the number of bins is unchanged) for the neutral channel, keeping the uncertainties unchanged for the charged channel.

The results gathered in Table 2 show that the current statistical uncertainties yield a relative uncertainty on $a_0^0 - a_0^2$ of around 80 % for D2 and 40 % for E2. For D2, the charged parameters are only very poorly constrained, but this does not prevent the fit to be reasonable. Reducing the statistical uncertainties by 10 (for the neutral part) yields a significant reduction in the uncertainties, leading to a relative uncertainty on $a_0^0 - a_0^2$ of 27 % for D2 and 10 % for E2. At this level of accuracy, there is no significant bias in the value of $a_0^0 - a_0^2$ extracted through these various approaches. As expected, no relevant information can be obtained on a_0^2 , due to the very small sensitivity of the neutral-pion channel to this quantity. To illustrate this point, if instead we fix the value of a_0^2 to its central value in Eq. (3.16), our results concerning the uncertainty on $a_0^0 - a_0^2$ and the quality of the fit remain unchanged.

From this discussion, one can hope to get some information on $a_0^0 - a_0^2$ using model 2, should a larger data set become available for K_{e4}^{00} in the future. One has, however, to keep in mind that we have assumed the equality between the charged and neutral normalizations in the polynomials for the neutral scalar form factors in the case of method D, as well as the equality between the charged parameters in the polynomials for the charged and neutral scalar form factors in the case of method E. These assumptions are certainly reasonable considering the current uncertainties involved, but one might need to reassess them in the presence of more accurate data. In this context, it is also interesting to notice that the current result from the DIRAC experiment Ref. [31] is $|a_0^0 - a_0^2| = 0.253 \pm 0.011$, i.e. a 4.3 % uncertainty, so that a substantial increase of the statistical sample of K_{e4}^{00} decays is needed in order to reach a comparable accuracy.

4 Radiative corrections to the K_{e4}^{00} decay rate

In this section we now discuss radiative corrections, which were addressed differently in the analyses of the K_{e4}^{+-} and K_{e4}^{00} channels so far. In the latter case, no radiative corrections were applied to the decay rate measured in Ref. [11].

This accounts for the unspecified factor δ_{EM} in Eq. (1.1). It is thus natural to ask how much of the observed 6.5 % discrepancy [see Eq. (1.2)] in the normalization of the form factor measured in the two channels is due to this correction factor. Our aim here is not to provide a complete discussion of radiative corrections in the K_{e4} decay channels at a level of sophistication that would match the treatment of isospin breaking due to the difference between masses of the charged and neutral pions. We rather want to work out these corrections in a somewhat simpler framework, trying to reproduce a treatment of radiative corrections in the neutral channel similar to the one that was applied in the charged channel, in order to make the comparison as meaningful as possible.

4.1 Treatment of radiative corrections in K_{e4}^{+-} data

Let us recall how radiative corrections are treated in the charged channel [2, 3]. First, virtual photon exchange between all possible pairs of charged external lines are considered, and the corresponding Sommerfeld–Gamow–Sakharov factors are applied. The corrections induced by emission of real photons are treated with PHOTOS [32–35]. The latter also implements the wave-function renormalization on the external charged legs. The couplings of photons to mesons are treated as point-like interactions, given by scalar QED. The result is then free from infrared singularities. Furthermore, one neglects the contributions that vanish when the electron mass goes to zero, which is a sensible limit to consider for the K_{e4} decay channels.

Apart from the Sommerfeld–Gamow–Sakharov factors, some contributions that would arise within a more systematic approach, provided by the effective low-energy theory of QCD and QED for light quarks and leptons [37–39], as applied in Refs. [12–14] to the channel with two charged pions, are not considered. These include, for instance, all structure-dependent corrections, where the photon is emitted from the tree-level K_{e4} vertices or from internal charged lines. The outcome of such a truncated calculation is affected by an ultraviolet divergence, which is removed by renormalizing the coupling $|V_{us}|^2 G_F^2$ [note that in Eq. (12) of Ref. [35] the factor (α/π) has been inadvertently omitted],

$$\left(|V_{us}|^2 G_F^2\right)^{\text{bare}} \left(1 - \frac{9}{4} \frac{\alpha}{\pi} \ln \frac{\Lambda^2}{M_\pi^2}\right) = |V_{us}|^2 G_F^2. \quad (4.1)$$

This same correction factor also appears in Ref. [27], with the ultraviolet cut-off Λ taken equal to M_W . From this last reference, we also see that the factor $9/4$ decomposes as $9/4 = 3 \times (1/2) - 1/4 + 1/2 + 1/2$, where the first contribution comes from the wave-function renormalization of the three charged mesons, the second from the (charged) lepton wave-function renormalization, and the last two ones from the virtual photon loops between the charged kaon and the charged lepton on the one hand, and between the two charged

pions on the other hand (the remaining divergent contributions of this type, i.e. a photon line connecting the external kaon to the charged-pion lines, or the charged lepton with each of the two pions, cancel pairwise).

In the case of the K_{e4}^{00} channel, we therefore expect that this factor becomes $1 \times (1/2) - 1/4 + 1/2 = 3/4$. Since it differs from the previous one, it cannot be absorbed by the renormalization of the same prefactor $|V_{us}|^2 G_F^2$ as before. It seems more natural instead to absorb these ultraviolet divergences into the normalizations of the form factors

$$\begin{aligned} f_s^{\text{bare}} \left(1 - \frac{9}{8} \frac{\alpha}{\pi} \ln \frac{\Lambda^2}{M_\pi^2}\right) &= f_s, \\ f_{s0}^{\text{bare}} \left(1 - \frac{3}{8} \frac{\alpha}{\pi} \ln \frac{\Lambda^2}{M_\pi^2}\right) &= f_{s0}. \end{aligned} \quad (4.2)$$

This is also more in line with the effective theory approach mentioned above, where the form factors are also corrected by (different) contributions from the low-energy constants K_i [38] or X_i [39], which are renormalized by the ultraviolet divergences coming from the photon loops. Using instead Eq. (4.1) in both cases would leave a remaining cut-off dependent contribution to the K_{e4}^{00} amplitude. For a typical value of $\Lambda = 1$ GeV, this would modify Eq. (5.1) at the per mille level.

4.2 Radiative corrections à la PHOTOS for the K_{e4}^{00} decay rate

In the following, we will try to estimate the potential impact of PHOTOS on K_{e4}^{00} rather than pursuing an effective field theory approach. If we want to reproduce the analog of the PHOTOS treatment [35] of radiative corrections for the K_{e4}^{00} decay rate, we need to consider the wave-function renormalization of the charged lepton and of the kaon in (scalar) QED, and the vertex correction corresponding to diagram (a) in Fig. 2. Using a Pauli–Villars regularization, and taking the photon propagator in the Feynman gauge, we reproduce the expressions of Eq. (6) in Ref. [27] for the former. In order to evaluate and discuss the contribution from diagram (a) in Fig. 2, we choose to describe the tree-level K_{e4}^{00} vertex as

$$A_\mu = -i \frac{1}{M_K} \left[F^{00}(p_1 + p_2)_\mu + R^{00}(k - p_1 - p_2)_\mu \right] \quad (4.3)$$

with constant form factors [Bose symmetry forbids a contribution of the form $G^{00}(p_1 - p_2)_\mu$ with G^{00} constant], so that the lowest-order amplitude reads

$$\begin{aligned} \mathcal{A}_0(K_{e4}^{00}) &\equiv \frac{i}{M_K} \frac{G_F}{\sqrt{2}} V_{us} \bar{u}(p_e) \gamma^\mu (1 - \gamma_5) v(p_\nu) \\ &\quad \times \left[F^{00}(p_1 + p_2)_\mu + R^{00}(k - p_1 - p_2)_\mu \right]. \end{aligned} \quad (4.4)$$

In the limit $m_e \rightarrow 0$, we obtain

$$\begin{aligned} \mathcal{A}(K_{e4}^{00}; 2(a)) = & e^2 \mathcal{A}_0(K_{e4}^{00}) \left[\frac{1}{16\pi^2} \left(\ln \frac{\Lambda^2}{M_K^2} + 1 \right) \right. \\ & + 4(k \cdot p_e) C \left((k - p_e)^2; M_K, m_e \right) \\ & - 2M_K^2 C_{11} \left((k - p_e)^2; M_K, m_e \right) \\ & \left. - 4(k \cdot p_e) C_{12} \left((k - p_e)^2; M_K, m_e \right) \right] \\ & + i \frac{e^2}{M_K} \frac{G_F}{\sqrt{2}} V_{us} R^{00} (p_1 + p_2)_\nu \\ & \times \bar{u}(p_e) \gamma^\nu (1 - \gamma_5) v(p_{\nu_e}) \\ & \times \left[\frac{1}{32\pi^2} \left(3 \ln \frac{\Lambda^2}{M_K^2} + \frac{1}{2} \right) \right. \\ & - 4(k \cdot p_e) C_{11} \left((k - p_e)^2; M_K, m_e \right) \\ & \left. + 2\bar{J} \left((k - p_e)^2; M_K, m_e \right) \right]. \quad (4.5) \end{aligned}$$

The various loop functions occurring in this expression are defined in Appendix B. Adding to it the wave-function renormalizations on the charged external lines gives the following result, in the framework adopted here, for the radiatively corrected amplitude (m_γ denotes a small photon mass, introduced as an infrared regulator, to be sent to zero once an infrared-safe observable has been constructed):

$$\begin{aligned} \sqrt{Z_e} \sqrt{Z_K} \mathcal{A}_0(K_{e4}^{00}) + \mathcal{A}(K_{e4}^{00}; 2(a)) = & \mathcal{A}_0(K_{e4}^{00}) \\ & \times \left[1 + \frac{e^2}{(4\pi)^2} \left(\frac{3}{2} \ln \frac{\Lambda^2}{M_\pi^2} - \frac{1}{2} \ln \frac{m_e^2}{M_\pi^2} + 2 \ln \frac{m_e^2}{m_\gamma^2} \right. \right. \\ & - \ln \frac{M_K^2}{M_\pi^2} - 2 \left. \right) + e^2 \left(4(k \cdot p_e) C \left((k - p_e)^2; M_K, m_e \right) \right. \\ & - 2M_K^2 C_{11} \left((k - p_e)^2; M_K, m_e \right) \\ & \left. \left. - 4(k \cdot p_e) C_{12} \left((k - p_e)^2; M_K, m_e \right) \right) \right] \\ & + i \frac{e^2}{M_K} \frac{G_F}{\sqrt{2}} V_{us} R^{00} (p_1 + p_2)_\nu \times u(p_e) \gamma^\nu (1 - \gamma_5) v(p_{\nu_e}) \\ & \times \left[\frac{1}{32\pi^2} \left(3 \ln \frac{\Lambda^2}{M_K^2} + \frac{1}{2} \right) + 2\bar{J} \left((k - p_e)^2; M_K, m_e \right) \right. \\ & \left. - 4(k \cdot p_e) C_{11} \left((k - p_e)^2; M_K, m_e \right) \right]. \quad (4.6) \end{aligned}$$

We make a few comments about this result:

- Although the result (4.5) holds in the Feynman gauge $\xi = 1$, we have also computed the wave-function renormalizations and $\mathcal{A}(K_{e4}^{00}; 2(a))$ in an arbitrary linear and covariant ξ -gauge, and we have checked that the final result (4.6) does actually not depend on the gauge-fixing parameter ξ .
- In order to reproduce the analog of the PHOTOS treatment [27,35] of radiative corrections for the K_{e4}^{00} decay rate, one needs to add the emission of soft photons from the charged external lines, diagrams (a) and (c) of Fig. 3 (in the K_{e4}^{+-} case, there are two more diagrams where the photon is emitted from the charged-pion lines) so that the result is free of infrared singularities at order α . These corrections will be discussed later on. At this stage, we simply note that the infrared-divergence of Eq. (4.6) is equal to

$$\frac{e^2}{(4\pi)^2} \mathcal{A}_0(K_{e4}^{00}) \times \ln m_\gamma [-4 - 2(k \cdot p_e) \tau(k, p_e)], \quad (4.7)$$

with the function $\tau(p_1, p_2)$ defined in Eq. (B.21).

- The factor $\frac{3}{8} \frac{\alpha}{\pi} \ln \frac{\Lambda^2}{M_\pi^2}$ discussed in Eq. (4.2) is indeed to be found in Eq. (4.6), provided that one sets R^{00} to zero. The only remaining contribution comes from F^{00} , which is proportional to f_{s0} at this level. This indeed corresponds to the situation considered in Ref. [27]. In the absence of radiative corrections, the form factor R^{00} (or R^{+-} in the charged channel) does not contribute to the decay distribution for $m_e = 0$. In this case, one may as well take $R^{00} = 0$ from the beginning. But once radiative corrections are switched on, taking $m_e = 0$ or $R^{00} = 0$ are no longer equivalent options. As shown by the second contribution in Eq. (4.6), there is a correction to F^{00} that is induced by R^{00} , and this contribution is not considered in Ref. [27], and is hence also missing in Ref. [35].
- At lowest order, and in the isospin limit, one has [36]

$$\begin{aligned} R^{00} = & \frac{F^{00}}{2} \left[1 + \frac{s_\pi}{s_e - M_K^2} \right] \\ = & F^{00} \left[\frac{2}{3} - \frac{1}{2} \left(\frac{1}{3} - \frac{s_\pi}{s_e - M_K^2} \right) \right]. \quad (4.8) \end{aligned}$$

Actually, as shown in the second expression, the vertex in the diagram (a) in Fig. 2 only accounts for the contribution $R^{00} = (2/3) \cdot F^{00}$. The second factor comes from the diagram (f) in Fig. 2.

4.3 Additional non-factorizable radiative corrections to the K_{e4}^{00} decay rate

After these preliminary remarks concerning the PHOTOS-type treatment of radiative corrections in the K_{e4}^{+-} and K_{e4}^{00} channels, let us now address radiative corrections in the channel with two neutral pions in a somewhat more systematic manner. This will allow us to estimate the size of the radiative corrections that are not included in the experimental analysis, as described in the previous section. We keep on considering the limit where m_e vanishes, so that in the absence of radiative corrections the amplitude reads simply

$$\mathcal{A}_0(K_{e4}^{00}) = i \frac{G_F}{\sqrt{2}} V_{us} \bar{u}(p_e) (\not{p}_1 + \not{p}_2) (1 - \gamma_5) v(p_\nu) \times \frac{F^{00}}{M_K} + \mathcal{O}(m_e). \quad (4.9)$$

For our purpose, it is convenient to distinguish between two types of radiative corrections, which we call factorizable and non-factorizable. Factorizable radiative corrections are defined by the contributions where both ends of the virtual photon line connect to a charged mesonic line or to the vertex with the leptonic current, or when both ends connect to the electron line. These factorizable contributions will not modify the structure of the matrix element, but will change the form factors F^{00} and R^{00} . We find it convenient to express them as⁴

$$\mathcal{A}_{\text{fact}}(K_{e4}^{00}) \equiv i \frac{G_F}{\sqrt{2}} V_{us} \bar{u}(p_e) (\not{p}_1 + \not{p}_2) (1 - \gamma_5) v(p_\nu) \times \frac{F^{00}}{M_K} \times \sqrt{Z_e} \sqrt{Z_K} + \mathcal{O}(m_e), \quad (4.10)$$

where we have factored out the wave-function renormalization factors computed in QED for Z_e , and in scalar QED for Z_K :

$$\begin{aligned} \sqrt{Z_e} &= 1 + e^2 \left[\bar{\lambda} - \frac{1}{(4\pi)^2} \left(\frac{3}{2} - \frac{1}{2} \ln \frac{m_e^2}{\mu^2} - \ln \frac{m_e^2}{m_\gamma^2} \right) \right] \\ \sqrt{Z_K} &= 1 + e^2 \left[-2\bar{\lambda} - \frac{1}{(4\pi)^2} \left(1 + \ln \frac{M_K^2}{\mu^2} - \ln \frac{M_K^2}{m_\gamma^2} \right) \right]. \end{aligned} \quad (4.11)$$

In contrast to the preceding subsection, we use now dimensional regularization, with the minimal subtraction of the combination

$$\bar{\lambda} = \frac{1}{16\pi^2} \left[\frac{1}{d-4} - \frac{1}{2} (\ln(4\pi) + \Gamma'(1) + 1) \right]. \quad (4.12)$$

⁴ In the K_{e4}^{+-} case, there are additional structures in $\mathcal{A}_{\text{fact}}(K_{e4}^{+-})$, due to the possibility, already at tree level, for a virtual photon to emit a pair of charged pions; see [12, 14]. Notice in this respect that the contributions in Fig. 6 and in Fig. 8 of Ref. [14] vanish in our case.

It is understood that F^{00} in Eq. (4.10) now includes all the remaining factorizable photonic corrections, together with the contributions from the low-energy constants L_i [37] and K_i [38]. These will take care of the UV divergences due to the meson loops and to the photon loops, respectively, so that the product $F^{00} \sqrt{Z_K}$ is actually UV finite. It, however, inherits the infrared divergence contained in $\sqrt{Z_K}$.

Let us next consider the non-factorizable contributions. As far as the corrections to F^{00} are concerned, they are represented by the diagrams shown in Fig. 2. One finds that the contributions coming from the diagrams (b), (c), and (d) are proportional to the lepton mass, and thus vanish in the limit $m_e \rightarrow 0$. There are therefore only three diagrams to compute in this approximation. Consistently dropping terms that vanish as $m_e \rightarrow 0$, one finds that

$$\begin{aligned} \mathcal{A}(K_{e4}^{00}; 2(a)) &= \mathcal{A}_0(K_{e4}^{00}) \times e^2 \\ &\times \left[4(k \cdot p_e) C \left((k - p_e)^2; M_K, m_e \right) \right. \\ &- 2(k \cdot p_e) C_{12} \left((k - p_e)^2; M_K, m_e \right) \\ &- \bar{J} \left((k - p_e)^2; M_K, m_e \right) - 4\bar{\lambda} \\ &\left. - \frac{1}{16\pi^2} \left(\ln \frac{M_K^2}{\mu^2} + \ln \frac{m_e^2}{\mu^2} - \frac{10}{3} \right) \right], \end{aligned} \quad (4.13)$$

$$\begin{aligned} \mathcal{A}(K_{e4}^{00}; 2(e)) &= \mathcal{A}_0(K_{e4}^{00}) \times \frac{e^2}{3} \\ &\times \left[\bar{J} \left((k - p_1 - p_2)^2, m_\gamma, M_K \right) - 2\bar{\lambda} \right. \\ &\left. - \frac{1}{16\pi^2} \left(\ln \frac{M_K^2}{\mu^2} - 1 \right) \right], \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} \mathcal{A}(K_{e4}^{00}; 2(f)) &= \mathcal{A}_0(K_{e4}^{00}) \times \frac{e^2}{2} \\ &\times \left[2(p_1 + p_2)^2 C \left((p_1 + p_2)^2, (k - p_1 - p_2)^2; M_K, M_K \right) \right. \\ &- (p_1 + p_2)^2 C_{11} \left((p_1 + p_2)^2, (k - p_1 - p_2)^2; M_K, M_K \right) \\ &- \frac{2}{3} \bar{J} \left((k - p_1 - p_2)^2; m_\gamma, M_K \right) + \frac{7}{3} \bar{\lambda} \\ &\left. + \frac{1}{16\pi^2} \left(\frac{7}{6} \ln \frac{M_K^2}{\mu^2} - \frac{2}{3} \right) \right]. \end{aligned} \quad (4.15)$$

Apart from the change of regularization, the expression for $\mathcal{A}(K_{e4}^{00}; 2(a))$ in Eq. (4.13) reproduces the one of Eq. (4.5) obtained previously, provided one takes $R^{00} = (2/3) \cdot F^{00}$, as discussed at the end of Sect. 4.2, and makes use of the identities

Fig. 2 The set of one-loop virtual photon exchange diagrams of the non-factorizable type to consider for the K_{e4}^{00} decay

$$\begin{aligned}
 (k \cdot p_e) C_{11} \left((k - p_e)^2; M_K, m_e \right) &= -\frac{1}{32\pi^2} \\
 &+ \frac{1}{2} \bar{J} \left((k - p_e)^2; M_K, m_e \right) + \dots, \\
 M_K^2 C_{11} \left((k - p_e)^2; M_K, m_e \right) &= -\frac{1}{32\pi^2} \left(1 + \ln \frac{M_K^2}{m_e^2} \right) \\
 &+ \frac{1}{2} \bar{J} \left((k - p_e)^2; M_K, m_e \right) \\
 &- (k \cdot p_e) C_{12} \left((k - p_e)^2; M_K, m_e \right) + \dots, \quad (4.16)
 \end{aligned}$$

where the ellipses denote terms that vanish in the limit $m_e \rightarrow 0$.

Adding up the contributions discussed so far, one obtains an expression for the radiative corrections at order $\mathcal{O}(\alpha)$ that still contains both infrared and ultraviolet divergences. The latter will be taken care of by the contributions from the counterterms X_i introduced in [39]. Their contribution reads

$$\mathcal{A}(K_{e4}^{00}; \text{cts}) = \mathcal{A}_0(K_{e4}^{00}) \times e^2 \left(-\frac{8}{3} X_1 - \frac{1}{2} X_6 \right). \quad (4.17)$$

The low-energy constant X_1 is not renormalized, whereas $X_6 = X_6^r(\mu) - 5\bar{\lambda}$. Collecting the divergent pieces from the various contributions leads to (we recall that at this stage $F^{00}\sqrt{Z_K}$ has already been made UV-finite through the contributions of the low-energy constants L_i and K_i)

$$\begin{aligned}
 \mathcal{A}(K_{e4}^{00}; \text{UV-div}) &= e^2 \mathcal{A}_0(K_{e4}^{00}) \\
 &\times \frac{\bar{\lambda}}{6} \left[\underbrace{-24}_{2(a)} \underbrace{-4}_{2(e)} \underbrace{+7}_{2(f)} \underbrace{+6}_{\sqrt{Z_e}} \underbrace{+15}_{X_6} \right] = 0, \quad (4.18)
 \end{aligned}$$

which vanishes, as it should.

As to the infrared divergences, collecting the IR-divergent pieces contained in the contributions computed so far, one obtains

$$\begin{aligned}
 \mathcal{A}(K_{e4}^{00}; \text{IR-div}) &= \frac{e^2}{(4\pi)^2} \mathcal{A}_0(K_{e4}^{00}) \\
 &\times \ln m_\gamma \times \left[\underbrace{(-2)}_{\sqrt{Z_e}} + \underbrace{(-2)}_{\sqrt{Z_K}} + \underbrace{(-2)(k \cdot p_e) \tau(k, p_e)}_{2(a)} \right], \quad (4.19)
 \end{aligned}$$

with the function $\tau(p_1, p_2)$ defined in Eq. (B.21). Besides the wave-function renormalizations, such divergences only arise from the contribution of $C((k - p_e)^2; M_K, m_e)$ in $\mathcal{A}(K_{e4}^{00}; 2(a))$. Notice that this infrared divergence coincides with the one of the result (4.6), given in Eq. (4.7). The construction of an infrared-safe observable at order $\mathcal{O}(\alpha)$ requires also to consider the process with the emission of one soft photon. The corresponding differential decay rate is given by

$$\begin{aligned}
 d\Gamma(K_{e4\gamma}^{00}) &= \frac{1}{2M_K} \frac{1}{2} \sum_{\text{spins, pol.}} \left| \mathcal{A}(K_{e4\gamma}^{00}) \right|^2 \\
 &\times (2\pi)^4 \delta^4(k - p_1 - p_2 - p_\ell - p_\nu - q) \\
 &\times \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} \frac{d^3\mathbf{p}_\ell}{(2\pi)^3 2E_\ell} \frac{d^3\mathbf{p}_\nu}{(2\pi)^3 2|\mathbf{p}_\nu|} \frac{d^3\mathbf{q}}{(2\pi)^3 2|\mathbf{q}|}. \quad (4.20)
 \end{aligned}$$

The amplitude $\mathcal{A}(K_{e4\gamma}^{00})$ for the radiative decay $K^\pm \rightarrow \pi^0 \pi^0 e^\pm \nu_e \gamma$ can be expanded in powers of the photon energy,

$$\mathcal{A}(K_{e4\gamma}^{00}) = \mathcal{A}_{-1}(K_{e4\gamma}^{00}) + \mathcal{A}_0(K_{e4\gamma}^{00}) + \dots \quad (4.21)$$

The Low approximation consists in keeping $\mathcal{A}_{-1}(K_{e4\gamma}^{00})$ alone. This is enough in order to study the emission of only soft photons and to discuss the issue of infrared divergences. Explicitly, one has [q_μ is the momentum of the emitted (real) photon, $\varepsilon^*(q)$ the corresponding polarization vector]

$$\mathcal{A}_{-1}(K_{e4\gamma}^{00}) = e \mathcal{A}_0(K_{e4}^{00}) \left(\frac{p_\ell \cdot \varepsilon^*(q)}{p_\ell \cdot q + \frac{m_\gamma^2}{2}} - \frac{k \cdot \varepsilon^*(q)}{k \cdot q - \frac{m_\gamma^2}{2}} \right). \quad (4.22)$$

Fig. 3 The one-photon emission contributions to the $K_{e4\gamma}^{00}$ decay

Then

$$\sum_{\text{spins, pol.}} \left| \mathcal{A}_{-1}(K_{e4\gamma}^{00}) \right|^2 = -e^2 \sum_{\text{spins}} \left| \mathcal{A}(K_{e4}^{00}) \right|^2 \times \left[\frac{m_\ell^2}{\left(p_\ell \cdot q + \frac{m_\gamma^2}{2} \right)^2} + \frac{M_K^2}{\left(k \cdot q - \frac{m_\gamma^2}{2} \right)^2} - 2 \frac{k \cdot p_\ell}{\left(p_\ell \cdot q + \frac{m_\gamma^2}{2} \right) \left(k \cdot q - \frac{m_\gamma^2}{2} \right)} \right]. \quad (4.23)$$

One may then perform the integration over the undetected soft photon. In the soft-photon approximation, the photon momentum in the delta-function of the phase-space integration is neglected, and one takes

$$\begin{aligned} d\Gamma^{\text{soft}}(K_{e4\gamma}^{00}) &= (2\pi)^4 \delta^4(k - p_1 - p_2 - p_\ell - p_\nu) \\ &\times \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} \frac{d^3\mathbf{p}_\ell}{(2\pi)^3 2E_\ell} \frac{d^3\mathbf{p}_\nu}{(2\pi)^3 2|\mathbf{p}_\nu|} \\ &\times \frac{1}{2M_K} \frac{1}{2} \int_{|\mathbf{q}| \leq \Delta E} \frac{d^3\mathbf{q}}{(2\pi)^3 2|\mathbf{q}|} \sum_{\text{spins, pol.}} \left| \mathcal{A}_{-1}(K_{e4\gamma}^{00}) \right|^2 \\ &= d\Gamma_0(K_{e4}^{00}) \times (-e^2) \int_{|\mathbf{q}| \leq \Delta E} \frac{d^3\mathbf{q}}{(2\pi)^3 2|\mathbf{q}|} \\ &\times \left[\frac{m_\ell^2}{\left(p_\ell \cdot q + \frac{m_\gamma^2}{2} \right)^2} + \frac{M_K^2}{\left(k \cdot q - \frac{m_\gamma^2}{2} \right)^2} - 2 \frac{k \cdot p_\ell}{\left(p_\ell \cdot q + \frac{m_\gamma^2}{2} \right) \left(k \cdot q - \frac{m_\gamma^2}{2} \right)} \right]. \end{aligned} \quad (4.24)$$

Expressions for the corresponding integrals can be found in [12, 14]. The integration is limited to photon energies $|\mathbf{q}|$ below the experimental detection threshold ΔE in the kaon rest frame. As far as the infrared divergences are concerned, one has

$$d\Gamma_{\text{IR-div}}^{\text{soft}}(K_{e4\gamma}^{00}) = d\Gamma_0(K_{e4}^{00}) \times \frac{e^2}{8\pi^2} \times \ln m_\gamma \times [2 + 2 + 2(k \cdot p_e)\tau(k, p_e)]. \quad (4.25)$$

Therefore, the contributions proportional to $\ln m_\gamma$ cancel in the sum $d\Gamma(K_{e4}^{00}) + d\Gamma(K_{e4\gamma}^{00})$. For later convenience, we rewrite Eq. (4.24) in a way that explicitly displays the IR-singular part:

$$d\Gamma^{\text{soft}}(K_{e4\gamma}^{00}) = d\Gamma_0(K_{e4}^{00}) \times \frac{e^2}{8\pi^2} [2 + (k \cdot p_e)\tau(k, p_e)] \times 2 \ln \frac{m_\gamma}{2\Delta E} + d\bar{\Gamma}^{\text{soft}}(K_{e4\gamma}^{00}). \quad (4.26)$$

We then add the contribution

$$\mathcal{A}_0(K_{e4}^{00}) \times \frac{e^2}{8\pi^2} [2 + (k \cdot p_e)\tau(k, p_e)] \times \ln \frac{m_\gamma}{2\Delta E} \quad (4.27)$$

to the amplitudes involving virtual photons, such as to make them infrared finite. To this end, we define the function

$$\begin{aligned} \bar{C}\left((k - p_e)^2; M_K, m_e; \Delta E\right) &= C\left((k - p_e)^2; M_K, m_e\right) \\ &+ \frac{1}{32\pi^2} \tau(k, p_e) \times \ln \frac{m_\gamma}{2\Delta E}. \end{aligned} \quad (4.28)$$

4.4 Discussion of radiative corrections for K_{e4}^{00}

We can now add the virtual and real contributions described up to now which should be involved in a PHOTOS-like treatment of this decay. We include them as a correction of the form $(1 + \delta_{\text{EM}})$ to the determination of the form factor from the measurement of the branching ratio. To this end, we compute the total decay rate including the soft-photon emission

$$\Gamma^{\text{tot}} = \Gamma(K_{e4}^{00}) + \bar{\Gamma}^{\text{soft}}(K_{e4\gamma}^{00}), \quad (4.29)$$

where $\Gamma(K_{e4}^{00})$ includes corrections at first order in the fine-structure constant α , and write it in terms of the decay rate $\Gamma_0(K_{e4}^{00})$ without radiative corrections in the form

$$\Gamma^{\text{tot}} = \Gamma_0(K_{e4}^{00}) \times (1 + 2\delta_{\text{EM}}). \quad (4.30)$$

Let us first discuss the corrections computed in subsection B above. In order to obtain a result that is as close as possible to the treatment of radiative corrections in the K_{e4}^{+-} channel, we absorb the UV-divergent factor of Eq. (4.6) in F^{00} and take

R^{00} equal to zero. Then $\Gamma(K_{e4}^{00})$ is computed by performing the phase-space integration of

$$\mathcal{A}_0(K_{e4}^{00}) \left[\mathcal{A}_0(K_{e4}^{00}) + 2e^2 \Delta \mathcal{A}(K_{e4}^{00}) \right], \quad (4.31)$$

where

$$\begin{aligned} \Delta \mathcal{A}(K_{e4}^{00}) = & \mathcal{A}_0(K_{e4}^{00}) \left[\frac{1}{(4\pi)^2} \left(-\frac{1}{2} \ln \frac{m_e^2}{M_\pi^2} + 4 \ln \frac{m_e}{2\Delta E} \right. \right. \\ & \left. \left. - \ln \frac{M_K^2}{M_\pi^2} - 2 \right) \right. \\ & + 4(k \cdot p_e) \bar{C} \left((k - p_e)^2; M_K, m_e; \Delta E \right) \\ & - 2M_K^2 C_{11} \left((k - p_e)^2; M_K, m_e \right) \\ & \left. - 4(k \cdot p_e) C_{12} \left((k - p_e)^2; M_K, m_e \right) \right]. \quad (4.32) \end{aligned}$$

We take $\Delta E = 11.7$ MeV, the value corresponding to the NA48/2 experiment [11], for the real-photon detection threshold in the kaon rest frame. This gives then

$$\delta_{EM} = 0.018. \quad (4.33)$$

This value has the expected size. Moreover, it goes into the right direction, in the sense that it reduces the discrepancy in Eq. (1.2) from 6.5 to 4.6 %.

As a test of the stability of the result (4.33) we may also evaluate the non-factorizable radiative corrections corresponding to all the diagrams in Fig. 2. This amounts to taking the expressions in Eqs. (4.13), (4.14), and (4.15) for the evaluation of $\Gamma(K_{e4}^{00})$ (let us stress again that these equations have been obtained in a regularization scheme differing from the one discussed in Sect. 4.2). We absorb the ultraviolet divergences, as well as the contribution (4.17) into F^{00} , in order to build a quantity both UV and IR finite. Constant terms have been discarded as they could be included in the contribution of the counterterms X_1 and X_6 . The resulting expression for $\Delta \mathcal{A}(K_{e4}^{00})$ then reads

$$\begin{aligned} \Delta \mathcal{A}(K_{e4}^{00}) = & \mathcal{A}_0(K_{e4}^{00}) \left[\frac{1}{(4\pi)^2} \left(-\frac{7}{4} \ln \frac{M_K^2}{M_\pi^2} - \frac{1}{2} \ln \frac{m_e^2}{M_\pi^2} \right. \right. \\ & \left. \left. + 2 \ln \frac{m_e}{2\Delta E} + 2 \ln \frac{M_K}{2\Delta E} \right) \right. \\ & + 4(k \cdot p_e) \bar{C} \left((k - p_e)^2; M_K, m_e; \Delta E \right) \\ & - 2(k \cdot p_e) C_{12} \left((k - p_e)^2; M_K, m_e \right) \\ & - \frac{2}{3} \bar{J} \left((k - p_e)^2; M_K, m_e \right) \\ & + (p_1 + p_2)^2 C \left((p_1 + p_2)^2, (k - p_1 - p_2)^2; M_K, M_K \right) \\ & - \frac{1}{2} (p_1 + p_2)^2 \\ & \times C_{11} \left((p_1 + p_2)^2, (k - p_1 - p_2)^2; M_K, M_K \right) \\ & \left. - \frac{1}{3} \bar{J} \left((k - p_1 - p_2)^2; m_\gamma, M_K \right) \right], \quad (4.34) \end{aligned}$$

instead of the expression in Eq. (4.32). For $\Delta E = 11.7$ MeV, we obtain now

$$\delta_{EM} = 0.017. \quad (4.35)$$

This value is quite close to the one obtained in Eq. (4.33), so that in the present case the treatment of radiative corrections *à la* PHOTOS seems to yield stable results even after the inclusion of non-factorizable contributions.

5 Summary and conclusion

The present study is devoted to isospin-breaking effects in the semileptonic decay of the charged kaon into two neutral pions, $K^\pm \rightarrow \pi^0 \pi^0 e^\pm \bar{\nu}_e$. Because of the smallness of the electron mass and of the limited experimental precision, this decay can be described in terms of a single form factor. This form factor also occurs in the description of the decay into two charged pions, $K^\pm \rightarrow \pi^+ \pi^- e^\pm \bar{\nu}_e$, and up to isospin-breaking contributions, the two determinations should agree. The present study focuses mainly on two aspects related to this issue: (1) to ascertain quantitatively to which extent the phenomenological parameterizations used in order to analyze the data could impinge on the resulting value of the normalization or on the shape of the form factor measured in the decay $K^\pm \rightarrow \pi^0 \pi^0 e^\pm \bar{\nu}_e$, and (2) to obtain a quantitative estimate of the radiative corrections to the total decay rate, which again might affect the normalization of the form factor.

Concerning the first issue, we have considered the form factors of the pion as a case study. As a first step, we have discussed the structure of the form factors, and their properties linked to the presence of a cusp, using exact expressions of the form factors valid up to two loops in the low-energy expansion. We have clearly established that the phenomenological parameterizations used in order to analyze the data did not agree with the general properties that can be inferred from these exact expressions. In a second step, we have generated pseudo-data from these form factors, which we have then analyzed using several phenomenological parameterizations. The outcome of this study is that the determination of the normalization of the form factor is actually not sensitive to the parameterizations used. As a side product, we see that the higher orders in the Taylor expansion of form factors are not accurately determined by a direct fit to simplified (polynomial) formulas, as shown in Table 1. Although our study was carried out for the scalar form factor of the neutral pion, we expect that the conclusion also holds for the K_{e4}^{00} form factor. This expectation rests on the following observations: (1) the two form factors have a similar shape, in particular as far as the cusp is concerned, the curvature of the scalar form factor being somewhat more pronounced, see Fig. 1; however, as just stated, the coefficients of the higher powers

in q^2 are only weakly constrained by the fits. (2) In both form factors the cusp is described by a constant times the function $\hat{\sigma}(s)$ at first approximation. Refining the description of the cusp, as in model 2 for instance, does not impinge on the determination of the normalization of the form factor. This is related to the fact that, in both cases, the region between threshold and cusp is very narrow.

We have also considered the possibility to constrain the $\pi\pi$ S -wave scattering lengths from the measurement of the decay distribution. We have found that, unfortunately, with the sample of events presently available, the statistical uncertainties remain large. A statistical sample comparable to the one available in the K_{e4}^{+-} channel would be required in order to reach a precision close to that obtained by the Dirac experiment.

The second issue addressed in this paper consists in radiative corrections. We have determined the correction factor δ_{EM} to the total decay rate in Eq. (1.2). In order to make a meaningful comparison with the value of the normalization of the form factor extracted from the K_{e4}^{+-} channel, we have used a simplified framework, including only those corrections that were also included in the latter case (one-loop photonic corrections on the wave functions and tree-level vertex). Our result $\delta_{EM} = 0.018$ leads to the replacement of Eq. (1.2) by

$$\frac{f_s[K_{e4}^{00}]}{f_s[K_{e4}^{+-}]} = 1.046(10). \quad (5.1)$$

Note that the error bar in this equation is purely from experimental origin, and does not include the systematic uncertainties from the methods used for the evaluation of radiative corrections in both channels. Such additional uncertainties can stem, for instance, from the regularization dependence of the PHOTOS(-like) treatment of radiative corrections, and from neglecting the dependence in the cut-off Λ discussed in Sect. 4.1.

We have also considered additional photonic corrections estimated within a different regularization scheme, and we have found that they do not modify the previous estimate in a significant way. A few comments are in order:

- The analysis of radiative corrections we have performed provides an adequate estimate of the global factor δ_{EM} that modifies the total decay rate. It need not be suitable for an analysis of radiative corrections to the phase-space distribution itself.
- Other isospin-breaking corrections, among them factorizable exchanges of virtual photons, but also effects due to $m_u - m_d$ or to the mass differences between charged and neutral pions and/or kaons, are not covered by our analysis. They could affect the normalization of the form factors measured in the two channels in different ways.

A more elaborate study is needed in order to reach a quantitatively meaningful interpretation of the result in Eq. (5.1).

- At lowest order in the chiral expansion, these additional isospin-breaking corrections are given by Eq. (1.3). For $R = 35.8(1.9)(1.8)$ [17], and adding errors in quadrature, we obtain

$$\frac{f_s[K_{e4}^{00}]}{f_s[K_{e4}^{+-}]} \Big|_{\text{LO}} = 1.042(3). \quad (5.2)$$

In view of the value given in Eq. (5.1), the corrections from higher orders to this ratio should therefore be small.

- Conversely, using the relation (1.3) in regard to the result (5.1), and discarding yet to be computed corrections to the former, we obtain $R = 32_{-6}^{+9}$.

The discussion of the radiative corrections presented here is clearly only a first step. In view of the statistical accuracy of the data, a full model-independent calculation of these corrections in the neutral as well as in the charged channels is certainly mandatory before a definite conclusion can be reached concerning the observed difference in the normalization of the form factors between the neutral and the charged channels. This task is clearly beyond the scope of the present note, and is left for future work.

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Appendix A: Properties of the functions $\bar{K}_n^\alpha(s)$

In this appendix, we wish to summarize the properties of the functions $\bar{K}_n^\alpha(s)$ that are needed in the discussion of the cusp in Sect. 2. Let us start with the functions $\bar{K}_n^0(s)$ and $\bar{K}_n^\nabla(s)$, defined by dispersive integral as in Eq. (2.26), with $s_{\text{thr}} = 4M_{\pi^0}^2$ and [40]

$$\begin{aligned} k_0^0(s) &= \frac{1}{16\pi} \sigma_0(s), \\ k_1^0(s) &= \frac{1}{8\pi} L_0(s), \\ k_2^0(s) &= \frac{1}{8\pi} \left(1 - \frac{4M_{\pi^0}^2}{s} \right) L_0(s), \end{aligned}$$

$$\begin{aligned}
k_3^0(s) &= \frac{3}{16\pi} \frac{M_{\pi_0}^2}{s\sigma_0(s)} L_0^2(s), \\
k_1^\nabla(s) &= \frac{1}{8\pi} \frac{\sigma_0(s)}{\sigma(s+4\Delta_\pi)} L(s+4\Delta_\pi), \\
k_3^\nabla(s) &= \frac{3}{16\pi} \frac{M_\pi^2}{s\sigma_0(s)} L^2(s+4\Delta_\pi).
\end{aligned} \quad (\text{A.1})$$

The definitions of the functions $\sigma_0(s)$ and $L_0(s)$ for $s \geq 4M_{\pi_0}^2$ can be found in Eqs. (2.8) and (2.11), respectively. The function $\sigma(s)$ is also to be found in Eq. (2.8), whereas $L(s)$ is defined as

$$L(s) = \begin{cases} \ln\left(\frac{1-\sigma(s)}{1+\sigma(s)}\right) \equiv \hat{L}(s) & [s \geq 4M_\pi^2] \\ \ln\left(\frac{\sigma(s)-1}{\sigma(s)+1}\right) \equiv \hat{L} + i\pi & [4M_{\pi_0}^2 \leq s \leq 4M_\pi^2] \end{cases} \quad (\text{A.2})$$

according to the definitions (2.14) and (2.15). For $s \geq 4M_{\pi_0}^2$, the functions $k_n^0(s)$ and $k_n^\nabla(s)$ are real and smooth. In the same range of s , the functions $\bar{K}_n^0(s)$ and $\bar{K}_n^\nabla(s)$ have smooth real and imaginary parts, with $\text{Im } \bar{K}_n^0(s) = k_n^0(s)$, $\text{Im } \bar{K}_n^\nabla(s) = k_n^\nabla(s)$. Finally, the functions $\bar{K}_n^0(s)$ for $n \geq 1$ can be expressed in terms of $\bar{J}_0(s) \equiv \bar{K}_0^0(s)$. The explicit expressions and their derivation were given in [40].

There is not much to add as far as the functions $\bar{K}_n(s)$ are concerned: it is sufficient to replace everywhere in Eq. (A.1) M_{π_0} by the charged-pion mass M_π , and hence $\sigma_0(s)$ by $\sigma(s)$, and $L_0(s)$ by $L(s)$. In the dispersive representation (2.26), the integration starts at $s_{\text{thr}} = 4M_\pi^2$. In the case of $\bar{K}_0(s) \equiv \bar{J}(s)$, the decomposition (2.12) then follows from (2.28) by noticing that $k_0(s)/\sigma(s)$ is a constant, and that [25, 40]

$$\text{Re } \bar{J}(s) = \frac{1}{16\pi^2} [2 + \sigma(s)L(s)] \quad [s \geq 0]. \quad (\text{A.3})$$

For the remaining functions $\bar{K}_n(s)$, it is most convenient to use their expressions in terms of $\bar{J}(s)$. One then finds

$$\begin{aligned}
16\pi^2 \bar{K}_1^{[0]} &= \hat{L}(s)^2 - \pi^2, \\
16\pi^2 \bar{K}_1^{[1]} &= -2\pi \frac{\hat{L}(s)}{\sigma(s)}, \\
16\pi^2 \bar{K}_2^{[0]} &= \sigma^2(\hat{L}^2(s) - \pi^2) - 4, \\
16\pi^2 \bar{K}_2^{[1]} &= -2\pi \hat{L}(s)\sigma(s), \\
16\pi^2 \bar{K}_3^{[0]} &= \hat{L}(s)(\hat{L}^2(s) - 2\pi^2) \frac{M_\pi^2}{s\sigma(s)} - \frac{\pi^2}{2}, \\
16\pi^2 \bar{K}_3^{[1]} &= -3\pi \frac{M_\pi^2}{s} \frac{\hat{L}^2(s)}{\sigma^2(s)}.
\end{aligned} \quad (\text{A.4})$$

One may check that all these functions are real and smooth for $s \geq 4M_{\pi_0}^2$ (actually, for $s \geq 0$).

The functions $\bar{K}_n^x(s)$, $n = 1, 2, 3$, are defined by

$$\bar{K}_n^x(s) = \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dx}{x} \frac{k_n^x(x)}{x-s-i0}, \quad (\text{A.5})$$

with [the quantities appearing in these formulas are defined in Ref. [25]; see Eqs. (2.13), (4.13), (4.16), and (4.17) therein]

$$\begin{aligned}
k_1^x(s) &= \frac{1}{8\pi} \frac{1}{s\sigma_0(s)} \\
&\quad \times \left[\lambda^{1/2}(t_-(s)) \mathcal{L}_-(s) - \lambda^{1/2}(t_+(s)) \mathcal{L}_+(s) \right], \\
k_2^x(s) &= \frac{1}{8\pi} \sigma(s) \sigma_0(s) L_0(s), \\
k_3^x(s) &= \frac{3}{16\pi} \frac{M_\pi^2}{s\sigma_0(s)} \left[\mathcal{L}_-^2(s) - \mathcal{L}_+^2(s) \right].
\end{aligned} \quad (\text{A.6})$$

The three functions $k_n^x(s)$, $n = 1, 2, 3$, are real and smooth for $s \geq M_\pi^2$, and they become purely imaginary for $4M_{\pi_0}^2 \leq s \leq 4M_\pi^2$. The functions $\hat{k}_n^x(s) \equiv k_n^x(s)/\sigma(s)$ are then smooth in the range $4M_{\pi_0}^2 \leq s \leq M_K^2$, so that one obtains

$$\bar{K}_n^{x[1]}(s) = -\frac{k_n^x(s)}{\sigma(s)}, \quad (\text{A.7})$$

and

$$\begin{aligned}
&\bar{K}_n^{x[0]}(s) \\
&= \begin{cases} \text{Re } \bar{K}_n^x(s) & [s \geq 4M_\pi^2] \\ \text{Re } \bar{K}_n^x(s) + \frac{\hat{\sigma}(s)}{\sigma(s)} k_n^x(s) & [4M_{\pi_0}^2 \leq s \leq 4M_\pi^2] \end{cases}.
\end{aligned} \quad (\text{A.8})$$

Since analytical expressions for $\text{Re } \bar{K}_n^x(s)$ are not available, one has to use the integral representation given in Eq. (A.5) for numerical applications.

Finally, there remains to discuss the function $\mathcal{K}^x(s)$ whose discontinuity along the real s axis for $s \geq 4M_\pi^2$ reads

$$\begin{aligned}
k^x(s) &= \frac{1}{16\pi} \frac{M_\pi^2}{\Delta_\pi} \frac{\sigma(s)}{\sigma_0(s)} \frac{1}{s} \\
&\quad \times \left[(\sigma(s) - \sigma_0(s)) \lambda^{1/2}(t_-(s)) \mathcal{L}_-(s) \right. \\
&\quad \left. - (\sigma(s) + \sigma_0(s)) \lambda^{1/2}(t_+(s)) \mathcal{L}_+(s) \right].
\end{aligned} \quad (\text{A.9})$$

The function $k^x(s)/\sigma(s)$ is real and smooth for $s \geq 4M_{\pi_0}^2$. Thus one infers

$$\bar{\mathcal{K}}^{x[1]}(s) = -\frac{k^x(s)}{\sigma(s)} \quad (\text{A.10})$$

and

$$\begin{aligned}
&\bar{\mathcal{K}}^{x[0]}(s) \\
&= \begin{cases} \text{Re } \bar{\mathcal{K}}^x(s) & [s \geq 4M_\pi^2] \\ \text{Re } \bar{\mathcal{K}}^x(s) + \frac{\hat{\sigma}(s)}{\sigma(s)} k^x(s) & [4M_{\pi_0}^2 \leq s \leq 4M_\pi^2] \end{cases}.
\end{aligned} \quad (\text{A.11})$$

Appendix B: Loop functions

The computation, in Sect. 4 of the diagrams describing the virtual photon corrections involves a certain number of loop functions, which are briefly discussed here, in order to make the calculation in Sect. 4 self-contained.

At the level of the two-point one-loop diagrams, one has

$$J(p^2; m_1, m_2) = \frac{1}{i} \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 - m_1^2)[(\ell - p)^2 - m_2^2]} \quad (\text{B.1})$$

and

$$\begin{aligned} J_\mu(p; m_1, m_2) &= \frac{1}{i} \int \frac{d^4 \ell}{(2\pi)^4} \frac{\ell_\mu}{(\ell^2 - m_1^2)[(\ell - p)^2 - m_2^2]} \\ &= \frac{p_\mu}{2p^2} \left[(p^2 + m_1^2 - m_2^2) J(p^2; m_1, m_2) \right. \\ &\quad \left. + i A(m_1^2) - i A(m_2^2) \right], \end{aligned} \quad (\text{B.2})$$

where $A(m^2)$ is related to the tadpole graph,

$$\begin{aligned} A(m^2) &= \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{\ell^2 - m^2} = -im^2 \left[2\bar{\lambda} + \frac{1}{16\pi^2} \ln \frac{m^2}{\mu^2} \right. \\ &\quad \left. + \mathcal{O}(d-4) \right]. \end{aligned} \quad (\text{B.3})$$

Other useful relations are

$$\begin{aligned} J(p^2; m_1, m_2) &= \bar{J}(p^2; m_1, m_2) - 2\bar{\lambda} \\ &\quad - \frac{1}{16\pi^2} \frac{m_1^2 \ln \frac{m_1^2}{\mu^2} - m_2^2 \ln \frac{m_2^2}{\mu^2}}{m_1^2 - m_2^2} \end{aligned} \quad (\text{B.4})$$

and

$$J(m^2; 0, m) = -2\bar{\lambda} + \frac{1}{16\pi^2} \left[1 - \ln \frac{m^2}{\mu^2} \right]. \quad (\text{B.5})$$

Explicit expressions of the function $\bar{J}(p^2; m_1, m_2)$ can be found in Ref. [37]. Moreover, the link with the functions $\bar{J}(s)$ and $\bar{J}_0(s)$ encountered in Sect. 2 is given by $\bar{J}(s) \equiv \bar{J}(s; M_\pi, M_\pi)$ and $\bar{J}_0(s) \equiv \bar{J}(s; M_{\pi^0}, M_{\pi^0})$.

As far as the three-point one-loop functions are concerned, one has

$$\begin{aligned} C((p_1 - p_2)^2, p_2^2; m_1, m_2) \\ = \frac{1}{i} \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 - 2\ell \cdot p_1)[(\ell - p_2)^2 - m_2^2](\ell^2 - m_\gamma^2)} \end{aligned} \quad (\text{B.6})$$

and

$$C_\mu(p_1, p_2; m_1, m_2) \quad (\text{B.7})$$

$$\begin{aligned} &= \frac{1}{i} \int \frac{d^4 \ell}{(2\pi)^4} \frac{\ell_\mu}{(\ell^2 - 2\ell \cdot p_1)[(\ell - p_2)^2 - m_2^2](\ell^2 - m_\gamma^2)} \\ &= p_{1\mu} C_{11}((p_1 - p_2)^2, p_2^2; m_1, m_2) \\ &\quad + p_{2\mu} C_{12}((p_1 - p_2)^2, p_2^2; m_1, m_2), \end{aligned} \quad (\text{B.8})$$

where $p_1^2 = m_1^2$. Explicitly, one has $[\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz]$

$$\begin{aligned} C_{11}((p_1 - p_2)^2, p_2^2; m_1, m_2) \\ = \frac{2}{\lambda((p_1 - p_2)^2, m_1^2, p_2^2)} \\ \times \{ p_2^2 [J(p_2^2; m_\gamma, m_2) - J((p_1 - p_2)^2; m_1, m_2)] \\ - (p_1 \cdot p_2) [J(m_1^2; m_\gamma, m_1) - J((p_1 - p_2)^2; m_1, m_2)] \\ + (p_1 \cdot p_2)(p_2^2 - m_2^2) C((p_1 - p_2)^2, p_2^2; m_1, m_2) \} \\ = \frac{2}{m_1^2 - m_2^2} \frac{(p_1 \cdot p_2)m_2^2 - p_2^2 m_1^2}{\lambda((p_1 - p_2)^2, m_1^2, p_2^2)} \frac{1}{16\pi^2} \ln \frac{m_2^2}{m_1^2} \\ + 2 \frac{(p_1 \cdot p_2) - p_2^2}{\lambda((p_1 - p_2)^2, m_1^2, p_2^2)} \\ \times \left[\bar{J}((p_1 - p_2)^2; m_1, m_2) - \frac{1}{16\pi^2} \right] \\ + 2 \frac{p_2^2}{\lambda((p_1 - p_2)^2, m_1^2, p_2^2)} \left[\bar{J}(p_2^2; m_\gamma, m_2) - \frac{1}{16\pi^2} \right] \\ + 2 \frac{(p_1 \cdot p_2)(p_2^2 - m_2^2)}{\lambda((p_1 - p_2)^2, m_1^2, p_2^2)} C((p_1 - p_2)^2, p_2^2; m_1, m_2), \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} C_{12}((p_1 - p_2)^2, p_2^2; m_1, m_2) \\ = \frac{2}{\lambda((p_1 - p_2)^2, m_1^2, p_2^2)} \\ \times \{ m_1^2 [J(m_1^2; m_\gamma, m_1) - J((p_1 - p_2)^2; m_1, m_2)] \\ - (p_1 \cdot p_2) [J(p_2^2; m_\gamma, m_2) - J((p_1 - p_2)^2; m_1, m_2)] \\ - m_1^2(p_2^2 - m_2^2) C((p_1 - p_2)^2, p_2^2; m_1, m_2) \} \\ = \frac{2m_1^2}{m_1^2 - m_2^2} \frac{(p_1 \cdot p_2) - m_2^2}{\lambda((p_1 - p_2)^2, m_1^2, p_2^2)} \frac{1}{16\pi^2} \ln \frac{m_2^2}{m_1^2} \\ + 2 \frac{(p_1 \cdot p_2) - m_1^2}{\lambda((p_1 - p_2)^2, m_1^2, p_2^2)} \\ \times \left[\bar{J}((p_1 - p_2)^2; m_1, m_2) - \frac{1}{16\pi^2} \right] \\ - 2 \frac{(p_1 \cdot p_2)}{\lambda((p_1 - p_2)^2, m_1^2, p_2^2)} \left[\bar{J}(p_2^2; m_\gamma, m_2) - \frac{1}{16\pi^2} \right] \\ - 2 \frac{m_1^2(p_2^2 - m_2^2)}{\lambda((p_1 - p_2)^2, m_1^2, p_2^2)} C((p_1 - p_2)^2, p_2^2; m_1, m_2). \end{aligned} \quad (\text{B.10})$$

In the case where also $p_2^2 = m_2^2$, these expressions simplify further, and one obtains

$$C_{11}(t; m_1, m_2) = \frac{1}{16\pi^2} \frac{m_2^2 - m_1^2 - t}{\lambda(t, m_1^2, m_2^2)} \frac{m_2^2}{m_1^2 - m_2^2} \ln \frac{m_2^2}{m_1^2} + \frac{1}{16\pi^2} \frac{m_1^2 - m_2^2 - t}{\lambda(t, m_1^2, m_2^2)} \left[16\pi^2 \bar{J}(t; m_1, m_2) - 1 \right], \quad (\text{B.11})$$

$$C_{12}(t; m_1, m_2) = \frac{1}{16\pi^2} \frac{m_1^2 - m_2^2 - t}{\lambda(t, m_1^2, m_2^2)} \frac{m_1^2}{m_1^2 - m_2^2} \ln \frac{m_2^2}{m_1^2} + \frac{1}{16\pi^2} \frac{m_2^2 - m_1^2 - t}{\lambda(t, m_1^2, m_2^2)} \left[16\pi^2 \bar{J}(t; m_1, m_2) - 1 \right]. \quad (\text{B.12})$$

For $p_2^2 = m_2^2$, the function $C(t; m_1, m_2)$ itself reads [12, 15]

$$C(t; m_1, m_2) = \frac{(-1)}{32\pi^2} \int_0^1 dy \frac{1}{P_y^2} \ln \left(\frac{P_y^2 - i\epsilon}{m_\gamma^2} \right), \quad (\text{B.13})$$

with $P_y^\mu = yp_1^\mu + (1-y)p_2^\mu$. One may write

$$P_y^2 = y^2 t - 2aty + p_2^2 \equiv t(y - y_+)(y - y_-). \quad (\text{B.14})$$

The two roots of P_y^2 are then given by

$$y_\pm = a \pm b, \quad (\text{B.15})$$

with

$$a \equiv \frac{1}{2} + \frac{p_2^2 - p_1^2}{2t}, \quad b^2 \equiv a^2 - \frac{p_2^2}{t} = \frac{1}{4t^2} \lambda(t, p_1^2, p_2^2). \quad (\text{B.16})$$

In the case under consideration, we have $p_1^\mu = k^\mu$, $p_2^\mu = p_e^\mu$, $m_1 = M_K$, $m_2 = m_e$, with $k^2 = M_K^2$, $p_e^2 = m_e^2$, $t = (k - p_e)^2 > 0$. Then $\lambda((k - p_e)^2, M_K^2, m_e^2) \geq 0$, and

$$a = \frac{1}{2} \left[1 - \frac{(M_K - m_e)(M_K + m_e)}{t} \right], \quad b = \frac{1}{2t} \lambda^{1/2}(t, M_K^2, m_e^2), \quad (\text{B.17})$$

with $a < 0$ and $|a| > |b|$. Therefore,

$$C(t; m_1, m_2) = \frac{1}{64\pi^2} \frac{1}{bt} \ln \left(\frac{m_\gamma^2}{t} \right) \times \ln \left(\frac{y_+ - 1}{y_- - 1} \cdot \frac{y_-}{y_+} \right) - \frac{1}{128\pi^2} \frac{1}{bt} \left[\ln^2(1 - y_+) - \ln^2(-y_+) - \ln^2(1 - y_-) + \ln^2(-y_-) \right] + \frac{1}{64\pi^2} \frac{1}{bt} \left[\ln(1 - y_+) \ln(1 - y_-) - \ln(-y_+) \ln(-y_-) - 2 \ln(2b) \ln \left(\frac{y_+ - 1}{y_+} \right) \right] + \frac{1}{32\pi^2} \frac{1}{bt} \left[\text{Li}_2 \left(\frac{y_+ - 1}{2b} \right) - \text{Li}_2 \left(\frac{y_+}{2b} \right) \right]. \quad (\text{B.18})$$

Notice that

$$\frac{y_+ - 1}{y_- - 1} \cdot \frac{y_-}{y_+} = \frac{a^2 - b^2 - a + b}{a^2 - b^2 - a - b} = \frac{p_1^2 + p_2^2 - t + 2tb}{p_1^2 + p_2^2 - t - 2tb}, \quad (\text{B.19})$$

so that the infrared divergent piece of $C(t; m_1, m_2)$ is given by

$$C_{\text{IR-div}}(t; m_1, m_2) = -\frac{1}{32\pi^2} \tau(p_1, p_2) \times \ln m_\gamma, \quad (\text{B.20})$$

with

$$\tau(p_1, p_2) \equiv \frac{1}{bt} \ln \left[\frac{p_1^2 + p_2^2 - t - 2tb}{p_1^2 + p_2^2 - t + 2tb} \right] = \frac{1}{\sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}} \times \ln \frac{(p_1 \cdot p_2) - \sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}}{(p_1 \cdot p_2) + \sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}}. \quad (\text{B.21})$$

Let us now consider the case where $p_2^2 \neq m_2^2$, but with $p_1^2 = m_1^2$ as before. Going through the same steps as in the previous case, one obtains

$$C(t, p_2^2; m_1, m_2) = \frac{(-1)}{16\pi^2} \int_0^1 dy \frac{1}{P_y^2} \times \ln \left[\frac{P_y^2 + (1-y)(m_2^2 - p_2^2) - i\epsilon}{(1-y)(m_2^2 - p_2^2) - i\epsilon} \right], \quad (\text{B.22})$$

with $P_y^\mu = yp_1^\mu + (1-y)p_2^\mu$. As before, one has

$$P_y^2 = y^2 t - 2aty + p_2^2 \equiv t(y - y_+)(y - y_-), \quad (\text{B.23})$$

with

$$y_\pm = a \pm b, \quad a \equiv \frac{1}{2} + \frac{p_2^2 - p_1^2}{2t}, \quad b^2 \equiv a^2 - \frac{p_2^2}{t} = \frac{1}{4t^2} \lambda(t, p_1^2, p_2^2). \quad (\text{B.24})$$

The case under consideration here corresponds to $p_1^\mu = k^\mu$, $p_1^2 = k^2 = M_K^2$, whereas $p_2^\mu = (k - p_1 - p_2)^\mu$, $p_2^2 = s_e$, $t \equiv (p_1 - p_2)^2 = s_\pi$, with $m_e^2 \leq s_e \leq (M_K - 2M_\pi)^2$, $4M_\pi^2 \leq s_\pi \leq (M_K - \sqrt{s_e})^2$. Then $\lambda(t, p_1^2, p_2^2) = \lambda(s_\pi, s_e, M_K^2) \geq 0$, so that b is real, with $|a| > |b|$. On the other hand, one has

$$P_y^2 + (1-y)(m_2^2 - p_2^2) = y^2 t - 2\tilde{a}ty + m_2^2 \equiv t(y - \tilde{y}_+)(y - \tilde{y}_-), \quad (\text{B.25})$$

with $\tilde{y}_{\pm} = \tilde{a} \pm \tilde{b}$, and

$$\tilde{a} \equiv \frac{1}{2} + \frac{m_2^2 - p_1^2}{2t}, \quad \tilde{b}^2 \equiv \tilde{a}^2 - \frac{m_2^2}{t} = \frac{1}{4t^2} \lambda(t, p_1^2, m_2^2). \quad (\text{B.26})$$

In the case at hand, this gives $\tilde{a} = 1/2$, and $\tilde{b}^2 = (s_{\pi} - 4M_K^2)/(4s_{\pi}) < 0$, so that

$$\tilde{y}_{\pm} = \frac{1}{2} \left[1 \pm i \sqrt{\frac{4M_K^2}{s_{\pi}} - 1} \right]. \quad (\text{B.27})$$

Then one obtains

$$\begin{aligned} & C(t, p_2^2; m_1, m_2) \\ &= \frac{1}{32\pi^2} \frac{1}{bt} \ln \left(\frac{m_2^2 - p_2^2}{t} \right) \times \ln \left[\frac{(y_+ - 1)y_-}{(y_- - 1)y_+} \right] \\ & - \frac{1}{32\pi^2} \frac{1}{bt} \ln \left(\frac{y_+ - 1}{y_+} \right) [\ln(y_+ - \tilde{y}_+) + \ln(y_+ - \tilde{y}_-)] \\ & + \frac{1}{32\pi^2} \frac{1}{bt} \ln \left(\frac{y_- - 1}{y_-} \right) [\ln(y_- - \tilde{y}_+) + \ln(y_- - \tilde{y}_-)] \\ & + \frac{1}{32\pi^2} \frac{1}{bt} \left[\text{Li}_2 \left(\frac{1 - y_+}{\tilde{y}_+ - y_+} \right) - \text{Li}_2 \left(\frac{-y_+}{\tilde{y}_+ - y_+} \right) \right. \\ & \left. + \text{Li}_2 \left(\frac{1 - y_+}{\tilde{y}_- - y_+} \right) - \text{Li}_2 \left(\frac{-y_+}{\tilde{y}_- - y_+} \right) \right] \\ & - \frac{1}{32\pi^2} \frac{1}{bt} \left[\text{Li}_2 \left(\frac{1 - y_-}{\tilde{y}_+ - y_-} \right) - \text{Li}_2 \left(\frac{-y_-}{\tilde{y}_+ - y_-} \right) \right. \\ & \left. + \text{Li}_2 \left(\frac{1 - y_-}{\tilde{y}_- - y_-} \right) - \text{Li}_2 \left(\frac{-y_-}{\tilde{y}_- - y_-} \right) \right] \\ & - \frac{1}{32\pi^2} \frac{1}{bt} \left[\text{Li}_2 \left(\frac{1}{1 - y_+} \right) - \text{Li}_2 \left(\frac{1}{1 - y_-} \right) \right]. \quad (\text{B.28}) \end{aligned}$$

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