Approximate Noether gauge symmetries of the Bardeen model

U. Camci^a

Department of Physics, Faculty of Science, Akdeniz University, 07058 Antalya, Turkey

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Abstract We investigate the approximate Noether gauge symmetries of the geodesic Lagrangian for the Bardeen spacetime model. This is accommodated by a set of new approximate Noether gauge symmetry relations for the perturbed geodesic Lagrangian in the spacetime. A detailed analysis of the spacetime of the Bardeen model up to third-order approximate Noether gauge symmetries is presented.

1 Introduction

The black holes predicted in the framework of Einstein's gravity theory appear to exist in the Universe [1]. The existence of singularities inside the black holes was predicted by singularity theorems, and they indicate a breakdown of the classical theory [2]. A regular (i.e. non-singular) black hole which is characterized by the fact that the spacetime metric as well as the curvature invariants $R, R_{ab}, R^{ab}, R_{abcd}R^{abcd}$ do not present singularities anywhere was firstly derived from Einstein's field equations by Bardeen [3] which was initially approximately obtained. As shown in [4], the metric of the Bardeen model solves the Einstein equations with a self-gravitating nonlinear magnetic monopole as a source. The Bardeen model includes two parameters in the solution, namely the parameters m and e . As usual for the Schwarzschild metric, the parameter m can be interpreted as the mass parameter for the system. The other parameter e can be interpreted as the magnetic charge of the geometry. Although the Bardeen model is always a regular black hole at $r = 0$ for $e \neq 0$, it represents a regular black-hole spacetime only when the inequality $27e^2 \leq 16m^2$ holds.

Let us briefly give some information as regards spacetime symmetries. A conformal Killing vector (CKV) $\mathbf{K}$ has to satisfy $\mathcal{L}_{\mathbf{K}}g_{ab} = 2\psi(x^a)g_{ab}$, where g_{ab} is the metric tensor, $\mathcal{L}_{\mathbf{K}}$ is the Lie derivative operator along $\mathbf{K}$, and $\psi(x^a)$ is a conformal factor. When $\psi_{;ab} \neq 0$, the CKV field is said to be

proper [5]. The vector field $\mathbf{K}$ is called the special conformal Killing vector (SCKV) field if $\psi_{;ab} = 0$, the homothetic Killing vector (HKV) field if $\psi_{;a} = 0$, e.g. ψ is a constant on the manifold, and the Killing vector (KV) field which gives the isometry if $\psi = 0$. The set of all CKV (respectively SCKV, HKV, and KV) form a finite-dimensional Lie algebra. The maximum dimension of the CKV algebra on the manifold M is 15 if M is conformally flat, and it is seven if the spacetime is not conformally flat.

If a Lagrangian L for a given dynamical system admits some symmetry, this property should strongly be related with Noether symmetries which describe physical features of differential equations possessing a Lagrangian L in terms of first integrals admitted by them [6, 7]. This can actually be seen in two ways. First of all, one can consider a strict Noether symmetry approach [8–10], which yields $\mathcal{L}_{\mathbf{X}}L = 0$, where $\mathcal{L}_{\mathbf{X}}$ is the Lie derivative operator along $\mathbf{X}$. On the other side, one could use the Noether gauge symmetry (NGS) approach [11–17], which is a generalization of the strict Noether symmetry approach in the sense that the Noether symmetry equation includes a gauge term in such a way that the vector field $\mathbf{X} = \xi\partial_s + \eta^a\frac{\partial}{\partial x^a}$ is called a NGS of a Lagrangian L if there exists a gauge function, $A(s, x^a)$, such that the Noether symmetry condition holds [6, 7],

$$\mathbf{X}^{[1]}L + L(D_s\xi) = D_sA, \quad (1)$$

where $\mathbf{X}^{[1]} = \mathbf{X} + \eta_s^a\frac{\partial}{\partial x^a}$ is the first prolongation operator of $\mathbf{X}$, $\eta_s^a = D_s\eta^a - \dot{x}^a D_s\xi$, and $D_s = \frac{\partial}{\partial s} + \dot{x}^a\frac{\partial}{\partial x^a}$ is the total derivative operator. The approximate symmetries are as useful as exact ones in a variety of problems arising from physics and applied mathematics. We will discuss how the approximate Noether gauge symmetries (ANGSs) of the perturbed geodesic Lagrangian are related to the geometrical symmetries of spacetimes in the following section.

The geodesic Lagrangian describing the motion of the massive or massless particles under the influence of a potential function $V(x^c)$ is

^a e-mail: ucamci@akdeniz.edu.tr

$$L(s, x^c, \dot{x}^c) = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b - V(x^c), \quad (2)$$

where the dot represents the derivative with respect to the geodesic parameter s . The significance of an ANGSS comes from the fact that a manifold having no exact symmetry can possess approximate ones. In this context, using the approximate symmetries of perturbed geodesic Lagrangian for spacetimes, one can recover the *lost* symmetries and conservation laws corresponding to the perturbed Lagrangian [18–22].

For the Schwarzschild metric, Kara et al. [20] have considered the approximate symmetries and the conservation laws of geodesic equations. Following the use of approximate symmetries for the Schwarzschild spacetime, Hussain et al. [21] have investigated the exact and approximate symmetries of the system of geodesic equations for the Reissner–Nordström (RN) spacetime. Using the ANGSS for the Schwarzschild and Kerr metrics, Hussain et al. [22] have recovered all the lost conservation laws as trivial first-order approximate conservation laws in the first-order approximate case. Using approximate Lie symmetries the energy content of colliding plane waves have been studied by Sharif and Waheed [23]. By analyzing the behavior of the effective potential for the particles and photons, one has investigated the time-like and null geodesic structures for the Bardeen metric [24]. In a recent paper, Sharif and Waheed [25] have applied the approximate Lie symmetry method to the investigation of the energy re-scaling factor for Bardeen’s regular black-hole solution. They only evaluated the third-order approximate symmetries of the orbital and geodesic equations using approximate Lie symmetry methods for the differential equations. Our main aim here is to extend our mathematical understanding of (1) by exploring its geometrical properties. We have obtained the geometrical nature of the ANGSSs up to second-order perturbed Lagrangian and applied this method to the Schwarzschild and RN spacetimes [26]. It should be stressed that we deal with third-order ANGSSs of the perturbed geodesic Lagrangian which have not been studied previously as far as we know.

The rest of the paper is organized as follows. In the following section, we present an analysis of third-order ANGSSs of the first-order geodesic Lagrangian according to perturbed spacetime geometry. In Sect. 3, we apply the obtained results of Sect. 2 to the spacetime of the Bardeen model. Conclusions and discussions are presented in Sect. 4.

2 Third-order approximate Noether gauge symmetries

The calculation method of exact symmetries and the first-order approximate symmetries of the Lagrangian are available in the literature [18–20]. Hussain et al. [21,22] have extended the procedure of calculating the approximate sym-

metries of a perturbed Lagrangian to the second order. In this study we introduce a new geometrical method to calculate the ANGSSs of the first-order perturbed Lagrangian extending the procedure of obtaining ANGSSs until the third order.

The third-order ANGSSs of the first-order perturbed Lagrangian,

$$L(s, x^a, \dot{x}^a, \varepsilon) = L_0(s, x^a, \dot{x}^a) + \varepsilon L_1(s, x^a, \dot{x}^a) + \varepsilon^2 L_2(s, x^a, \dot{x}^a) + \varepsilon^3 L_3(s, x^a, \dot{x}^a) + O(\varepsilon^4), \quad (3)$$

is given by the ANGSS generator

$$\mathbf{X} = \mathbf{X}_0 + \varepsilon \mathbf{X}_1 + \varepsilon^2 \mathbf{X}_2 + \varepsilon^3 \mathbf{X}_3, \quad (4)$$

up to the gauge function $A(s, x^a) = A_0(s, x^a) + \varepsilon A_1(s, x^a) + \varepsilon^2 A_2(s, x^a) + \varepsilon^3 A_3(s, x^a)$ if the ANGSS generator satisfies the approximate Noether gauge symmetry conditions

$$\mathbf{X}_0^{[1]} L_0 + L_0(D_s \xi_0) = D_s A_0, \quad (5)$$

$$\mathbf{X}_1^{[1]} L_0 + \mathbf{X}_0^{[1]} L_1 + L_0(D_s \xi_1) + L_1(D_s \xi_0) = D_s A_1, \quad (6)$$

$$\mathbf{X}_2^{[1]} L_0 + \mathbf{X}_1^{[1]} L_1 + \mathbf{X}_0^{[1]} L_2 + L_0(D_s \xi_2) + L_1(D_s \xi_1) + L_2(D_s \xi_0) = D_s A_2, \quad (7)$$

$$\mathbf{X}_3^{[1]} L_0 + \mathbf{X}_2^{[1]} L_1 + \mathbf{X}_1^{[1]} L_2 + \mathbf{X}_0^{[1]} L_3 + L_0(D_s \xi_3) + L_1(D_s \xi_2) + L_2(D_s \xi_1) + L_3(D_s \xi_0) = D_s A_3, \quad (8)$$

where $\mathbf{X}_0$ is the exact NGS generator, $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ are the first-order, second-order, and third-order ANGSS generators, respectively, which are defined as

$$\mathbf{X}_j = \xi_j \frac{\partial}{\partial s} + \eta_j^a \frac{\partial}{\partial x^a}, \quad (j = 0, 1, 2, 3), \quad (9)$$

$$\mathbf{X}_j^{[1]} = \mathbf{X}_j + \eta_{j(s)}^a \frac{\partial}{\partial \dot{x}^a}, \quad \eta_{j(s)}^a = D_s \eta_j^a - \dot{x}^a D_s \xi_j. \quad (10)$$

The above perturbed Lagrangian (3) yields a third-order (in ε) perturbed system of ordinary differential equations (ODEs). If $\mathbf{X}_j$ are the ANGSSs corresponding to the perturbed Lagrangians $L_j(s, x^a, \dot{x}^a)$, then

$$I_0 = \xi_0 L_0 + (\eta_0^a - \xi_0 \dot{x}^a) \frac{\partial L_0}{\partial \dot{x}^a} - A_0, \quad (11)$$

$$I_1 = \xi_0 L_1 + \xi_1 L_0 + (\eta_0^a - \xi_0 \dot{x}^a) \frac{\partial L_1}{\partial \dot{x}^a} + (\eta_1^a - \xi_1 \dot{x}^a) \frac{\partial L_0}{\partial \dot{x}^a} - A_1, \quad (12)$$

$$I_2 = \xi_0 L_2 + \xi_1 L_1 + \xi_2 L_0 + (\eta_0^a - \xi_0 \dot{x}^a) \frac{\partial L_2}{\partial \dot{x}^a} + (\eta_1^a - \xi_1 \dot{x}^a) \frac{\partial L_1}{\partial \dot{x}^a} + (\eta_2^a - \xi_2 \dot{x}^a) \frac{\partial L_0}{\partial \dot{x}^a} - A_2, \quad (13)$$

$$I_3 = \xi_0 L_3 + \xi_1 L_2 + \xi_2 L_1 + \xi_3 L_0 + (\eta_0^a - \xi_0 \dot{x}^a) \frac{\partial L_3}{\partial \dot{x}^a} + (\eta_1^a - \xi_1 \dot{x}^a) \frac{\partial L_2}{\partial \dot{x}^a} + (\eta_2^a - \xi_2 \dot{x}^a) \frac{\partial L_1}{\partial \dot{x}^a} + (\eta_3^a - \xi_3 \dot{x}^a) \frac{\partial L_0}{\partial \dot{x}^a} - A_3 \quad (14)$$

are the first integrals associated with ANGSSs $\mathbf{X}_j$ ($j = 0, 1, 2, 3$).

The spacetime metric g_{ab} can be decomposed as follows:

$$g_{ab} = \gamma_{ab} + \varepsilon h_{ab} + \varepsilon^2 \sigma_{ab} + \varepsilon^3 k_{ab}, \quad (15)$$

which means by (2) and (3) that the exact and perturbed geodesic Lagrangians of motion have the form

$$L_0(s, x^a, \dot{x}^a) = \frac{1}{2} \gamma_{ab} \dot{x}^a \dot{x}^b - V(x^c), \quad (16)$$

$$L_1(s, x^a, \dot{x}^a) = \frac{1}{2} h_{ab} \dot{x}^a \dot{x}^b, \quad (17)$$

$$L_2(s, x^a, \dot{x}^a) = \frac{1}{2} \sigma_{ab} \dot{x}^a \dot{x}^b, \quad (18)$$

$$L_3(s, x^a, \dot{x}^a) = \frac{1}{2} k_{ab} \dot{x}^a \dot{x}^b, \quad (19)$$

where γ_{ab} , h_{ab} , σ_{ab} , and k_{ab} are the exact, the first-order, the second-order, and the third-order perturbed metrics, respectively. The metric γ_{ab} should be non-degenerate [i.e., $\det(\gamma_{ab}) \neq 0$]. But the other metrics h_{ab} , σ_{ab} , and k_{ab} can be degenerate [i.e., $\det(h_{ab}) = 0$, $\det(\sigma_{ab}) = 0$, $\det(k_{ab}) = 0$] or non-degenerate, and they represent the slight deviations from flat spacetime geometry if the metric γ_{ab} represents the flat geometry.

For the ANGSS conditions (5–8) of the perturbed geodesic Lagrangians (16–19) we obtain

$$\begin{aligned} \mathbf{X}_j^{[1]} L_0 &= (\gamma_{ad} \eta_{j,s}^d) \dot{x}^a + \frac{1}{2} (\mathfrak{L}_{\eta_j} \gamma_{ab} - 2\xi_{j,s} \gamma_{ab}) \dot{x}^a \dot{x}^b \\ &\quad - (\gamma_{bd} \xi_{j,a}) \dot{x}^a \dot{x}^b \dot{x}^d - \eta_j^d V_{,d}, \end{aligned} \quad (20)$$

$$\begin{aligned} \mathbf{X}_j^{[1]} L_1 &= (h_{ad} \eta_{j,s}^d) \dot{x}^a + \frac{1}{2} (\mathfrak{L}_{\eta_j} h_{ab} - 2\xi_{j,s} h_{ab}) \dot{x}^a \dot{x}^b \\ &\quad - (h_{bd} \xi_{j,a}) \dot{x}^a \dot{x}^b \dot{x}^d, \end{aligned} \quad (21)$$

$$\begin{aligned} \mathbf{X}_j^{[1]} L_2 &= (\sigma_{ad} \eta_{j,s}^d) \dot{x}^a + \frac{1}{2} (\mathfrak{L}_{\eta_j} \sigma_{ab} - 2\xi_{j,s} \sigma_{ab}) \dot{x}^a \dot{x}^b \\ &\quad - (\sigma_{bd} \xi_{j,a}) \dot{x}^a \dot{x}^b \dot{x}^d, \end{aligned} \quad (22)$$

$$\begin{aligned} \mathbf{X}_j^{[1]} L_3 &= (k_{ad} \eta_{j,s}^d) \dot{x}^a + \frac{1}{2} (\mathfrak{L}_{\eta_j} k_{ab} - 2\xi_{j,s} k_{ab}) \dot{x}^a \dot{x}^b \\ &\quad - (k_{bd} \xi_{j,a}) \dot{x}^a \dot{x}^b \dot{x}^d, \end{aligned} \quad (23)$$

where $\mathfrak{L}_{\eta_0}$, $\mathfrak{L}_{\eta_1}$, $\mathfrak{L}_{\eta_2}$, and $\mathfrak{L}_{\eta_3}$ are the geometrical derivative, i.e. the Lie derivative, the operators along $\eta_0 = \eta_0^a \partial_{x^a}$, $\eta_1 = \eta_1^a \partial_{x^a}$, $\eta_2 = \eta_2^a \partial_{x^a}$, and $\eta_3 = \eta_3^a \partial_{x^a}$, respectively. Putting the above expressions into (5–8) together with $D_s \xi_j = \xi_{j,s} + \xi_{j,d} \dot{x}^d$ and $D_s A_j = A_{j,s} + A_{j,d} \dot{x}^d$, ($j = 0, 1, 2, 3$), we find the ANGSS conditions for (5):

$$\xi_{0,a} = 0, \quad \gamma_{ab} \eta_{0,s}^b = A_{0,a} \quad (24)$$

$$\mathfrak{L}_{\eta_0} \gamma_{ab} = \xi_{0,s} \gamma_{ab} \quad (25)$$

$$\mathfrak{L}_{\eta_0} V = -\xi_{0,s} V - A_{0,s}, \quad (26)$$

for (6):

$$\xi_{1,a} = 0, \quad \gamma_{ab} \eta_{1,s}^b + h_{ab} \eta_{0,s}^b = A_{1,a} \quad (27)$$

$$\mathfrak{L}_{\eta_1} \gamma_{ab} + \mathfrak{L}_{\eta_0} h_{ab} = \xi_{1,s} \gamma_{ab} + \xi_{0,s} h_{ab} \quad (28)$$

$$\mathfrak{L}_{\eta_1} V = -\xi_{1,s} V - A_{1,s}, \quad (29)$$

for (7):

$$\xi_{2,a} = 0, \quad \gamma_{ab} \eta_{2,s}^b + h_{ab} \eta_{1,s}^b + \sigma_{ab} \eta_{0,s}^b = A_{2,a} \quad (30)$$

$$\begin{aligned} \mathfrak{L}_{\eta_2} \gamma_{ab} + \mathfrak{L}_{\eta_1} h_{ab} + \mathfrak{L}_{\eta_0} \sigma_{ab} \\ = \xi_{2,s} \gamma_{ab} + \xi_{1,s} h_{ab} + \xi_{0,s} \sigma_{ab} \end{aligned} \quad (31)$$

$$\mathfrak{L}_{\eta_2} V = -\xi_{2,s} V - A_{2,s}, \quad (32)$$

and for (8):

$$\begin{aligned} \xi_{3,a} = 0, \quad \gamma_{ab} \eta_{3,s}^b + h_{ab} \eta_{2,s}^b + \sigma_{ab} \eta_{1,s}^b + k_{ab} \eta_{0,s}^b = A_{3,a} \\ \mathfrak{L}_{\eta_3} \gamma_{ab} + \mathfrak{L}_{\eta_2} h_{ab} + \mathfrak{L}_{\eta_1} \sigma_{ab} + \mathfrak{L}_{\eta_0} k_{ab} \\ = \xi_{3,s} \gamma_{ab} + \xi_{2,s} h_{ab} + \xi_{1,s} \sigma_{ab} + \xi_{0,s} k_{ab} \end{aligned} \quad (33)$$

$$\mathfrak{L}_{\eta_3} V = -\xi_{3,s} V - A_{3,s}. \quad (34)$$

$$\mathfrak{L}_{\eta_3} V = -\xi_{3,s} V - A_{3,s}. \quad (35)$$

Thus we find the geometrical form of the ANGSS equations in terms of the Lie derivatives of exact and perturbed metrics up to the third order. Equations (24–26) are, of course, the conditions used to determine NGSs of exact geodesic Lagrangian. We have shown that the exact (5), the first-order (6), the second-order (7), and the third-order (8) form of the ANGSS conditions are equivalent to the sets of equations (24–26), (27–29), (30–32), and (33–35), respectively.

3 Application to Bardeen model

The Bardeen model describes a singularity-free black-hole spacetime, which means it is a regular spacetime. In this section, we will apply the above geometrical method to the third-order perturbed spacetime of the Bardeen model for finding the ANGSSs of the perturbed geodesic Lagrangian. The line element representing the Bardeen spacetime is given by

$$\begin{aligned} ds^2 &= \left[1 - \frac{2mr^2}{(r^2 + e^2)^{3/2}} \right] dt^2 - \left[1 - \frac{2mr^2}{(r^2 + e^2)^{3/2}} \right]^{-1} dr^2 \\ &\quad - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \end{aligned} \quad (36)$$

where m is the mass of configuration and e represents the magnetic charge of the nonlinear self-gravitating monopole, and we have taken the gravitational units, i.e. $G = c = 1$. The Bardeen black-hole metric reduces to the Schwarzschild metric for $e = 0$ and is flat for $m = 0$. Using the expansions

$$g_{tt} = 1 - \frac{\varepsilon}{r} + \frac{3k\varepsilon^3}{2r^3} + O(\varepsilon^4) \quad (37)$$

and

$$g_{rr} = - \left[1 + \frac{\varepsilon}{r} + \frac{\varepsilon^2}{r^2} + \left(1 - \frac{3k}{2} \right) \frac{\varepsilon^3}{r^3} \right] + O(\varepsilon^4), \quad (38)$$

the first-order perturbed geodesic Lagrangian of the Bardeen model is given by

$$L = \frac{1}{2} \left[\dot{t}^2 - \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right] - \frac{\varepsilon}{2r} (\dot{t}^2 + \dot{r}^2) - \frac{\varepsilon^2}{2r^2} \dot{r}^2 + \frac{\varepsilon^3}{2r^3} \left[\frac{3k}{2} \dot{t}^2 - \left(1 - \frac{3k}{2} \right) \dot{r}^2 \right] - V(t, r, \theta, \phi) + O(\varepsilon^4), \quad (39)$$

where we take $\varepsilon = 2m$ and $e^2 = k\varepsilon^2$, $0 < k \leq 4/27$, in the limit of a small mass of a gravitating point source. The above Lagrangian (39) yields

$$L_0 = \frac{1}{2} \left[\dot{t}^2 - \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right], \quad (40)$$

$$L_1 = -\frac{1}{2r} (\dot{t}^2 + \dot{r}^2), \quad (41)$$

$$L_2 = -\frac{1}{2r^2} \dot{r}^2, \quad (42)$$

$$L_3 = \frac{1}{2r^3} \left[\frac{3k}{2} \dot{t}^2 - \left(1 - \frac{3k}{2} \right) \dot{r}^2 \right], \quad (43)$$

where we used $\gamma_{ab} = \text{diag}(1, -1, -r^2, -r^2 \sin^2 \theta)$ which is the Minkowski metric, $h_{ab} = \text{diag}(-1/r, -1/r, 0, 0)$, $\sigma_{ab} = \text{diag}(-1/r^2, 0, 0, 0)$, and $k_{ab} = \text{diag}(3k/2r^3, (3k/2 - 1)/r^3, 0, 0)$. Also, the Lagrangian (39) reduces to the geodesic Lagrangian of the Minkowski metric in the limit $\varepsilon \rightarrow 0$. Thus, we can apply the new set of the ANGSS obtained in the previous section to these exact metric γ_{ab} and perturbed metrics h_{ab} , σ_{ab} , and k_{ab} for the Bardeen space-time. For the metric γ_{ab} , we find from the exact ANGSS equations (24–26) the following 19 equations:

$$\begin{aligned} \xi_{0,t} &= 0, \quad \xi_{0,r} = 0, \quad \xi_{0,\theta} = 0, \quad \xi_{0,\phi} = 0, \\ \eta_{0,s}^0 - A_{0,t} &= 0, \quad \eta_{0,s}^1 + A_{0,r} = 0, \quad r^2 \eta_{0,s}^2 + A_{0,\theta} = 0, \\ r^2 \sin^2 \theta \eta_{0,s}^3 + A_{0,\phi} &= 0, \quad 2\eta_{0,t}^0 - \xi_{0,s} = 0, \\ 2\eta_{0,r}^1 - \xi_{0,s} &= 0, \quad 2\eta_{0,\theta}^2 + \frac{2}{r} \eta_{0,t}^1 - \xi_{0,s} = 0, \\ \frac{2}{r} \eta_{0,t}^1 + 2 \cot \theta \eta_{0,\theta}^2 + 2\eta_{0,\phi}^3 - \xi_{0,s} &= 0, \\ \eta_{0,r}^0 - \eta_{0,t}^1 &= 0, \quad \eta_{0,\theta}^0 - r^2 \eta_{0,t}^2 = 0, \\ \eta_{0,\phi}^0 - r^2 \sin^2 \theta \eta_{0,t}^3 &= 0, \quad \eta_{0,\theta}^1 + r^2 \eta_{0,r}^2 = 0, \\ \eta_{0,\phi}^1 + r^2 \sin^2 \theta \eta_{0,r}^3 &= 0, \quad \eta_{0,\phi}^2 + \sin^2 \theta \eta_{0,\theta}^3 = 0, \\ V_{,t} \eta_{0,t}^0 + V_{,r} \eta_{0,t}^1 + V_{,\theta} \eta_{0,t}^2 + V_{,\phi} \eta_{0,t}^3 + V \xi_{0,s} + A_{0,s} &= 0. \end{aligned} \quad (44)$$

Considering both γ_{ab} and h_{ab} , the first-order ANGSS equations (27–29) yield again 19 equations:

$$\begin{aligned} \xi_{1,t} &= 0, \quad \xi_{1,r} = 0, \quad \xi_{1,\theta} = 0, \quad \xi_{1,\phi} = 0, \\ \eta_{1,s}^0 - \frac{1}{r} \eta_{0,s}^1 - A_{1,t} &= 0, \quad \eta_{1,s}^1 + \frac{1}{r} \eta_{0,s}^2 + A_{1,r} = 0, \\ r^2 \eta_{1,s}^2 + A_{1,\theta} &= 0, \quad r^2 \sin^2 \theta \eta_{1,s}^3 + A_{1,\phi} = 0, \\ \frac{1}{r} \eta_{0,t}^1 - 2\eta_{0,t}^2 + 2r \eta_{1,t}^0 + \xi_{0,s} - r \xi_{1,s} &= 0, \\ \frac{1}{r} \eta_{0,t}^1 - 2\eta_{0,r}^1 - 2r \eta_{1,r}^1 + \xi_{0,s} + \xi_{1,s} &= 0, \\ \frac{2}{r} \eta_{1,t}^1 + 2 \cot \theta \eta_{1,\theta}^2 + 2\eta_{1,\phi}^3 - \xi_{1,s} &= 0, \\ \frac{2}{r} \eta_{1,t}^1 + \eta_{1,\theta}^2 - \xi_{1,s} &= 0, \quad \eta_{1,r}^0 - \frac{1}{r} (\eta_{0,r}^0 + \eta_{0,t}^1) - \eta_{1,t}^1 = 0, \\ \eta_{1,\theta}^0 - \frac{1}{r} \eta_{0,\theta}^0 - r^2 \eta_{1,t}^2 &= 0, \quad \eta_{1,\theta}^1 + \frac{1}{r} \eta_{0,\theta}^1 + r^2 \eta_{1,r}^2 = 0, \\ \eta_{1,\phi}^0 - \frac{1}{r} \eta_{0,\phi}^0 - r^2 \sin^2 \theta \eta_{1,t}^3 &= 0, \\ \eta_{1,\phi}^1 + \frac{1}{r} \eta_{0,\phi}^1 + r^2 \sin^2 \theta \eta_{1,r}^3 &= 0, \\ \eta_{1,\phi}^2 + \sin^2 \theta \eta_{1,\theta}^3 &= 0, \\ V_{,t} \eta_{1,t}^0 + V_{,r} \eta_{1,t}^1 + V_{,\theta} \eta_{1,t}^2 + V_{,\phi} \eta_{1,t}^3 + V \xi_{1,s} + A_{1,s} &= 0. \end{aligned} \quad (45)$$

Using the metrics γ_{ab} , h_{ab} , and σ_{ab} , the second-order ANGSS equations (30–32) give the following equations:

$$\begin{aligned} \xi_{2,t} &= 0, \quad \xi_{2,r} = 0, \quad \xi_{2,\theta} = 0, \quad \xi_{2,\phi} = 0, \\ \frac{1}{r} \eta_{1,s}^0 - \eta_{2,s}^0 + A_{2,t} &= 0, \\ \frac{1}{r^2} (r \eta_{1,s}^1 + \eta_{0,s}^1) + \eta_{2,s}^1 + A_{2,r} &= 0, \\ r^2 \eta_{2,s}^2 + A_{2,\theta} &= 0, \quad r^2 \sin^2 \theta \eta_{2,s}^3 + A_{2,\phi} = 0, \\ \frac{1}{r^2} (\eta_{1,t}^1 - 2r \eta_{1,t}^2 + r \xi_{1,s}) + 2\eta_{2,t}^0 - \xi_{2,s} &= 0, \\ \frac{1}{r^3} (2\eta_{0,t}^1 + r \eta_{1,t}^1 - 2r^2 \eta_{1,r}^1 - 2r^2 \eta_{0,r}^1 + r \xi_{0,s} + r^2 \xi_{1,s}) \\ - 2\eta_{2,r}^1 + \xi_{2,s} &= 0, \quad \frac{2}{r} \eta_{2,t}^1 + 2\eta_{2,\theta}^2 - \xi_{2,s} = 0, \\ \frac{2}{r} \eta_{2,t}^1 + 2 \cot \theta \eta_{2,\theta}^2 + 2\eta_{2,\phi}^3 - \xi_{2,s} &= 0, \quad \eta_{2,\phi}^2 + \sin^2 \theta \eta_{2,\theta}^3 = 0, \\ \eta_{2,\theta}^0 - r^2 \eta_{2,t}^2 - \frac{1}{r} \eta_{1,\theta}^0 &= 0, \quad \eta_{2,\phi}^0 - r^2 \sin^2 \theta \eta_{2,t}^3 - \frac{1}{r} \eta_{1,\phi}^0 = 0, \\ \eta_{2,\theta}^1 + r^2 \eta_{2,r}^2 + \frac{1}{r} \eta_{1,\theta}^1 + \frac{1}{r^2} \eta_{0,\theta}^1 &= 0, \\ \eta_{2,\phi}^1 + r^2 \sin^2 \theta \eta_{2,r}^3 + \frac{1}{r} \eta_{1,\phi}^1 + \frac{1}{r^2} \eta_{0,\phi}^1 &= 0, \\ V_{,t} \eta_{2,t}^0 + V_{,r} \eta_{2,t}^1 + V_{,\theta} \eta_{2,t}^2 + V_{,\phi} \eta_{2,t}^3 + V \xi_{2,s} + A_{2,s} &= 0. \end{aligned} \quad (46)$$

Finally, the third-order ANGSS equations (33–35) of the metrics γ_{ab} , h_{ab} , σ_{ab} , and k_{ab} are explicitly written as

$$\begin{aligned}
&\xi_{3,t} = 0, \quad \xi_{3,r} = 0, \quad \xi_{3,\theta} = 0, \quad \xi_{3,\phi} = 0, \\
&\eta_{3,s}^0 - \frac{1}{r}\eta_{2,s}^0 \frac{3k}{2r^3}\eta_{0,s}^0 - A_{3,t} = 0, \\
&\frac{1}{r^3} \left[\left(1 - \frac{3k}{2}\right) \eta_{0,s}^1 + r \eta_{1,s}^1 + r^2 \eta_{2,s}^1 \right] + \eta_{3,s}^1 + A_{3,r} = 0, \\
&r^2 \eta_{3,s}^2 + A_{3,\theta} = 0, \quad r^2 \sin^2 \theta \eta_{3,s}^3 + A_{3,\phi} = 0, \\
&2\eta_{3,t}^0 + \frac{1}{r^2}(\eta_2^1 - 2r \eta_{2,t}^0 + r \xi_{2,s}) \\
&\quad - \frac{9k}{2r^4}\eta_0^1 + \frac{3k}{2r^3}(2\eta_{0,t}^0 - \xi_{0,s}) - \xi_{3,s} = 0, \\
&2\eta_1^1 + r \eta_2^1 - 2r^2 \eta_{2,r}^1 - 2r \eta_{1,r}^1 + r \xi_{1,s} + r^2 \xi_{2,s} - \frac{3k}{2}\xi_{0,s} \\
&\quad + \frac{3(3k-2)}{2r}\eta_0^1 + (3k-2)\eta_{0,r}^1 + r^3(-2\eta_{3,r}^1 + \xi_{3,s}) = 0, \\
&\frac{2}{r}\eta_3^1 + 2\eta_{3,\theta}^2 - \xi_{3,s} = 0, \quad \eta_{3,\phi}^2 + \sin^2 \theta \eta_{3,\phi}^3 = 0, \\
&\frac{2}{r}\eta_3^1 + 2 \cot \theta \eta_3^2 + 2\eta_{3,\phi}^3 - \xi_{3,s} = 0, \\
&\eta_{3,r}^0 - \eta_{3,t}^1 - \frac{1}{r}(\eta_{2,r}^0 + \eta_{2,t}^1) - \frac{1}{r^2}\eta_{1,t}^1 \\
&\quad + \frac{3k}{2r^3} \left[\eta_{0,r}^0 + (3k-2)\eta_{0,t}^1 \right] = 0, \\
&\eta_{3,\theta}^0 - r^2 \eta_{3,t}^2 - \frac{1}{r}\eta_{2,\theta}^0 + \frac{3k}{2r^3}\eta_{0,\theta}^0 = 0, \\
&\eta_{3,\phi}^0 - r^2 \sin^2 \theta \eta_{3,t}^3 - \frac{1}{r}\eta_{2,\phi}^0 + \frac{3k}{2r^3}\eta_{0,\phi}^0 = 0, \\
&\eta_{3,\theta}^1 + r^2 \eta_{2,r}^3 + \frac{1}{r}\eta_{2,\theta}^1 + \frac{1}{r^2}\eta_{1,\theta}^1 + \frac{(2-3k)}{2r^3}\eta_{0,\theta}^1 = 0, \\
&\eta_{3,\phi}^1 + r^2 \sin^2 \theta \eta_{3,r}^3 + \frac{1}{r}\eta_{2,\phi}^1 + \frac{1}{r^2}\eta_{1,\phi}^1 + \frac{(2-3k)}{2r^3}\eta_{0,\phi}^1 = 0, \\
&V_{,t}\eta_3^0 + V_{,r}\eta_3^1 + V_{,\theta}\eta_3^2 + V_{,\phi}\eta_3^3 + V\xi_{3,s} + A_{3,s} = 0.
\end{aligned} \tag{47}$$

We simultaneously solved the exact, the first-order, the second-order, and the third-order ANGSS equations (44–47) for the constant potential, e.g. $V(t, r, \theta, \phi) = V_0$. From (44–47), we find for the exact ANGSS

$$\xi_0 = a_1, \quad \eta_0^0 = a_2, \quad \eta_0^1 = 0, \quad \eta_0^2 = -a_3 \cos \phi + a_4 \sin \phi, \tag{48}$$

$$\eta_0^3 = \cot \theta (a_3 \sin \phi + a_4 \cos \phi) + a_5, \quad A_0 = a_6, \tag{49}$$

for the first-order ANGSS

$$\xi_1 = b_1, \quad \eta_1^0 = b_2, \quad \eta_1^1 = 0, \quad \eta_1^2 = -b_3 \cos \phi + b_4 \sin \phi, \tag{50}$$

$$\eta_1^3 = \cot \theta (b_3 \sin \phi + b_4 \cos \phi) + b_5, \quad A_1 = b_6, \tag{51}$$

for the second-order ANGSS

$$\xi_2 = c_1, \quad \eta_2^0 = c_2, \quad \eta_2^1 = 0, \quad \eta_2^2 = -c_3 \cos \phi + c_4 \sin \phi, \tag{52}$$

$$\eta_2^3 = \cot \theta (c_3 \sin \phi + c_4 \cos \phi) + c_5, \quad A_2 = c_6, \tag{53}$$

and for the third-order ANGSS

$$\xi_3 = d_1 \frac{s^2}{2} + d_2 s + d_3, \tag{54}$$

$$\eta_3^0 = \frac{s}{2}(d_1 t + 2d_4) + d_2 \frac{t}{2} + d_5, \tag{55}$$

$$\eta_3^1 = \frac{r}{2}(d_1 s + d_2), \quad \eta_3^2 = d_6 \cos \phi + d_7 \sin \phi, \tag{56}$$

$$\eta_3^3 = \cot \theta (d_6 \cos \phi - d_7 \sin \phi) + d_8, \tag{57}$$

$$A_3 = \frac{1}{4}d_1(t^2 - r^2 - 2V_0 s^2) - d_2 V_0 s + d_4 t + d_9, \tag{58}$$

where $a_\ell, b_\ell, c_\ell (\ell = 1, \dots, 6)$, and $d_1, \dots, d_9$ are the arbitrary constants of integration associated with the ANGSS.

It is easily seen from the above solutions that we have five exact, first-order, and second-order ANGSSs and eight third-order ANGSSs, which includes also exact ones. It is well known that the spacetime of the Bardeen model has four KVs, i.e.,

$$\mathbf{K}_0 = \partial_t, \quad \mathbf{K}_1 = \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi, \tag{59}$$

$$\mathbf{K}_2 = \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi, \quad \mathbf{K}_3 = \partial_\phi, \tag{60}$$

which correspond to the conservation of energy and angular momentum only.

For all orders of the ANGSSs, we point out here that there is the vector field $\mathbf{Y}_0 = \partial_s$, which gives a translation in the geodesic parameter s and it always exists for a geodesic Lagrangian of the type (2), and the remaining ones are the four KVs given in (59) and (60). We found that the exact, the first-order, and the second-order ANGSS algebras admitted by the Bardeen model are *five* dimensional. In addition to those of five ANGSSs, there are three additional third-order ANGSSs, namely the following:

$$\mathbf{Y}_1 = s \mathbf{Y}_0, \quad \text{with gauge term } A_3 = t, \tag{61}$$

$$\mathbf{Y}_2 = s \partial_s + \frac{1}{2}(t \partial_t + r \partial_r), \tag{62}$$

$$\mathbf{Y}_3 = s(s \partial_s + t \partial_t + r \partial_r), \tag{63}$$

with gauge term $A_3 = \frac{1}{2}(t^2 - r^2 - 2V_0 s^2)$. Thus, it is seen that for the Bardeen model the conservation laws of linear momentum and spin angular momentum in the third-order approximation are lost. In the third-order approximate symmetry case, we compute the ANGSSs of the perturbed geodesic Lagrangian for the Bardeen metric with non-constant potential. In order to determine the ANGSSs associated with a given non-constant potential $V(t, r, \theta, \phi)$, one has to use the new ANGSS equations (44–47). We have collected our results in Table 1.

Now, we intend to recover the lost conservation laws. Therefore, we will consider the lower than third order, i.e. the second-order, approximate symmetry of the Bardeen model. Thus we will retain the second-order terms in ε and neglect

Table 1 Third-order approximate Noether gauge symmetries and associated gauge functions A_0 , A_1 , A_2 , and A_3 of the Bardeen metric for some non-constant potentials $V(t, r, \theta, \phi)$. Here V_0 is a non-zero constant

Potential	ANGSs	A_0	A_1	A_2	A_3
$V = \frac{V_0}{r}$	$\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{Y}_0$	0	0	0	0
	$\mathbf{Y}_1$	0	0	0	t
$V = \frac{V_0}{r^2}$	$\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{Y}_0, \mathbf{Y}_2$	0	0	0	0
	$\mathbf{Y}_1, \mathbf{Y}_3$	0	0	0	$t, \frac{1}{2}(t^2 - r^2)$
$V = \frac{V_0}{r \sin \theta}$	$\mathbf{K}_0, \mathbf{K}_3, \mathbf{Y}_0$	0	0	0	0
	$\mathbf{Y}_1$	0	0	0	t
$V = \frac{V_0}{r^2 \sin \theta}$	$\mathbf{K}_0, \mathbf{K}_3, \mathbf{Y}_0, \mathbf{Y}_2$	0	0	0	0
	$\mathbf{Y}_1, \mathbf{Y}_3$	0	0	0	$t, \frac{1}{2}(t^2 - r^2)$
$V = V_0 r^2$	$\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{Y}_0$	0	0	0	0
	$\mathbf{Y}_1$	0	0	0	t
$V = V_0 t^2$	$\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{Y}_0$	0	0	0	0
	$\sin(\sqrt{2V_0}s)\mathbf{K}_0, \cos(\sqrt{2V_0}s)\mathbf{K}_0$	0	0	0	$\sqrt{2V_0}\cos(\sqrt{2V_0}s),$ $-\sqrt{2V_0}\sin(\sqrt{2V_0}s)$
$V = \frac{V_0}{8}(r^2 - t^2)$	$\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{Y}_0$	0	0	0	0
	$e^{V_0s}[(1 - v_0s)\mathbf{Y}_0 + V_0\mathbf{Y}_2],$ $e^{V_0s}[(1 + v_0s)\mathbf{Y}_0 - V_0\mathbf{Y}_2],$ $e^{V_0s/2}\mathbf{Y}_0, e^{-V_0s/2}\mathbf{Y}_0$	0	0	0	$-\frac{V_0^2}{4}(t^2 - r^2)e^{V_0s}, -\frac{V_0^2}{4}(t^2 - r^2)e^{-V_0s},$ $-\frac{V_0}{2}te^{-V_0s/2}, \frac{V_0}{2}te^{V_0s/2}$
$V = V_0 e^{\lambda\phi}$	$\mathbf{K}_0, \mathbf{Y}_0, \mathbf{Y}_2 - \frac{1}{\lambda}\mathbf{K}_3 (\lambda \neq 0)$	0	0	0	0
	$\mathbf{Y}_1$	0	0	0	t
$V = V_0\phi$	$\mathbf{K}_0, \mathbf{Y}_0$	0	0	0	0
	$\mathbf{K}_3, \mathbf{Y}_1$	$-V_0s, 0$	$-V_0s, 0$	$-V_0s, 0$	$-V_0s, t$
$V = V_0t$	$\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{Y}_0$	0	0	0	0
	$\mathbf{K}_0, \mathbf{Y}_2 - \frac{3V_0s^2}{4}\mathbf{Y}_0, \mathbf{Y}_3 - \frac{V_0s^3}{4}\mathbf{Y}_0$	$-V_0s, 0, 0$	$-V_0s, 0, 0$	$-V_0s, 0, 0$	$-V_0s, \frac{V_0s}{2}(\frac{s^2}{2} - 3t),$ $\frac{1}{2}(t^2 - r^2) + \frac{V_0s^2}{2}(\frac{V_0s^2}{4} - 3t)$

$O(\varepsilon^3)$ in the geodesic Lagrangian (39) for the Bardeen model. It is noted that the metric of the Bardeen model as the second-order perturbed spacetime reduces to the second-order perturbed Schwarzschild solution. So we will use the new set of determining equations up to second order to check whether the conservation laws are lost or not. Solving the exact, first-order and second-order ANGSequations (44–46) for the constant potential the same components follow as given by (48–49) and (50–51) for the exact and first-order ANGSequations, respectively. But for the second-order ANGSequations we have the following components:

$$\xi_2 = c_1 \frac{s^2}{2} + c_2s + c_3, \quad (64)$$

$$\eta_2^0 = \frac{t}{2}(c_1s + c_2) - r \sin \theta (c_4 \cos \phi - c_5 \sin \phi) - c_6r \cos \theta + c_7s + c_8, \quad (65)$$

$$\eta_2^1 = \frac{r}{2}(c_1s + c_2) + t \sin \theta (-c_4 \cos \phi + c_5 \sin \phi) - \cos \theta (c_6t + c_9s + c_{10}) - \sin \theta [s(c_{11} \sin \phi - c_{12} \cos \phi) + c_{13} \cos \phi - c_{14} \sin \phi], \quad (66)$$

$$\eta_2^2 = \frac{\cos \theta}{r} [t(-c_4 \cos \phi + c_5 \sin \phi) - s(c_{11} \sin \phi + c_{12} \cos \phi) - c_{13} \cos \phi + c_{14} \sin \phi] + \frac{\sin \theta}{r} (c_6t + c_9s + c_{10}) - c_{15} \cos \phi + c_{16} \sin \phi, \quad (67)$$

$$\eta_2^3 = \frac{1}{r \sin \theta} [t(c_4 \sin \phi + c_5 \cos \phi) + s(-c_{11} \cos \phi + c_{12} \sin \phi) + c_{13} \sin \phi + c_{14} \cos \phi] + \cot \theta (c_{15} \sin \phi + c_{16} \cos \phi) + c_{17}, \quad (68)$$

$$A_2 = \frac{1}{4}c_1(t^2 - r^2 - 2V_0s^2) - c_2V_0s + c_7t + c_9r \cos \theta + r \sin \theta (c_{11} \sin \phi + c_{12} \cos \phi) + c_{18}, \quad (69)$$

where $c_1, \dots, c_{18}$ are constants of integration. It is immediately seen from these solutions that the “lost” symmetries for the Bardeen metric have appeared in the second-order ANGSequations, and the second-order ANGSequations generators are the four KVs (59–60) and $\mathbf{Y}_0, \dots, \mathbf{Y}_3$, and $\mathbf{Z}_1, \dots, \mathbf{Z}_9$ given by (70–82) in the appendix, where for the second-order ANGSequations of the Bardeen model, the gauge functions are second order, $\mathbf{Y}_2$ has a gauge function such as $A_2 = -V_0s$, and the gauge function

of $\mathbf{Y}_3$ is also of the form $A_2 = \frac{1}{2}(t^2 - r^2 - 2V_0s^2)$. Hence, we have recovered 17 ANGSSs in the second-order approximate case of the perturbed Lagrangian for the Bardeen model, as is expected. We conclude that the first-order ANGSSs that have been recovered by the first-order perturbed Schwarzschild metric are the second-order ANGSSs for the Bardeen metric, which is also equivalent to the second-order perturbed Schwarzschild metric. The ANGSSs and corresponding gauge functions of the second-order perturbed Schwarzschild solution are also computed for some non-constant potentials, and the results of the calculations are the same as the ones shown in the Table 1 of [26], but there is only one difference: the gauge function A_1 of the first-order perturbation should be replaced by the gauge function A_2 of the second-order perturbation in Table 1 of [26].

4 Conclusions and discussions

In this paper, we studied the geometric nature of the ANGSSs up to third order (in ε) for the geodesic Lagrangian of spacetimes. We showed that the computation of third-order ANGSSs of the perturbed first-order geodesic Lagrangian (3) is reduced to the problem of finding solutions for the new geometrical set of exact differential conditions (24–26), the first-order differential conditions (27–29), the second-order differential conditions (30–32), and the third-order differential conditions (33–35), which involve the exact, first-order, second-order and third-order ANGSSs and the potential function defining the dynamical system. For the spacetime of the Bardeen model, this new method of finding ANGSSs enables one to get an ANGSS with non-zero potential.

It is pointed out in [25] that one cannot recognize the parameter e of the Bardeen spacetime as the Coulomb charge Q of the RN spacetime. We note that the Bardeen model with magnetic charge e is the analog of the RN spacetime with electric charge Q . For the Bardeen model, although there is no term of g_{tt} quadratic in ε , g_{rr} includes a quadratic term [see (37) and (38)]. Therefore, getting the lost symmetries, it is sufficient to calculate the ANGSSs of the RN spacetime for the second-order term of the perturbed Lagrangian, but this may not true for the Bardeen model in which the calculation is needed of the ANGSSs up to third-order term in the perturbed Lagrangian.

Applying our new method to the Bardeen model up to the third-order approximation, we found that the number of exact, first-order, and second-order ANGSSs is five. The number of third-order ANGSSs was found to be eight—not 17 as is expected. That is, in the third-order approximate symmetry case, the linear and spin angular momentum conservation laws for the Bardeen spacetime are lost. Retaining only terms of first and second order in the metric coefficients g_{tt} and g_{rr} given in (37) and (38), and neglecting $O(\varepsilon^3)$, it reduces to

the second-order perturbed spacetime for the Schwarzschild solution. Thus, for the Bardeen model as the second-order perturbed spacetime, we reconsider our new ANGSS equations to check whether those of the lost conservation laws can be recovered. Then we find that for the Bardeen model all the lost conservation laws are recovered in the second-order approximation.

It would be interesting to analyze the other special spacetimes according to the ANGSSs in which one can find the answer of the question “what is the order of the ANGSSs to give all conservation laws?”. This issue will be a subject of research in the future.

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Appendix: ANGSSs for Schwarzschild spacetime

The number of the ANGSS generators of the first-order (in ε) geodesic Lagrangian for the Schwarzschild spacetime with vanishing potential is 17 [22]: the four KVs given in (59) and (60), and

$$\mathbf{Y}_0 = \partial_s, \quad (70)$$

$$\mathbf{Y}_1 = s\mathbf{Y}_0, \text{ with gauge term } A_1 = t, \quad (71)$$

$$\mathbf{Y}_2 = s\partial_s + \frac{1}{2}(t\partial_t + r\partial_r), \quad (72)$$

$$\mathbf{Y}_3 = s(s\partial_s + t\partial_t + r\partial_r), \quad (73)$$

with gauge term $A_1 = \frac{1}{2}(t^2 - r^2)$,

$$\mathbf{Z}_1 = \sin\theta \cos\phi \partial_r + \frac{\cos\theta \cos\phi}{r} \partial_\theta - \frac{\csc\theta \sin\phi}{r} \partial_\phi, \quad (74)$$

$$\mathbf{Z}_2 = \sin\theta \sin\phi \partial_r + \frac{\cos\theta \sin\phi}{r} \partial_\theta + \frac{\csc\theta \cos\phi}{r} \partial_\phi, \quad (75)$$

$$\mathbf{Z}_3 = \cos\theta \partial_r - \frac{\sin\theta}{r} \partial_\theta, \quad (76)$$

$$\mathbf{Z}_4 = r \sin\theta \cos\phi \partial_t + t\mathbf{Z}_1, \quad (77)$$

$$\mathbf{Z}_5 = r \sin\theta \sin\phi \partial_t + t\mathbf{Z}_2, \quad (78)$$

$$\mathbf{Z}_6 = r \cos\theta \partial_t + t\mathbf{Z}_3, \quad (79)$$

$$\mathbf{Z}_7 = s\mathbf{Z}_1, \text{ with gauge term } A_1 = -r \sin\theta \cos\phi, \quad (80)$$

$$\mathbf{Z}_8 = s\mathbf{Z}_2, \text{ with gauge term } A_1 = -r \sin\theta \sin\phi, \quad (81)$$

$$\mathbf{Z}_9 = s\mathbf{Z}_3, \text{ with gauge term } A_1 = -r \sin\theta, \quad (82)$$

where the generators $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3$ provide the conservation of linear momentum, and $\mathbf{Z}_4, \mathbf{Z}_5$, and $\mathbf{Z}_6$ yield the conservation

of spin angular momentum. Using the generator $\mathbf{Y}_2$ one can write $s = t^2$ or $s = r^2$ which gives rise to the interpretation of the four KVs $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$, and the 11 generators $\mathbf{Y}_0, \mathbf{Y}_1, \mathbf{Z}_1, \dots, \mathbf{Z}_9$ being CKVs with conformal factor $\psi = (c_1 t^2 + c_2)/2$. It is a very well-known fact that the Lie algebra of the CKVs for a conformally flat spacetime is 15 dimensional.

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