Quantum billiards in multidimensional models with branes

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Abstract A gravitational D -dimensional model with l scalar fields and several forms is considered. When a cosmological-type diagonal metric is chosen, an electromagnetic composite brane ansatz is adopted and certain restrictions on the branes are imposed; the conformally covariant Wheeler–DeWitt (WDW) equation for the model is studied. Under certain restrictions asymptotic solutions to WDW equation are found in the limit of the formation of the billiard walls which reduce the problem to the so-called quantum billiard on the $(D+l-2)$ -dimensional Lobachevsky space. Two examples of quantum billiards are considered. The first one deals with 9-dimensional quantum billiard for $D = 11$ model with 330 four-forms which mimic space-like $M2$ - and $M5$ -branes of $D = 11$ supergravity. The second one deals with the 9-dimensional quantum billiard for $D = 10$ gravitational model with one scalar field, 210 four-forms and 120 three-forms which mimic space-like $D2$ -, $D4$ -, $FS1$ - and $NS5$ -branes in $D = 10$ IIA supergravity. It is shown that in both examples wave functions vanish in the limit of the formation of the billiard walls (i.e. we get a quantum resolution of the singularity for 11 D model) but magnetic branes could not be neglected in calculations of quantum asymptotic solutions while they are irrelevant for classical oscillating behavior when all 120 electric branes are present.

1 Introduction

This paper deals with the quantum billiard approach for D -dimensional cosmological-type models defined on the manifold $(u_-, u_+) \times \mathbb{R}^{D-1}$, where $D \geq 4$.

The billiard approach in classical gravity originally appeared in the dissertation of Chitré [1] for the explanation of the BLK-oscillations [2] in the Bianchi-IX model [3, 4]. In

this approach a simple triangle billiard in the Lobachevsky space H^2 was used.

In [5] the billiard approach for $D = 4$ was extended to the quantum case. Namely, the solutions to the Wheeler–DeWitt (WDW) equation [6] were reduced to the problem of finding the spectrum of the Laplace–Beltrami operator on a Chitré’s triangle billiard. Such an approach was also used in [7] in the context of studying the large scale inhomogeneities of the metric in the vicinity of the singularity.

A straightforward generalization of the Chitré’s billiard to the multidimensional case was performed in [8–10], where a multidimensional cosmological model with a multicomponent “perfect” fluid and n Einstein factor spaces was studied. In [10] the search of oscillating behavior near the singularity was reduced to the problem of proving the finiteness of the billiard volume. This problem was reformulated in terms of the problem of the illumination of the sphere S^{n-2} by point-like sources. In [10] the inequalities on the Kasner parameters were found and the “quantum billiard” approach was considered; see also [11, 12]. The classical billiard approach for multidimensional models with fields of forms and scalar fields was suggested in [13], where the inequalities for the Kasner parameters were also written. For certain examples these inequalities have played a key role in the proof of the never-ending oscillating behavior near the singularity which takes place in effective gravitational models with forms and scalar fields induced by superstrings [14–16]. It was shown in [17] that in these models the parts of billiards are related to Weyl chambers of certain hyperbolic Kac–Moody (KM) Lie algebras [18–21]. This fact simplifies the proof of the finiteness of the billiard volume. Using this approach the well-known result from [22] on the critical dimension of pure gravity was explained using hyperbolic algebras in [23]. For reviews on the billiard approach see [16, 24].

In recent publications [25, 26] the quantum billiard approach for the multidimensional gravitational model with several forms was considered. The main motivation for this approach is coming from the quantum gravity paradigm;

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see [27,28] and references therein. It should be noted that the asymptotic solutions to the WDW equation presented in these papers are equivalent to the solutions obtained earlier in [10]. The wave function (Ψ_{KKN}) from [25,26] corresponds to the harmonic time gauge, while the wave function (Ψ_{IM}) from [10] is related to the “tortoise” time gauge. (These functions are connected by a certain conformal transformation $\Psi_{\text{KKN}} = \Omega \Psi_{\text{IM}}$.) In [10,25,26] a “semi-quantum” approach was used: the gravity (of a toy model) was quantized but the matter sources (e.g. fluids, forms) were considered at the classical level.¹ Such a semi-quantum form of the WDW equation for the model with fields of forms and a scalar field was suggested earlier in [33].

In our previous publication [34] we have used another form of the WDW equation with enlarged minisuperspace which includes the form potentials [35]. We have suggested another version of the quantum billiard approach by deducing the asymptotic solutions to WDW equation for the model with fields of forms when a non-composite electric brane ansatz has been adopted.

In [34] we have considered an example of a 9-dimensional quantum billiard for $D = 11$ model with 120 four-forms which mimic space-like $M2$ -brane solutions ($SM2$ -branes) in $D = 11$ supergravity.² It was shown in [34] that the wave function vanishes as $y^0 \rightarrow -\infty$ (i.e. at the singularity), where y^0 is the “tortoise” time-like coordinate in the minisuperspace.

In this paper we substantially generalize the approach of [34] to the case when scalar (dilaton) fields and dilaton couplings are added into consideration. Here the composite electromagnetic ansatz for branes is considered instead of non-composite electric one from [34]. We present new examples of quantum billiards with electric and magnetic S -branes in $D = 11$ and $D = 10$ models, which are non-composite analogs of truncated bosonic sectors of $D = 11$ and $D = 10$ supergravitational models. In both examples of billiards magnetic branes do not participate in the formation of the billiard walls since magnetic walls are hidden by electric ones. The adding of magnetic branes does not change the classical asymptotic oscillating behavior of scale factors and scalar field (for $D = 10$). In the quantum case adding of magnetic branes changes the asymptotic behavior of the wave function, but nevertheless, as in [34], the wave function vanishes as $y^0 \rightarrow -\infty$. For $D = 11$ this means a quantum resolution of the singularity for the model with electric and magnetic branes which mimic (space-like) $SM2$ - and $SM5$ -branes in $11D$ supergravity.

¹ Here, one should also mention the recent papers by Lecian (e.g. with co-authors) [29–32] devoted to the quantum billiard approach in the Mixmaster model, which were inspired by Refs. [25,26].

² For S -brane solutions see [36–39] and references therein.

2 The setup

Here we study the multidimensional gravitational model governed by the action

$$S_{\text{act}} = \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|g|} \{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_{a \in \Delta} \frac{\theta_a}{n_a!} \exp[2\lambda_a(\varphi)] (F^a)^2_g \} + S_{\text{YGH}}, \quad (2.1)$$

where $g = g_{MN}(z) dz^M \otimes dz^N$ is a metric on the manifold M , $\dim M = D$, $\varphi = (\varphi^\alpha) \in \mathbb{R}^l$ is a vector from dilatonic scalar fields, $(h_{\alpha\beta})$ is a non-degenerate symmetric $l \times l$ matrix ($l \in \mathbb{N}$), $\theta_a \neq 0$, and we have

$$F^a = dA^a = \frac{1}{n_a!} F^a_{M_1 \dots M_{n_a}} dz^{M_1} \wedge \dots \wedge dz^{M_{n_a}},$$

which is an n_a -form ($n_a \geq 2$) on M and λ_a is a 1-form on $\mathbb{R}^l$: $\lambda_a(\varphi) = \lambda_{a\alpha} \varphi^\alpha$, $a \in \Delta$, $\alpha = 1, \dots, l$. In (2.1) we denote $|g| = |\det(g_{MN})|$, $(F^a)^2_g = F^a_{M_1 \dots M_{n_a}} F^a_{N_1 \dots N_{n_a}} g^{M_1 N_1} \dots g^{M_{n_a} N_{n_a}}$, $a \in \Delta$, where Δ is some finite set of (color) indices and S_{YGH} is the standard (York–Gibbons–Hawking) boundary term [40,41]. In the models with one time and the usual fields of forms all $\theta_a > 0$ when the signature of the metric is $(-1, +1, \dots, +1)$. For such a choice of signature $\theta_b < 0$ corresponds to a “phantom” form field F^b .

2.1 Ansatz for composite brane configurations

We consider the manifold

$$M = (u_-, u_+) \times \mathbb{R}^n, \quad (2.2)$$

with the metric

$$g = w e^{2\gamma(u)} du \otimes du + \sum_{i=1}^n e^{2\phi^i(u)} \varepsilon(i) dx^i \otimes dx^i, \quad (2.3)$$

where $w = \pm 1$, $\varepsilon(i) = \pm 1$, $i = 1, \dots, n$. The dimension of M is $D = 1 + n$. Here one may replace $\mathbb{R}^n$ in (2.2) by $\mathbb{R}^k \times (S^1)^{n-k}$, $0 \leq k \leq n$, without any change of all the relations as presented below.

Although in what follows all examples deal with cosmological (S -brane) solutions with $w = -1$ and $\varepsilon(i) = +1$ for all i , we reserve here general notations for signs just keeping in mind possible future applications to static configurations with $w = 1$, $\varepsilon(1) = -1$ and $\varepsilon(k) = +1$ for $k > 1$ (e.g. fluxbranes, wormholes etc.) and solutions with several time-like directions.

By $\Omega = \Omega(n)$ we denote a set of all non-empty subsets of $\{1, \dots, n\}$. The number of elements in Ω is $|\Omega| = 2^n - 1$.

For any $I = \{i_1, \dots, i_k\} \in \Omega$, $i_1 < \dots < i_k$, we use the following notations:

$$\tau(I) \equiv dx^{i_1} \wedge \dots \wedge dx^{i_k}, \quad (2.4)$$

$$\varepsilon(I) \equiv \varepsilon(i_1) \dots \varepsilon(i_k), \quad (2.5)$$

$$d(I) = |I| \equiv k. \quad (2.6)$$

For fields of forms we consider the following composite electromagnetic ansatz:

$$F^a = \sum_{I \in \Omega_{a,e}} \mathcal{F}^{(a,e,I)} + \sum_{J \in \Omega_{a,m}} \mathcal{F}^{(a,m,J)}, \quad (2.7)$$

where

$$\mathcal{F}^{(a,e,I)} = d\Phi^{(a,e,I)} \wedge \tau(I), \quad (2.8)$$

$$\mathcal{F}^{(a,m,J)} = e^{-2\lambda_a(\varphi)} * (d\Phi^{(a,m,J)} \wedge \tau(J)) \quad (2.9)$$

are elementary forms of electric and magnetic types, respectively, $a \in \Delta$, $I \in \Omega_{a,e}$, $J \in \Omega_{a,m}$ and $\Omega_{a,v} \subset \Omega$, $v = e, m$.

In (2.9) $*$ is the Hodge operator on (M, g) .

For scalar functions we put

$$\varphi^\alpha = \varphi^\alpha(u), \quad \Phi^s = \Phi^s(u), \quad (2.10)$$

$s \in S$. Thus, φ^α and Φ^s are functions on (u_-, u_+) .

Here and below

$$S = S_e \sqcup S_m, \quad S_v = \sqcup_{a \in \Delta} \{a\} \times \{v\} \times \Omega_{a,v}, \quad (2.11)$$

$v = e, m$ and $\sqcup$ is the union of non-intersecting sets. The set S consists of elements $s = (a_s, v_s, I_s)$, where $a_s \in \Delta$ is the color index, $v_s = e, m$ is the electromagnetic index, and the set $I_s \in \Omega_{a_s, v_s}$ describes the location of the brane.

Due to (2.8) and (2.9)

$$d(I) = n_a - 1, \quad d(J) = D - n_a - 1, \quad (2.12)$$

for $I \in \Omega_{a,e}$ and $J \in \Omega_{a,m}$, $a \in \Delta$, i.e. in electric and magnetic case, respectively.

2.2 Sigma-model action

Here we present two restrictions on the sets of branes which guarantee the diagonal form of the energy-momentum tensor and the existence of the sigma-model representation (without additional constraints) [42] (see also [43]).

The first restriction deals with any pair of two (different) branes both electric (ee -pair) or magnetic (mm -pair) with coinciding color index:

$$(R1) \quad d(I \cap J) \leq d(I) - 2, \quad (2.13)$$

for any $I, J \in \Omega_{a,v}$, $a \in \Delta$, $v = e, m$ (here $d(I) = d(J)$).

The second restriction deals with any pair of two branes with the same color index, which include one electric and one magnetic brane (em -pair):

$$(R2) \quad d(I \cap J) \neq 0, \quad (2.14)$$

where $I \in \Omega_{a,e}$, $J \in \Omega_{a,m}$, $a \in \Delta$.

These restrictions are satisfied identically in the non-composite case, when there are no two branes corresponding to the same form F^a for any $a \in \Delta$.

It follows from [42] that the equations of motion for the model (2.1) and the Bianchi identities, $d\mathcal{F}^s = 0$, $s \in S_m$, for fields from (2.3), (2.7)–(2.10), when restrictions (R1) and (R2) are imposed, are equivalent to the equations of motion for the σ -model governed by the action

$$S_{\sigma 0} = \frac{\mu}{2} \int du \mathcal{N} \left\{ \hat{G}_{AB} \dot{\sigma}^A \dot{\sigma}^B + \sum_{s \in S} \varepsilon_s \exp(-2U_A^s \sigma^A) (\dot{\Phi}^s)^2 \right\}, \quad (2.15)$$

where $\dot{x} \equiv dx/du$, $(\sigma^A) = (\phi^i, \varphi^\alpha)$, $\mu \neq 0$, the index set S is defined in (2.11) and $\mathcal{N} = \exp(\gamma_0 - \gamma) > 0$ is a modified lapse function with $\gamma_0(\phi) \equiv \sum_{i=1}^n \phi^i$,

$$(\hat{G}_{AB}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix} \quad (2.16)$$

is a truncated target space metric with

$$G_{ij} = \delta_{ij} - 1, \quad (2.17)$$

and co-vectors

$$U_A^s \sigma^A = \sum_{i \in I_s} \phi^i - \chi_s \lambda_{a_s}(\varphi), \quad (U_A^s) = (\delta_{iI_s}, -\chi_s \lambda_{a_s \alpha}), \quad (2.18)$$

$s = (a_s, v_s, I_s)$.

Here $\chi_e = +1$ and $\chi_m = -1$;

$$\delta_{iI} = \sum_{j \in I} \delta_{ij} \quad (2.19)$$

is the indicator of i belonging to I : $\delta_{iI} = 1$ for $i \in I$ and $\delta_{iI} = 0$ otherwise; and

$$\begin{aligned} \varepsilon_s &= \varepsilon(I_s) \theta_{a_s} \quad \text{for } v_s = e; \\ \varepsilon_s &= -\varepsilon[g] \varepsilon(I_s) \theta_{a_s} \quad \text{for } v_s = m, \end{aligned} \quad (2.20)$$

$s \in S$, $\varepsilon[g] \equiv \text{sign det}(g_{MN})$.

In the electric case ($\mathcal{F}^{(a,m,I)} = 0$) when any factor space with the coordinate x^i is compactified to a circle of length L_i , the action (2.15) coincides with the action (2.1) if $\mu = -w/\kappa_0^2$, $\kappa^2 = \kappa_0^2 L_1 \dots L_n$.

In what follows we will use the scalar products of the U^s -vectors, $(U^s, U^{s'})$; $s, s' \in S$, where

$$(U, U') = \hat{G}^{AB} U_A U'_B, \quad (2.21)$$

for $U = (U_A)$, $U' = (U'_A) \in \mathbb{R}^{N_0}$, $N_0 = n + l$ and

$$(\hat{G}^{AB}) = \begin{pmatrix} G^{ij} & 0 \\ 0 & h^{\alpha\beta} \end{pmatrix} \quad (2.22)$$

is the matrix inverse to the matrix (2.16). Here

$$G^{ij} = \delta^{ij} + \frac{1}{2-D}, \quad (2.23)$$

$i, j = 1, \dots, n$.

The scalar products (2.21) read [42]

$$(U^s, U^{s'}) = d(I_s \cap I_{s'}) + \frac{d(I_s)d(I_{s'})}{2-D} + \chi_s \chi_{s'} \lambda_{a_s \alpha} \lambda_{a_{s'} \beta} h^{\alpha\beta}, \quad (2.24)$$

where $(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}$ and $s = (a_s, v_s, I_s)$, $s' = (a_{s'}, v_{s'}, I_{s'})$ belong to S .

The action (2.15) may also be written in the form

$$S_\sigma = \frac{\mu}{2} \int du \mathcal{N} \left\{ \mathcal{G}_{\hat{A}\hat{B}}(X) \dot{X}^{\hat{A}} \dot{X}^{\hat{B}} \right\}, \quad (2.25)$$

where $X = (X^{\hat{A}}) = (\phi^i, \varphi^\alpha, \Phi^s) \in \mathbb{R}^N$, $N = n + l + m$, $m = |S|$ is the number of branes and minisupermetric $\mathcal{G} = \mathcal{G}_{\hat{A}\hat{B}}(X) dX^{\hat{A}} \otimes dX^{\hat{B}}$ on the minisuperspace $\mathcal{M} = \mathbb{R}^N$ is defined as follows:

$$(\mathcal{G}_{\hat{A}\hat{B}}(X)) = \begin{pmatrix} G_{ij} & 0 & 0 \\ 0 & h_{\alpha\beta} & 0 \\ 0 & 0 & \varepsilon_s \exp(-2U^s(\sigma)) \delta_{ss'} \end{pmatrix}. \quad (2.26)$$

The minisuperspace metric (2.26) may also be written in the form

$$\mathcal{G} = \hat{G} + \sum_{s \in S} \varepsilon_s e^{-2U^s(\sigma)} d\Phi^s \otimes d\Phi^s, \quad (2.27)$$

where

$$\hat{G} = \hat{G}_{AB} d\sigma^A \otimes d\sigma^B = G_{ij} d\phi^i \otimes d\phi^j + h_{\alpha\beta} d\varphi^\alpha \otimes d\varphi^\beta \quad (2.28)$$

is a truncated minisupermetric and $U^s(\sigma) = U_A^s \sigma^A$ is defined in (2.18).

In what follows we denote

$$U^\Lambda(\sigma) = U_A^\Lambda \sigma^A = \gamma_0(\phi), \quad (2.29)$$

$$(U_A^\Lambda) = (U_i^\Lambda = 1, U_\alpha^\Lambda = 0).$$

This vector is time-like and all $(U^s, U^\Lambda) < 0$, since [42]

$$(U^\Lambda, U^\Lambda) = -\frac{D-1}{D-2}, \quad (U^s, U^\Lambda) = \frac{d(I_s)}{2-D}. \quad (2.30)$$

3 Quantum billiard approach

In this section we develop a quantum analog of the billiard approach which deals with asymptotical solutions to Wheeler–DeWitt (WDW) equation.

3.1 Restrictions

First we outline restrictions on parameters which will be used in derivation of the “quantum billiard”

$$(i) \quad (U^s, U^s) > 0, \quad (3.1)$$

$$(ii) \quad (h_{\alpha\beta}) > 0, \quad (3.2)$$

$$(iii) \quad \varepsilon_s > 0, \quad (3.3)$$

for all s . These restrictions are necessary conditions for the formation of infinite “wall” potential in certain limit (see below). The first restriction reads (see 2.24)

$$(U^s, U^s) = d(I_s) \left(1 + \frac{d(I_s)}{2-D} \right) + \lambda_{a_s \alpha} \lambda_{a_s \beta} h^{\alpha\beta} > 0. \quad (3.4)$$

The second restriction means that the matrix $(h_{\alpha\beta})$ is positive definite, i.e. the so-called phantom scalar fields are not considered.

3.2 Wheeler–DeWitt equation

Now we fix the temporal gauge as follows:

$$\gamma_0 - \gamma = 2f(X), \quad \mathcal{N} = e^{2f}, \quad (3.5)$$

where $f: \mathcal{M} \rightarrow \mathbb{R}$ is a smooth function. Then we obtain the Lagrange system with the Lagrangian

$$L_f = \frac{\mu}{2} e^{2f} \mathcal{G}_{\hat{A}\hat{B}}(X) \dot{X}^{\hat{A}} \dot{X}^{\hat{B}} \quad (3.6)$$

and the energy constraint

$$E_f = \frac{\mu}{2} e^{2f} \mathcal{G}_{\hat{A}\hat{B}}(X) \dot{X}^{\hat{A}} \dot{X}^{\hat{B}} = 0. \quad (3.7)$$

The set of Lagrange equations with the constraint (3.7) is equivalent to the set of Hamiltonian equations for the Hamiltonian

$$H^f = \frac{1}{2\mu} e^{-2f} \mathcal{G}^{\hat{A}\hat{B}}(X) P_{\hat{A}} P_{\hat{B}} \quad (3.8)$$

with the constraint

$$H^f = 0, \quad (3.9)$$

where $P_{\hat{A}} = \mu e^{2f} \mathcal{G}_{\hat{A}\hat{B}}(X) \dot{X}^{\hat{B}}$ are momenta (for fixed gauge) and $(\mathcal{G}^{\hat{A}\hat{B}}) = (\mathcal{G}_{\hat{A}\hat{B}})^{-1}$.

Here we use the prescriptions of covariant and conformally covariant quantization of the hamiltonian constraint $H^f = 0$ which was suggested initially by Misner [44] and considered afterwards in [45–48] and some other papers.

We obtain the Wheeler–DeWitt (WDW) equation³

$$\hat{H}^f \Psi^f \equiv \left(-\frac{1}{2\mu} \Delta \left[e^{2f} \mathcal{G} \right] + \frac{a}{\mu} R \left[e^{2f} \mathcal{G} \right] \right) \Psi^f = 0, \quad (3.10)$$

³ For the WDW equation in 4D gravity see [6].

where

$$a = a_N = \frac{(N-2)}{8(N-1)}, \quad (3.11)$$

$$N = n + l + m.$$

Here $\Psi^f = \Psi^f(X)$ is the wave function corresponding to the f -gauge (3.5) and satisfying the relation⁴

$$\Psi^f = e^{bf} \Psi^{f=0}, \quad b = b_N = (2-N)/2. \quad (3.12)$$

In (3.10) we denote by $\Delta[\mathcal{G}^f]$ and $R[\mathcal{G}^f]$ the Laplace–Beltrami operator and the scalar curvature corresponding to the metric

$$\mathcal{G}^f = e^{2f} \mathcal{G}, \quad (3.13)$$

respectively.

The choice of minisuperspace covariant form for the Hamiltonian operator $\hat{H}^f$ (3.10) with arbitrary real number a is one of the solutions to the operator ordering problem in multidimensional quantum cosmology [49–52]. The Laplace–Beltrami form of WDW equation was considered previously in [6, 53–56]. Similar prescription appears in quantization of a point-like particle moving in a curved background, for a review see [57, 58].

It was shown in [47, 48] by rigorous constraint quantization of parametrized relativistic gauge systems in curved spacetimes that the privileged choice for a in cosmological case is given by (3.11). For this value of a and $N > 1$ there is one-to-one correspondence between solutions to WDW equations for any two choices of temporal gauges given by (3.5) with smooth functions f_1 and f_2 instead of f , respectively. This fact follows from (3.12) and the following relation:

$$\hat{H}^f = e^{-2f} e^{bf} \hat{H}^{f=0} e^{-bf}. \quad (3.14)$$

We note that the coefficients a_N and b_N are the well-known ones in the conformally covariant theory of a scalar field [59].

Now we put $f = f(\sigma)$. Then we get

$$\Delta[\mathcal{G}^f] = e^{\bar{U}} |\bar{G}|^{-1/2} \frac{\partial}{\partial \sigma^A} \left(\bar{G}^{AB} e^{-\bar{U}} |\bar{G}|^{1/2} \frac{\partial}{\partial \sigma^B} \right) + \sum_{s \in S} e^{2\bar{U}^s(\sigma)} \left(\frac{\partial}{\partial \Phi^s} \right)^2, \quad (3.15)$$

where

$$\bar{U} = \sum_{s \in S} \bar{U}^s, \quad \bar{U}^s = U^s(\sigma) - f \quad (3.16)$$

and

$$\bar{G}_{AB} = e^{2f} \hat{G}_{AB}, \quad \bar{G}^{AB} = e^{-2f} \hat{G}^{AB}, \quad (3.17)$$

$$|\bar{G}| = |\det(\bar{G}_{AB})|.$$

⁴ We eliminate here a typo in a corresponding relation from [35].

Here we deal with a special class of asymptotical solutions to WDW equation. Due to restrictions (3.2) and (3.3) the (minisuperspace) metrics $\hat{G}$, $\mathcal{G}$ have a pseudo-Euclidean signatures $(-, +, \dots, +)$. We put

$$e^{2f} = -(\hat{G}_{AB} \sigma^A \sigma^B)^{-1}, \quad (3.18)$$

where we impose $\hat{G}_{AB} \sigma^A \sigma^B < 0$.

Here we use a diagonalization of σ -variables

$$\sigma^A = S_a^A z^a, \quad (3.19)$$

$a = 0, \dots, N_0 - 1$, with $N_0 = n + l$, obeying $\hat{G}_{AB} \sigma^A \sigma^B = \eta_{ab} z^a z^b$, where $(\eta_{ab}) = \text{diag}(-1, +1, \dots, +1)$.

We restrict the WDW equation to the lower light cone $V_- = \{z = (z^0, \vec{z}) | z^0 < 0, \eta_{ab} z^a z^b < 0\}$ and introduce the Misner–Chitre-like coordinates

$$z^0 = -e^{-y^0} \frac{1 + \vec{y}^2}{1 - \vec{y}^2}, \quad (3.20)$$

$$\vec{z} = -2e^{-y^0} \frac{\vec{y}}{1 - \vec{y}^2}, \quad (3.21)$$

where $y^0 < 0$ and $\vec{y}^2 < 1$. In these variables we have $f = y^0$.

Using the relation $f_{,A} = \bar{G}_{AB} \sigma^B$, following from (3.19), we obtain

$$\Delta[\bar{G}]f = 0, \quad \bar{G}^{AB} f_{,A} f_{,B} = -1. \quad (3.22)$$

These relations just follow from the relation

$$\bar{G} = -dy^0 \otimes dy^0 + h_L, \quad (3.23)$$

where

$$h_L = \frac{4\delta_{rs} dy^r \otimes dy^s}{(1 - \vec{y}^2)^2} \quad (3.24)$$

(the summation over $r, s = 1, \dots, N_0 - 1$ is assumed). Here the metric h_L is defined on the unit ball $D^{N_0-1} = \{\vec{y} \in \mathbb{R}^{N_0-1} | \vec{y}^2 < 1\}$. D^{N_0-1} with the metric h_L is a realization of the $(N_0 - 1)$ -dimensional Lobachevsky space H^{N_0-1} .

For the wave function we suggest the following ansatz:

$$\Psi^f = e^{C(\sigma)} \Psi_*, \quad (3.25)$$

where the prefactor $e^{C(\sigma)}$ is chosen for the sake of cancellation the terms linear in derivatives $\Psi_{*,A}$ arising in calculation of $\Delta[\mathcal{G}^f] \Psi^f$. This takes place for

$$C = C(\sigma) = \frac{1}{2} \bar{U} = \frac{1}{2} \left(\sum_{s \in S} U_A^s \sigma^A - m f \right). \quad (3.26)$$

With this choice of the prefactor we obtain

$$e^{\bar{U}} |\bar{G}|^{-1/2} \frac{\partial}{\partial \sigma^A} \left(\bar{G}^{AB} e^{-\bar{U}} |\bar{G}|^{1/2} \frac{\partial}{\partial \sigma^B} \right) (e^{C(\sigma)} \Psi_*) = e^{C(\sigma)} [\Delta[\bar{G}] \Psi_* + (\Delta[\bar{G}] C) \Psi_* - \bar{G}^{AB} C_{,A} C_{,B} \Psi_*]. \quad (3.27)$$

Using relations (3.22) we get

$$\Delta[\bar{G}]C = \frac{1}{2}(n+l-2) \sum_{s \in S} U_A^s \sigma^A, \quad (3.28)$$

$$\bar{G}^{AB}{}_{C,AC,B} = \frac{1}{4} \left[e^{-2f} \sum_{s,s' \in S} (U^s, U^{s'}) - 2m \sum_{s \in S} U_A^s \sigma^A - m^2 \right]. \quad (3.29)$$

The calculation of the scalar curvature $R[e^{2f}\mathcal{G}]$ gives us the following formula:

$$R[e^{2f}\mathcal{G}] = e^{-2f} \left[- \sum_{s \in S} (U^s, U^s) - \sum_{s,s' \in S} (U^s, U^{s'}) \right] + 2(N-1) \sum_{s \in S} U_A^s \sigma^A + (N-1)(m+2-n-l). \quad (3.30)$$

Collecting relations (3.28), (3.29), and (3.30) we obtain the following identity:

$$\begin{aligned} & \left(-\frac{1}{2} \Delta[e^{2f}\mathcal{G}] + aR[e^{2f}\mathcal{G}] \right) (e^{C(\sigma)} \Psi_*) \\ &= e^{C(\sigma)} \left(-\frac{1}{2} \Delta[\bar{G}] - \frac{1}{2} \sum_{s \in S} e^{2\bar{U}^s} \left(\frac{\partial}{\partial \Phi^s} \right)^2 + \delta V \right) \Psi_*, \end{aligned} \quad (3.31)$$

where

$$\delta V = A e^{-2f} - \frac{1}{8}(n+l-2)^2. \quad (3.32)$$

Here we denote

$$A = \frac{1}{8(N-1)} \left[\sum_{s,s' \in S} (U^s, U^{s'}) - (N-2) \sum_{s \in S} (U^s, U^s) \right]. \quad (3.33)$$

In what follows we call A as A -number.

It should be noted that linear in σ^A terms, which appear in (3.28), (3.29), and (3.30) are canceled in (3.31) due to our choice of conformal coupling $a = (N-2)/(8(N-1))$.

Now we put

$$\Psi^f = e^{C(\sigma)} e^{iQ_s \Phi^s} \Psi_{0,L}(\sigma), \quad (3.34)$$

where the parameters $Q_s \neq 0$ correspond to the charge densities of branes and $e^{iQ_s \Phi^s} = \exp(i \sum_{s \in S} Q_s \Phi^s)$. Using (3.31) we get

$$\begin{aligned} \hat{H}^f \Psi^f &= \mu^{-1} e^{C(\sigma)} e^{iQ_s \Phi^s} \\ &\times \left(-\frac{1}{2} \Delta[\bar{G}] + \frac{1}{2} \sum_{s \in S} Q_s^2 e^{-2f+2U^s(\sigma)} + \delta V \right) \Psi_{0,L} = 0. \end{aligned} \quad (3.35)$$

Here and in what follows $U^s(\sigma) = U_A^s \sigma^A$.

3.3 Asymptotic behavior of solutions for $y^0 \rightarrow -\infty$

Now we proceed with the studying the asymptotical solutions to WDW equation in the limit $y^0 \rightarrow -\infty$. Due to (3.34) and (3.35) this equation reads

$$\left(-\frac{1}{2} \Delta[\bar{G}] + \frac{1}{2} \sum_{s \in S} Q_s^2 e^{-2f+2U^s(\sigma)} + \delta V \right) \Psi_{0,L} = 0. \quad (3.36)$$

It was shown in [13] that

$$\frac{1}{2} \sum_{s \in S} Q_s^2 e^{-2f+2U^s(\sigma)} \rightarrow V_\infty, \quad (3.37)$$

as $y^0 = f \rightarrow -\infty$. Here V_∞ is the potential of infinite walls which are produced by branes

$$V_\infty = \sum_{s \in S} \theta_\infty(\vec{v}_s^2 - 1 - (\vec{y} - \vec{v}_s)^2), \quad (3.38)$$

where we denote $\theta_\infty(x) = +\infty$, for $x \geq 0$ and $\theta_\infty(x) = 0$ for $x < 0$. The vectors $\vec{v}_s$, $s \in S$, which belong to $\mathbb{R}^{N_0-1}$ ($N_0 = n+l$), are defined by

$$\vec{v}_s = -\vec{u}_s/u_{s0}, \quad (3.39)$$

where the N_0 -dimensional vectors $u_s = (u_{s0}, \vec{u}_s) = (u_{sa})$ are obtained from U^s -vectors using the diagonalization matrix (S_a^A) from (3.19)

$$u_{sa} = S_a^A U_A^s. \quad (3.40)$$

Due to (3.1) we get

$$(U^s, U^s) = -(u_{s0})^2 + (\vec{u}_s)^2 > 0 \quad (3.41)$$

for all s . In what follows we use a diagonalization (3.19) from [13] obeying

$$u_{s0} > 0 \quad (3.42)$$

for all $s \in S$. The inverse matrix $(S_A^a) = (S_a^A)^{-1}$ defines the map which is inverse to (3.19)

$$z^a = S_A^a \sigma^A, \quad (3.43)$$

$a = 0, \dots, N_0 - 1$. The inequalities (3.41) imply $|\vec{v}_s| > 1$ for all s . The potential V_∞ corresponds to the billiard B in the multidimensional Lobachevsky space (D^{N_0-1}, h_L) . This billiard is an open domain in D^{N_0-1} obeying the a set of inequalities:

$$|\vec{y} - \vec{v}_s| < \sqrt{\vec{v}_s^2 - 1} = r_s, \quad (3.44)$$

$s \in S$. The boundary of the billiard ∂B is formed by parts of hyper-spheres with centers in $\vec{v}_s$ and radii r_s .

The condition (3.42) is obeyed for the diagonalization (3.43) with

$$z^0 = U_A \sigma^A / \sqrt{|(U, U)|}, \quad (3.45)$$

where U is a time-like vector

$$(U, U) < 0, \quad (3.46)$$

and

$$(U, U^s) < 0, \quad (3.47)$$

for all $s \in S$. (Here the relation $(U, U^s) = -u_{s0}\sqrt{|(U, U)|}$ is used.) The inequalities (3.46) and (3.47) are satisfied identically if

$$U = kU^\Lambda, \quad k > 0, \quad (3.48)$$

see (2.30). This choice of U with $k = 1$ was done in [13].

Remark 1 Conditions (3.42) (or (3.47)) may be relaxed. In this case we obtain a more general definition of the billiard walls (e.g. for $u_{s0} < 0$ and $u_{s0} = 0$) described in [24].

Thus, we are led to the asymptotical relation for the function $\Psi_{0,L}(y^0, \vec{y})$

$$\left(-\frac{1}{2}\Delta[\bar{G}] + \delta V\right)\Psi_{0,L} = 0 \quad (3.49)$$

with the zero boundary condition $\Psi_{0,L}|_{\partial B} = 0$ imposed.

Due to (3.23) we get $\Delta[\bar{G}] = -(\partial_0)^2 + \Delta[h_L]$, where $\Delta[h_L] = \Delta_L$ is the Laplace–Beltrami operator corresponding to the Lobachevsky metric h_L .

By separating the variables,

$$\Psi_{0,L} = \Psi_0(y^0)\Psi_L(\vec{y}), \quad (3.50)$$

we obtain the following asymptotical relation (for $y^0 \rightarrow -\infty$):

$$\left(\left(\frac{\partial}{\partial y^0}\right)^2 + 2Ae^{-2y^0} + E - \frac{1}{4}(N_0 - 2)^2\right)\Psi_0 = 0, \quad (3.51)$$

where

$$\Delta_L \Psi_L = -E \Psi_L, \quad \Psi_L|_{\partial B} = 0. \quad (3.52)$$

We assume that the minus Laplace–Beltrami operator $(-\Delta_L)$ with the zero boundary conditions has a spectrum obeying the following inequality:

$$E \geq \frac{1}{4}(N_0 - 2)^2. \quad (3.53)$$

The examples of billiards obeying this restriction were considered in [25, 26] (see also the next section).

Here we restrict ourselves to the case of negative A -number

$$A < 0. \quad (3.54)$$

Solving Eq. (3.51) we get for $A < 0$ the following set of basis solutions:

$$\Psi_0 = \mathcal{B}_{i\omega}(\sqrt{2|A|}e^{-y^0}), \quad (3.55)$$

where $\mathcal{B}_{i\omega}(z) = I_{i\omega}(z)$, $K_{i\omega}(z)$ are the modified Bessel functions and

$$\omega = \sqrt{E - \frac{1}{4}(N_0 - 2)^2} \geq 0. \quad (3.56)$$

By using the asymptotical relations

$$I_\nu \sim \frac{e^z}{\sqrt{2\pi z}}, \quad K_\nu \sim \frac{e^{-z}}{\sqrt{2z}}, \quad (3.57)$$

for $z \rightarrow +\infty$, we find

$$\Psi_0 \sim C_\pm \exp\left(\pm\sqrt{2|A|}e^{-y^0} + \frac{1}{2}y^0\right) \quad (3.58)$$

for $y^0 \rightarrow -\infty$. Here $C_\pm$ are non-zero constants, “plus” corresponds to $\mathcal{B} = I$ and “minus” - to $\mathcal{B} = K$.

Now we evaluate the prefactor $e^{C(\sigma)}$ in (3.34), where

$$C(\sigma) = \frac{1}{2}(U(\sigma) - mf). \quad (3.59)$$

Here we denote

$$U(\sigma) = U_A \sigma^A = \sum_{s \in S} U_A^s \sigma^s, \quad U_A = \sum_{s \in S} U_A^s. \quad (3.60)$$

In what follows we use the vector $U = (U_A)$ as a time-like vector in the relation for z^0 in (3.45). Thus, we need to impose the restriction (3.46) $((U, U) < 0)$.

Using (3.20), (3.45), and $f = y^0$ we obtain

$$C(\sigma) = \frac{1}{2}(qz^0 - mf) = \frac{1}{2}\left(-qe^{-y^0}\frac{(1+\vec{y}^2)}{1-\vec{y}^2} - my^0\right), \quad (3.61)$$

where

$$q = \sqrt{-(U, U)} > 0. \quad (3.62)$$

Combining (3.34), (3.50), (3.58), and (3.61) we get

$$\Psi^f \sim C_\pm \exp\left(\theta^\pm(|\vec{y}|)e^{-y^0} - \frac{1}{2}(m-1)y^0\right)e^{iQ_s\Phi^s}\Psi_L(\vec{y}), \quad (3.63)$$

as $y^0 \rightarrow -\infty$ for any fixed $\vec{y} \in B$ and $C_\pm \neq 0$. Here we denote

$$\theta^\pm(|\vec{y}|) = -\frac{q}{2}\frac{(1+\vec{y}^2)}{(1-\vec{y}^2)} \pm \sqrt{-2A}, \quad (3.64)$$

where “plus” corresponds to the solution with $\mathcal{B} = I$ and “minus” to $\mathcal{B} = K$.

Relation (3.33) may be rewritten in the following form:

$$A = \frac{1}{8(N-1)}\left[(U, U) - (N-2)\sum_{s \in S}(U^s, U^s)\right], \quad (3.65)$$

where we have used the identity

$$(U, U) = \sum_{s, s' \in S}(U^s, U^{s'}) \quad (3.66)$$

following from the definition of U in (3.60). It should be noted that restrictions $(U, U) < 0$ and $(U^s, U^s) > 0, s \in S$, imply $A < 0$.

Now we study the asymptotical behavior of the wave function (3.34)

$$\Psi^f = e^{C(\sigma)} e^{iQ_s \Phi^s} \mathcal{B}_{i\omega} \left(\sqrt{2|A|} e^{-y^0} \right) \Psi_L(\vec{y}), \quad (3.67)$$

with $C(\sigma)$ from (3.61) and $(U, U) < 0, A < 0$.

A. Let $\mathcal{B} = K$. Then

$$\Psi^f \rightarrow 0 \quad (3.68)$$

as $y^0 \rightarrow -\infty$ for fixed $\vec{y} \in B$ and $\Phi^s \in \mathbb{R}, s \in S$.

This follows just from the asymptotic relation (3.63).

B. Now we consider the case $\mathcal{B} = I$.

B1. First we put

$$\frac{1}{2}q > \sqrt{2|A|}, \quad (3.69)$$

or, equivalently,

$$\sum_{s \in S} (U^s, U^s) < -(U, U). \quad (3.70)$$

We get

$$\Psi^f \rightarrow 0$$

as $y^0 \rightarrow -\infty$ for fixed $\vec{y} \in B$ and $\Phi^s \in \mathbb{R}, s \in S$. This also follows from (3.63). The equivalence of the conditions (3.69) and (3.70) could be readily verified using the relations (3.62) and (3.65).

B2. Let

$$\frac{1}{2}q = \sqrt{2|A|}, \quad (3.71)$$

or, equivalently,

$$\sum_{s \in S} (U^s, U^s) = -(U, U). \quad (3.72)$$

Then we also get

$$\Psi^f \rightarrow 0$$

as $y^0 \rightarrow -\infty$ for fixed $\vec{y} \in B \setminus \{\vec{0}\}$ and $\Phi^s \in \mathbb{R}, s \in S$. This also follows from (3.63).

B3. Now we consider the third case

$$\frac{1}{2}q < \sqrt{2|A|}, \quad (3.73)$$

or, equivalently,

$$\sum_{s \in S} (U^s, U^s) > -(U, U). \quad (3.74)$$

Let the point $\vec{0}$ belong to the billiard B (this is valid when relations (3.42) are satisfied) and $\Psi_L(\vec{0}) \neq 0$. Then there

exists $\varepsilon > 0$ such that for all $\vec{y}$ obeying $|\vec{y}| < \varepsilon$ and all $\Phi^s \in \mathbb{R}, s \in S$)

$$|\Psi^f| \rightarrow +\infty \quad (3.75)$$

as $y^0 \rightarrow -\infty$.

Indeed, since the billiard B is an open domain, $\vec{0} \in B$ and $\Psi_L(\vec{y})$ is continuous function there exists $\varepsilon > 0$ such that the ball $B_\varepsilon = \{\vec{y} | |\vec{y}| < \varepsilon\}$ belongs to B and $\Psi_L(\vec{y}) \neq 0$ for all $\vec{y} \in B_\varepsilon$. Due to (3.74) $\varepsilon > 0$ may be chosen such that

$$\frac{1}{2}q \frac{(1 + \vec{y}^2)}{1 - \vec{y}^2} < \sqrt{2|A|}, \quad (3.76)$$

for all $|\vec{y}| < \varepsilon$. Then relations (3.63), (3.76) and $\Psi_L(\vec{y}) \neq 0$ imply (3.75) for all $\vec{y} \in B_\varepsilon$.

Remark 2 It should be noted that solution (3.55) is similar to those which were found in quantum cosmological models with Λ -term, perfect fluid etc., see [60,61] and references therein. For these solutions we have $v = e^{qz^0}$ instead of e^{-y^0} , where v is the volume [60] or the “quasi-volume” [61] scale factor. Our restriction $A < 0$ corresponds to the restriction $\Lambda < 0$ for the solutions from [60].

4 Examples

Here we illustrate our approach by two examples of quantum billiards in dimensions $D = 11$ and $D = 10$. In what follows we use the notation $\Omega(n, k)$ for the set of all subsets of $\{1, \dots, n\}$, which contain k elements. Any element of $\Omega(n, k)$ has the form $I = \{i_1, \dots, i_k\}, 1 \leq i_1 < \dots < i_k \leq n$. The number of elements in $\Omega(n, k)$ is $C_n^k = \frac{n!}{k!(n-k)!}$.

In this section we deal with $(n + 1)$ -dimensional cosmological metric of Bianchi-I type,

$$g = -e^{2\gamma(u)} du \otimes du + \sum_{i=1}^n e^{2\phi^i(u)} dx^i \otimes dx^i, \quad (4.1)$$

where $u \in (u_-, u_+)$.

4.1 Nine-dimensional billiard in $D = 11$ model

Let us consider an 11-dimensional gravitational model with several 4-forms, which produce non-composite analogs of SM -brane solutions in $D = 11$ supergravity [62]. The action reads as follows:

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int_M d^{11}z \sqrt{|g|} \{R[g] + \mathcal{L}\} + S_{\text{YGH}}, \quad (4.2)$$

where first we put $\mathcal{L} = \mathcal{L}_e$, where

$$\mathcal{L}_e = -\frac{1}{4!} \sum_{I \in \Omega(10,3)} (F_{4,e}^I)_g^2. \quad (4.3)$$

Here $F_{4,e}^I$ is “electric” 4-form with the index $I \in \Omega(10, 3)$. The number of such forms is $C_{10}^3 = 120$.

The action (4.2) with $\mathcal{L}$ from (4.3) describes non-composite analogs of $SM2$ -brane solutions which are given by the metric (4.1) with $n = 10$ and

$$F_{4,e}^I = \mathcal{F}^{(a,e,I)}, \quad (4.4)$$

$I \in \Omega(10, 3)$, $a = (4, e, I)$, where electric monoms $\mathcal{F}^{(a,e,I)}$ are defined in (2.8).

Consider the non-trivial case when all charge densities of branes Q_s , $s \in S_e$, are non-zero. In the classical case we get a 9-dimensional billiard $B \in H^9$ with 120 “electric” walls [34]. This classical billiard coincides with the $9d$ billiard from [16,17]. B has a finite volume. It is a union of several identical “small” billiards which have finite volumes as they correspond to the Weyl chamber of the hyperbolic Kac–Moody algebra E_{10} [17].

Remark 3 In [34] we have used 120 form fields and a non-composite ansatz for branes to avoid the appearance of the set of 45 constraints which arise for composite solutions with diagonal metric [42,63]. These constraints are coming from the relations $T_{ij} = 0$, $1 \leq i < j \leq 10$, where T_{ij} are spatial components of the stress-energy tensor. We note that in [16,17] this problem was circumvented by considering non-diagonal metrics from the very beginning.

Let us calculate (U, U) , where $U = U_e = \sum_{s \in S_e} U^s$. We get $U_i = \sum_{I \in \Omega(10,3)} \delta_{iI}$, where δ_{iI} was defined in (2.19). Thus, U_i is the number of sets $I \in \Omega(10, 3)$ which contain i ($i = 1, \dots, 10$). It is obvious that $U_i = C_9^2 = 36$. Here $U = 36U^\Lambda$ (see 2.29) and hence we may use the z -variables from [10,13].

Then we get (see 2.23)

$$(U, U) = G^{ij} U_i U_j = \sum_{i,j=1}^{10} \left(\delta^{ij} - \frac{1}{9} \right) (36)^2 = -1440 < 0 \quad (4.5)$$

in agreement with our restriction (3.46). Since $N = 130$ ($m = 120$) and $(U^s, U^s) = 2$ we obtain from (3.65) the following value for the A -number [34]:

$$A = A_e(M2) = -\frac{1340}{43}. \quad (4.6)$$

In this case the inequality (3.70) is satisfied identically since [34]

$$240 = \sum_{s \in S_e} (U^s, U^s) < -(U, U) = 1440. \quad (4.7)$$

The minus Laplace–Beltrami operator $(-\Delta_L)$ on B with the zero boundary conditions imposed has a spectrum obeying restriction (3.53) with $N_0 = 10$ [26].

We get from the previous analysis the asymptotical vanishing of the wave function $\Psi^f \rightarrow 0$ as $y^0 \rightarrow -\infty$.

Now we consider the electromagnetic case, which mimics solutions with $SM2$ - and $SM5$ -branes.

We put in (4.2) $\mathcal{L} = \mathcal{L}_e + \mathcal{L}_m$, where

$$\mathcal{L}_m = -\frac{1}{4!} \sum_{J \in \Omega(10,6)} (F_{4,m}^J)_g^2. \quad (4.8)$$

Here $F_{4,m}^J$ is a “magnetic” 4-form with the index $J \in \Omega(10, 6)$. The number of such forms is $C_{10}^6 = 210$. We extend the cosmological electric ansatz by adding the following relations:

$$F_{4,m}^J = \mathcal{F}^{(a,m,J)}, \quad (4.9)$$

$J \in \Omega(10, 6)$, $a = (4, m, J)$, where $\mathcal{F}^{(a,m,J)}$ are defined in (2.9). For charge densities we put $Q_s \neq 0$, $s \in S$.

In the electromagnetic case we get the same 9-dimensional billiard $B \in H^9$ as in the electric case, since magnetic walls are hidden by electric ones. This could be readily verified using the billiard chamber

$$W = \left\{ \sigma | \hat{G}_{AB} \sigma^A \sigma^B < 0; U(\sigma) < 0; U^s(\sigma) < 0, s \in S \right\}$$

belonging to the lower light cone and the fact that any magnetic U -vector is the sum of two electric ones. The Lobachevsky space H^9 may be identified with the hypersurface $y^0 = 0$ in the lower light cone. Then the billiard B may be obtained just by the projection of W onto H^9 : $(y^0, \vec{y}) \mapsto \vec{y}$. Adding into our consideration a magnetic brane with $U_m = U_{1e} + U_{2e}$, where U_{1e} and U_{2e} correspond to electric branes, gives a new inequality in the definition of W : $U_m(\sigma) < 0$, which is satisfied identically due to the relations $U_{1e}(\sigma) < 0$ and $U_{2e}(\sigma) < 0$ from the definition of W in the electric case. Thus, the addition of any magnetic $SM5$ -brane does not change the electric billiard chamber nor the electric billiard.

A calculation similar to the electric case of $U = U_{em} = \sum_{s \in S} U^s$, $s \in S$, gives us $U_i = 36 + 126 = 162$, where now $126 = C_9^5$ is the number of sets $J \in \Omega(10, 6)$ which contain i ($i = 1, \dots, 10$).

We obtain

$$(U, U) = G^{ij} U_i U_j = -\frac{10}{9} (162)^2 = -29160 < 0. \quad (4.10)$$

Now we have $m = 330$, $N = 340$, and $(U^s, U^s) = 2$ for all s . We get from (3.65) the following value for the A -number:

$$A = A_{em}(M2, M5) = -\frac{12940}{113}. \quad (4.11)$$

In this case the inequality (3.70) is also obeyed

$$660 = \sum_{s \in S} (U^s, U^s) < -(U, U) = 29160. \quad (4.12)$$

The analysis carried out in the previous section implies the asymptotical vanishing of the wave function $\Psi^f \rightarrow 0$ as $y^0 \rightarrow -\infty$.

Thus, adding 210 magnetic $SM5$ -branes does not change the “electric” billiard B and the spectrum of the Laplace–Beltrami operator Δ_L on B (with zero boundary condition). At quantum level we get a quantitatively different behavior $\Psi^f \rightarrow 0$ as $y^0 \rightarrow -\infty$, since parameters of the solutions q and $\sqrt{2|A|}$ in electric and electromagnetic cases are different: $q_{em}/q_e = 9/2$ and $\sqrt{|A_{em}|}/\sqrt{|A_e|} \sim 1.9$. Hidden magnetic walls change the asymptotical behavior of Ψ^f (see 3.63) though at the classical level they could be neglected.

Remark 4 The wave function, corresponding to the harmonic gauge also vanishes, i.e. $\Psi = e^{-by^0}\Psi^f \rightarrow 0$ as $y^0 \rightarrow -\infty$, since the term $(-by^0)$ in the exponent is suppressed by the e^{-y^0} -term from (3.63).

4.2 Nine-dimensional billiard in $D = 10$ model

Now we consider a 10-dimensional gravitational model with one scalar field and several 4- and 3-forms. This model gives us non-composite analogs of space-like $D2$ -, $FS1$ -, $D4$ -, and $NS5$ -brane solutions in $D = 10$ IIA supergravity.

The action reads as follows:

$$S_{10} = \frac{1}{2\kappa_{10}^2} \int_M d^{11}z \sqrt{|g|} \{ R[g] - g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta + \mathcal{L} \} + S_{\text{YGH}}. \quad (4.13)$$

First we put $\mathcal{L} = \mathcal{L}_e$, where

$$\mathcal{L}_e = -\frac{1}{4!} e^{2\lambda_4 \varphi} \sum_{I_1 \in \Omega(9,3)} (F_{4,e}^{I_1})_g^2 - \frac{1}{3!} e^{2\lambda_3 \varphi} \sum_{I_2 \in \Omega(9,2)} (F_{3,e}^{I_2})_g^2. \quad (4.14)$$

Here $F_{4,e}^{I_1}$ is the “electric” 4-form, $I_1 \in \Omega(9,3)$, and $F_{3,e}^{I_2}$ is the “electric” 3-form, $I_2 \in \Omega(9,2)$. The number of 4-forms is $C_9^3 = 84$ and the number of 3-forms is $C_9^2 = 36$. In (4.14) $\lambda_4 = \frac{1}{2\sqrt{2}}$ and $\lambda_3 = -2\lambda_4$.

The action (4.13) with $\mathcal{L}$ from (4.14) describes non-composite $SD2$ -, $FS1$ -brane solutions which are given by the metric (4.1) with $n = 9$ and

$$F_{4,e}^{I_1} = \mathcal{F}^{(a_1,e,I_1)}, \quad F_{3,e}^{I_2} = \mathcal{F}^{(a_2,e,I_2)}, \quad (4.15)$$

with $I_1 \in \Omega(9,3)$, $a_1 = (4, e, I_1)$ and $I_2 \in \Omega(9,2)$, $a_2 = (3, e, I_2)$; see (2.8). We put $Q_s \neq 0$, $s \in S_e$.

In the classical case we get the same 9-dimensional billiard $B \in H^9$ with 120 “electric” walls as in the $SM2$ -brane case [15, 16].

Let us calculate (U, U) , where $U = U_e = \sum_{s \in S_e} U^s$. We get

$$U_i = \sum_{I_1 \in \Omega(9,3)} \delta_{iI_1} + \sum_{I_2 \in \Omega(9,2)} \delta_{iI_2} = C_8^2 + C_8^1 = 36. \quad (4.16)$$

The first term in the sum $C_8^2 = 28$ is the number of sets $I_1 \in \Omega(9,3)$ which contain i and the second term $C_8^1 = 8$ is the number of sets $I_2 \in \Omega(9,2)$ which contain i ($i = 1, \dots, 9$). Thus, $U_i = 36$ for all i . For the φ -component we get (see 2.18)

$$U_\varphi = -84\lambda_4 - 36\lambda_3 = -12\lambda_4. \quad (4.17)$$

Then we get the same value

$$(U, U) = G^{ij} U_i U_j + U_\varphi^2 = \sum_{i,j=1}^9 \left(\delta^{ij} - \frac{1}{8} \right) (36)^2 + \frac{(12)^2}{8} = -1440 < 0 \quad (4.18)$$

as in the $SM2$ -case.

Since $N = 130$ ($m = 120$) and $(U^s, U^s) = 2$, $s \in S_e$, we obtain from (3.65) the same value for the A -number as in the $SM2$ -case,

$$A = A_e(D2, FS1) = -\frac{1340}{43}, \quad (4.19)$$

and the asymptotical vanishing of the wave function $\Psi^f \rightarrow 0$ as $y^0 \rightarrow -\infty$.

Now we consider the electromagnetic case, which mimics solutions with $SD2$ -, $FS1$ -, $SD4$ - and $NS5$ -branes in $D = 10$ IIA supergravity.

We put in (4.13) $\mathcal{L} = \mathcal{L}_e + \mathcal{L}_m$, where

$$\mathcal{L}_m = -\frac{1}{4!} e^{2\lambda_4 \varphi} \sum_{J_1 \in \Omega(9,5)} (F_{4,m}^{J_1})_g^2 - \frac{1}{3!} e^{2\lambda_3 \varphi} \sum_{J_2 \in \Omega(9,6)} (F_{3,m}^{J_2})_g^2. \quad (4.20)$$

Here $F_{4,m}^{J_1}$, $J_1 \in \Omega(9,5)$, is the “magnetic” 4-form and $F_{3,m}^{J_2}$, $J_2 \in \Omega(9,6)$, is the “magnetic” 3-form.

The number of “magnetic” 4-forms is $C_9^5 = 126$, while the number of “magnetic” 3-forms is $C_9^6 = 84$. We extend the cosmological electric ansatz by adding the following relations:

$$F_{4,m}^{J_1} = \mathcal{F}^{(a_1,m,J_1)}, \quad F_{3,m}^{J_2} = \mathcal{F}^{(a_2,m,J_2)}, \quad (4.21)$$

$J_1 \in \Omega(9,5)$, $a_1 = (4, m, J_1)$, and $J_2 \in \Omega(9,6)$, $a_2 = (3, m, J_2)$, where $\mathcal{F}^{(a,m,J)}$ are defined in (2.9). For the charge densities we put $Q_s \neq 0$, $s \in S$.

In the electromagnetic case the 9-dimensional billiard is the same as in the pure electric case, i.e. $B = B_e \in H^9$,

since magnetic walls are hidden by electric ones. (This may be readily proved along a similar line to that was followed for M -branes in the previous subsection.)

The calculation of (U, U) in the electromagnetic case $U = U_{em} = \sum_{s \in S} U^s$ gives

$$U_i = 36 + C_8^4 + C_8^5 = 162. \quad (4.22)$$

Here $C_8^4 = 70$ is the number of sets $I_1 \in \Omega(9, 5)$ which contain i and $C_8^5 = 56$ is the number of sets $I_2 \in \Omega(9, 6)$ which contain i ($i = 1, \dots, 9$). Thus, $U_i = 162$ for all i . For the φ -component we get (see 2.18)

$$U_\varphi = -12\lambda_4 + 126\lambda_4 + 84\lambda_3 = -54\lambda_4. \quad (4.23)$$

We obtain

$$\begin{aligned} (U, U) &= G^{ij} U_i U_j + U_\varphi^2 \\ &= -\frac{9}{8}(162)^2 + \frac{1}{8}(54)^2 = -29160 < 0. \end{aligned} \quad (4.24)$$

Since $m = 330$, $N = 340$ and $(U^s, U^s) = 2$ for all s , we get from (3.65) the following value for the A -number:

$$A = A_{em}(D2, FS1, D4, NS5) = -\frac{12940}{113}. \quad (4.25)$$

Thus, we are led to the same values of the scalar product (U, U) and the A -number as for the model which mimics $SM2$ - and $SM5$ -branes. For the wave function we obtain the asymptotic vanishing $\Psi^f \rightarrow 0$ as $y^0 \rightarrow -\infty$.⁵

Remark 5 The coincidence of the A -numbers

$$A_{em}(D2, FS1, D4, NS5) = A_{em}(M2, M5)$$

is not surprising since there is a one-to-one correspondence between the sets of space-like branes: $(SM2, SM5)$ and $(SD2, SFS1, SD4, SNS5)$, which preserves the scalar products (U^s, U^s) . The number N is the same in both cases.

5 Conclusions

Here we have continued our approach from [34] by considering the quantum billiard for the cosmological-type model with n 1-dimensional factor spaces in the theory with several forms and l scalar fields. After adopting the electromagnetic composite brane ansatz with certain restrictions on brane intersections and parameters of the model we have deduced the Wheeler–DeWitt (WDW) equation for the model, written

in the conformally covariant form. It should be noted that in our previous paper [34] we were dealing with a gravitational model which contains fields of forms without scalar fields. In [34] only electric non-composite configurations of branes were considered. Thus, the generalization of the model from [34] is rather evident.

By imposing certain restrictions on the parameters of the model we have obtained the asymptotic solutions to the WDW equation, which are of the quantum billiard form since they are governed by the spectrum of the Laplace–Beltrami operator on the billiard with the zero boundary condition imposed. The billiard is a part of the $(N_0 - 1)$ -dimensional Lobachevsky space H^{N_0-1} , where $N_0 = n + l$.

Here we have presented two examples of quantum billiards: (a) the quantum $9d$ billiard for $11D$ gravitational model with 120 “electric” 4-forms and 210 “magnetic” 4-forms which mimics the quantum billiard with space-like $M2$ - and $M5$ -branes in $D = 11$ supergravity, (b) the quantum $9d$ billiard for $10D$ gravitational model with one scalar field, 84 “electric” 4-forms, 126 “magnetic” 4-forms, 36 “electric” 3-forms and 84 “magnetic” 3-forms, which mimics the quantum billiard with space-like $D2$ -, $D4$ -, $FS1$ - and $NS5$ -branes in $D = 10$ IIA supergravity.

In both cases we have shown the asymptotic vanishing of the basis wave functions $\Psi^f \rightarrow 0$, as $y^0 \rightarrow -\infty$, for any choice of the Bessel function $\mathcal{B} = K, I$. For $D = 11$ model this result may be interpreted as a quantum resolution of the singularity. It should be noted that in the approach of [25, 26] asymptotic (basis) solutions to WDW equation in the harmonic gauge are vanishing as $\rho = e^{-y^0} \rightarrow +\infty$.

In the examples presented above the magnetic walls change the asymptotical behavior of the wave function Ψ^f . Thus, hidden magnetic walls which do not contribute to the asymptotical behavior of the classical solutions for $y^0 \rightarrow -\infty$ should be taken into account in the quantum case. This is the first lesson from this paper. The second one is related to the use of the conformally covariant version of the WDW equation. Here we were able to develop the quantum billiard approach for the model with branes only for a special conformal choice of the parameter $a = (N - 2)/(8(N - 1))$ in the WDW equation, where $N = n + l + m$ and m is the number of branes. The study of the asymptotical behavior of the wave function (as $y^0 \rightarrow -\infty$) for the non-conformal choice of the parameter $a \neq (N - 2)/(8(N - 1))$ should be a subject of a separate publication.

It should be noted that in the two examples presented here we have considered non-composite branes while initially we had formulated the quantum billiard approach for the composite branes with rather severe restrictions on brane intersections. Unfortunately, these restrictions exclude the possibility of efficiently applying the formalism to cosmological models with diagonal metrics in $11D$ and $10D$ IIA supergravities (the relaxing of these restrictions will

⁵ Recently, analogous behavior of the wave function was obtained in [64] for $D = 4$ simple supergravity, when a tachyon case was considered.

lead to quadratic constraints on the brane charge densities Q_s [63]). In the classical case this obstacle was avoided in [16] by considering the ADM type approach for non-diagonal cosmological metrics and using the Iwasawa decomposition. In this case the Chern–Simons terms were irrelevant for the classical formation of the billiard walls [16]. But in the quantum case the consideration of the Chern–Simons contributions needs a separate investigation. This (and some other topics) may be a subject of future publications.

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