Escape from hypercube driven by multi-variate α -stable noises: role of independence

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Abstract. We explore properties of the escape kinetics from the d -dimensional hypercube driven by multi-variate α -stable noises. Using methods of stochastic dynamics we show complex dependence of the mean first passage time for the escape from the hypercube as a function of the hypercube dimension d . Finally, we show how the escape process can be used to quantify independence of components of multi-variate α -stable noises.

1 Introduction

In order to describe complicated interactions of a test particle with its environment one can use the concept of noise. Noise is a stochastic process which is used to approximate not fully known or very complicated interactions providing an effective description of complex realms. The idea of noise underlines stochastic (Langevin) dynamics. Systems described by the Langevin like equations are inherently stochastic. Therefore, a probabilistic framework for analysis of noise driven systems is provided by the theory of stochastic processes [1–4].

One of interesting and frequently studied properties of stochastic processes is the first passage time [5]. The first passage time quantifies the time which is needed to leave a given domain for the first time. In order to characterize this quantity one can study the first passage time distribution and its properties. In particular, the mean first passage time (MFPT) is considered as the most popular and important one. The mean first passage time has a very clear and intuitive physical interpretation. If one associates the value of the stochastic process with the particle position the MFPT represents the mean time which a particle needs to escape from a domain of motion for the first time. The MFPT is a quantifier which not only characterizes diffusive systems but also epidemiological, biological or ecological systems to name a few [5,6].

The study of noise induced effects [7–12] is often based on the analysis of the mean first passage time as a function of the noise parameters or external perturbations. In that context the effect of resonant activation [13–16] and the noise enhanced stability [17,18] are of the particular relevance. In the resonant activation combined ac-

tion of noise and barrier modulation process results in the optimal (fastest) escape kinetics as measured by the MFPT. On the contrary, in the noise-enhanced stability noise enhances life-time of unstable states as measured by the MFPT. Typically it is assumed that a particle moves in the 1D and the noise is of the Gaussian white type. Here, we relax the assumption that the noise is Gaussian. Instead of the white Gaussian noise, we consider more general (white) α -stable noises [19,20], which include the Gaussian white noise as a special limiting case.

In this paper we explore a d -dimensional escape problem of a random walker driven by a multi-variate α -stable (Lévy type) noise from a hypercube. Starting with known results for the 1D escape problem we study properties of Markovian escape processes in higher dimensions showing their non-trivial scaling. Finally, we discuss the problem of increments' independence of multi-variate α -stable motions in the context of the escape from the d -dimensional hypercube and order statistics.

2 Model and results

There is a significant difference in description of 1D and higher dimensional systems driven by α -stable noises. Therefore, we begin with the 1D escape problem (Sect. 2.1). Next, we focus on the escape from d ($d \geq 2$) dimensional hypercubes, which can be described in the unified way (Sect. 2.2). Finally, we use the survival probability in order to examine the problem of first escape in more details and to study the problem of independence of increments of multi-variate α -stable motions (Sect. 2.3).

2.1 Escape in 1D

In 1D, a motion of a free, overdamped particle driven by a symmetric α -stable Lévy type noise is described by the

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Langevin equation

$$\frac{dx}{dt} = \sigma \zeta_\alpha(t). \quad (1)$$

Equation (1) can be rewritten in the incremental form $dx = \sigma dL_\alpha(t)$, where $L_\alpha(t)$ is a symmetric Markovian α -stable motion [21]. Increments of $L_\alpha(t)$ are independent, stationary and distributed according to an 1D α -stable density [19,21]. $\zeta_\alpha(t)$ is a white α -stable noise which is a formal time derivative of a symmetric α -stable motion. The characteristic function of symmetric α -stable densities is [19,20]

$$\phi(k) = \mathbb{E}[e^{ikX}] = \exp[-\sigma^\alpha |k|^\alpha], \quad (2)$$

where $\alpha \in (0, 2]$ is the stability index, $\sigma > 0$ is the scale parameter. For $\alpha < 2$, symmetric α -stable densities have the power-law asymptotics of $|x|^{-(\alpha+1)}$ type resulting in divergence of fractional moments $\langle |x|^\nu \rangle$ of order $\nu > \alpha$.

Equation (1) is associated with the (space-fractional) Smoluchowski-Fokker-Planck equation [22–26]

$$\frac{\partial p(x, t|x_0, 0)}{\partial t} = \sigma^\alpha \frac{\partial^\alpha p(x, t|x_0, 0)}{\partial |x|^\alpha}, \quad (3)$$

where $\frac{\partial^\alpha f(x)}{\partial |x|^\alpha}$ is the Riesz-Weil fractional derivative defined by the Fourier transform, i.e. $\mathcal{F}[\frac{\partial^\alpha f(x)}{\partial |x|^\alpha}] = -|k|^\alpha \mathcal{F}[f(x)]$. For $\alpha = 2$, equation (2) is the characteristic function of the Gaussian distribution. Therefore, in the limit of $\alpha = 2$, the fractional Smoluchowski-Fokker-Planck equation (3) reduces to the normal (non-fractional) form, i.e. $\partial^\alpha / \partial |x|^\alpha \rightarrow \partial^2 / \partial x^2$.

The simplest setup to study a noise induced escape is a finite $[-L, L]$ interval restricted by two absorbing boundaries. For $\alpha < 2$, absorbing boundaries introduce non-local boundary conditions reflecting discontinuity of trajectories of α -stable motions. Consequently, for $\alpha < 2$, the fractional Smoluchowski-Fokker-Planck equation is associated with the non-local boundary conditions, i.e. $p(x, t|x_0, 0) = 0$ for $|x| \geq L$, see [27,28]. For $\alpha = 2$, the boundary conditions are defined at $|x| = L$ only. In such a case, for $x_0 = 0$, the solution of equation (3) is:

$$p(x, t|0, 0) = \sum_{n=1}^{\infty} \frac{1}{L} \sin\left[\frac{n\pi}{2}\right] \sin\left[\frac{n\pi(x+L)}{2L}\right] \times \exp\left[-\frac{n^2\pi^2\sigma^2}{4L^2}t\right], \quad (4)$$

see [29], equation (81). From equation (4) one can calculate the survival probability $S(t) = \int_{-L}^L p(x, t|0, 0)dx$

$$S(t) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin^3\left[\frac{n\pi}{2}\right] \exp\left[-\frac{n^2\pi^2\sigma^2}{4L^2}t\right], \quad (5)$$

which is the probability that at time t a random walker has not escaped from the $[-L, L]$ interval. From the survival probability the first passage time density $f(t)$ can be

calculated

$$f(t) = -\frac{dS}{dt} = \sum_{n=1}^{\infty} \frac{n\pi\sigma^2}{L^2} \sin^3\left[\frac{n\pi}{2}\right] \exp\left[-\frac{n^2\pi^2\sigma^2}{4L^2}t\right]. \quad (6)$$

The first passage time density has the exponential asymptotics determined by the lowest eigenvalue, see equation (6), and the mean first passage time $\mathcal{T} = \langle \tau \rangle$ from a bounded interval is finite

$$\mathcal{T} = \langle \tau \rangle = \int_0^\infty S(t)dt = \int_0^\infty f(t)tdt = \frac{L^2}{2\sigma^2}. \quad (7)$$

The mean first passage time (MFPT) is the average time which is needed to cross the (absorbing) boundary for the first time, i.e. to leave the domain of motion Ω . The exact value of the MFPT depends on the initial position. Here, it is assumed that initially a test particle is located in the center of the domain of motion (interval).

The mean first passage time $\mathcal{T} = \langle \tau \rangle$ can be also calculated from the backward Smoluchowski-Fokker-Planck equation [3,5], which for $\alpha = 2$ has the form

$$\sigma^2 \frac{\partial^2 \mathcal{T}(x)}{\partial x^2} = -1, \quad (8)$$

with the boundary condition $\mathcal{T}|_{\partial\Omega} = 0$ and the initial condition $x(0) \in \Omega$. The backward Smoluchowski-Fokker-Planck equation (8) can be easily generalized to the fractional case [28]

$$\sigma^\alpha \frac{\partial^\alpha \mathcal{T}(x)}{\partial |x|^\alpha} = -1. \quad (9)$$

For $\alpha < 2$, the generalization of equation (8) into equation (9) introduces non-local boundary conditions. For $\alpha < 2$, due to long jumps, the whole exterior of Ω is absorbing.

For the escape from the $[-L, L]$ interval restricted by two absorbing boundaries and $x(0) = 0$ the mean first passage time [28,30,31] is

$$\mathcal{T} = \langle \tau \rangle = \frac{L^\alpha}{\Gamma(1+\alpha)\sigma^\alpha}. \quad (10)$$

Equation (7) is a special, $\alpha = 2$, case of the more general equation (10). The MFPT can be a non-monotonous function of the stability index α , see equation (10) and [32].

2.2 Escape in ≥ 2 D

The multi-variate Langevin equation has the form

$$\frac{d\mathbf{x}}{dt} = \sigma \zeta_\alpha(t), \quad (11)$$

where $\zeta_\alpha(t)$ is the multi-variate α -stable Lévy type noise which is the formal time derivative of the symmetric, multi-variate α -stable motion $\mathbf{L}_\alpha(t)$. Equation (11) can be rewritten in the incremental form $d\mathbf{x} = \sigma d\mathbf{L}_\alpha(t)$,

$$\phi(\mathbf{k}) = \begin{cases} \exp \left\{ -\int_{S_d} |\langle \mathbf{k}, \mathbf{s} \rangle|^\alpha \left[1 - i \operatorname{sign}(\langle \mathbf{k}, \mathbf{s} \rangle) \tan \frac{\pi\alpha}{2} \right] \Lambda(d\mathbf{s}) + i \langle \mathbf{k}, \boldsymbol{\mu}^0 \rangle \right\} & \text{for } \alpha \neq 1, \\ \exp \left\{ -\int_{S_d} |\langle \mathbf{k}, \mathbf{s} \rangle| \left[1 + i \frac{2}{\pi} \operatorname{sign}(\langle \mathbf{k}, \mathbf{s} \rangle) \ln(\langle \mathbf{k}, \mathbf{s} \rangle) \right] \Lambda(d\mathbf{s}) + i \langle \mathbf{k}, \boldsymbol{\mu}^0 \rangle \right\} & \text{for } \alpha = 1, \end{cases} \quad (12)$$

where $d\mathbf{L}_\alpha(t)$ represents increments of the multi-variate α -stable motion [19]. Analogously, like in 1D, the multi-variate α -stable motion $\mathbf{L}_\alpha(t)$ is a Markovian process with increments that are stationary, independent and distributed according to a multi-variate α -stable density. General d -variate α -stable densities have the characteristic function

see equation (12) above

where $\langle \mathbf{k}, \mathbf{s} \rangle$ is the scalar product, $\Lambda(\cdot)$ represents the spectral measure on the unit sphere S_d of $\mathbb{R}^d$ and $\boldsymbol{\mu}^0$ is a vector in $\mathbb{R}^d$, see [19]. The spectral measure $\Lambda(\cdot)$ replaces skewness and scale parameters (β and σ) which characterize general 1D α -stable densities. A multi-variate α -stable density is symmetric if the spectral measure is symmetric, see references [19,33,34]. The multi-variate α -stable motion $\mathbf{L}_\alpha(t)$ can be generated by general methods described in reference [19,35,36].

The Langevin equation (11) is a simple generalization of equation (1). Nevertheless, the description of 2D escape kinetics significantly differs from 1D. Bi-variate α -stable noises are defined by the spectral measure $\Lambda(\cdot)$ which determines the escape kinetics along with the associated fractional diffusion equation [37–39]. Here, we use two most natural escape scenarios, i.e. we compare escape from the hypercube driven by isotropic multi-variate α -stable noises and Cartesian α -stable noises. In the force free case, these noises result in α -stable motions with uniform continuous spectral measures (spherical Lévy flights) and uniform discrete spectral measures concentrated on the intersections of the unit hypersphere with axes (Cartesian Lévy flights) [40].

The uniform continuous spectral measure $\Lambda(\cdot)$ corresponds to the situation when α -stable densities are spherically symmetric (isotropic). In such a case, the Langevin equation (11) is associated with the following fractional diffusion equation [24,25,37]

$$\frac{\partial p(\mathbf{x}, t | \mathbf{x}_0, t)}{\partial t} = -\sigma^\alpha (-\Delta)^{\alpha/2} p(\mathbf{x}, t | \mathbf{x}_0, t), \quad (13)$$

where $-(-\Delta)^{\alpha/2}$ is the fractional Riesz-Weil derivative (Laplacian) defined by the Fourier transform [37]

$$\mathcal{F} \left[-(-\Delta)^{\alpha/2} p(\mathbf{x}, t | \mathbf{x}_0, t) \right] = -|\mathbf{k}|^\alpha \mathcal{F} [p(\mathbf{x}, t | \mathbf{x}_0, t)]. \quad (14)$$

In 2D, the Cartesian Lévy flights, i.e. α -stable motions described by discrete uniform spectral measures located on the intersections of a unit circle with axes, are described by the following fractional diffusion equation

$$\frac{\partial p(\mathbf{x}, t | \mathbf{x}_0, t)}{\partial t} = \sigma^\alpha \left[\frac{\partial^\alpha}{\partial |x|^\alpha} + \frac{\partial^\alpha}{\partial |y|^\alpha} \right] p(\mathbf{x}, t | \mathbf{x}_0, 0), \quad (15)$$

which is a 2D analog of equation (3). Equation (15) can be easily generalized into higher dimensions.

We use the stochastic dynamics to investigate properties of the mean first passage time from the hypercube (of the dimension d , i.e. $[-L, L]^d$) with absorbing boundaries. If the trajectory leaves the hypercube the particle is removed and the first passage time $\tau = t$ is recorded. Such an approach guarantees proper implementation of boundary conditions because the whole exterior of the hypercube is absorbing. Initially a random walker is located in the center of the hypercube, i.e. $\mathbf{x}(0) = \mathbf{0}$. The first passage time τ is estimated as:

$$\tau = \min\{t > 0 : \mathbf{x}(0) = \mathbf{0} \text{ and } \mathbf{x}(t) \notin \Omega\}, \quad (16)$$

where Ω is the domain of motion. The mean first passage time $\langle \tau \rangle$ is the average of first passage times τ .

In d dimensions, for $\alpha = 2$ with the correlation matrix proportional to the identity matrix, the solution of the diffusion equation

$$\frac{\partial p(\mathbf{x}, t | \mathbf{x}_0, 0)}{\partial t} = \nabla^2 p(\mathbf{x}, t | \mathbf{x}_0, 0) \quad (17)$$

can be constructed using Fourier series, see [29], equation (81), and for $\mathbf{x}_0 = \mathbf{0}$, it reads

$$\begin{aligned} p(\mathbf{x}, t | \mathbf{0}, 0) = & \sum_{\{i_1, \dots, i_d\}=1}^{\infty} \frac{1}{L^d} \sin \left[\frac{i_1 \pi}{2} \right] \sin \left[\frac{i_1 \pi (x_1 + L)}{2L} \right] \\ & \times \dots \times \sin \left[\frac{i_d \pi}{2} \right] \sin \left[\frac{i_d \pi (x_d + L)}{2L} \right] \\ & \times \exp \left[-\frac{\pi^2 \sigma^2}{4L^2} (i_1^2 + \dots + i_d^2) t \right]. \end{aligned} \quad (18)$$

From equation (18) one gets the survival probability $S(t) = \int_{\Omega} p(\mathbf{x}, t | \mathbf{0}, 0) d\mathbf{x}$

$$\begin{aligned} S(t) = & \frac{4^d}{\pi^d} \sum_{\{i_1, \dots, i_d\}=1}^{\infty} \frac{\sin^3 [i_1 \pi / 2]}{i_1} \dots \frac{\sin^3 [i_d \pi / 2]}{i_d} \\ & \times \exp \left[-\frac{\pi^2 \sigma^2}{4L^2} (i_1^2 + \dots + i_d^2) t \right]. \end{aligned} \quad (19)$$

Finally, the formula for the mean first passage time is:

$$\begin{aligned} \mathcal{T} = \langle \tau \rangle = & \int_0^\infty S(t) dt \\ = & \frac{4^{d+1} L^2}{\pi^{2+d} \sigma^2} \sum_{\{i_1, \dots, i_d\}=1}^{\infty} \frac{\sin^3 [i_1 \pi / 2]}{i_1} \times \dots \\ & \times \frac{\sin^3 [i_d \pi / 2]}{i_d} \frac{1}{i_1^2 + \dots + i_d^2}. \end{aligned} \quad (20)$$

Equation (20) can be approximated by the combinatorial methods. The dominating contribution comes from small values of indices, e.g. when they are equal to $\{1, 3, 5, 7, 9\}$

Fig. 1. The mean first passage time $\langle \tau \rangle$ for escape from the d -dimensional hypercube for $\alpha = 2$. Points represent results of numerical simulations while solid lines present theoretical values given by equation (21). Error bars are smaller than the symbol size.

and they are repeated $d-k-l-m-n$, k , l , m and n times respectively, consequently

$$\begin{aligned} \mathcal{T} \approx & \frac{4^{d+1} L^2}{\pi^{2+d} \sigma^2} \sum_{\{k,l,m,n\}=0}^{k+l+m+n \leq d} \frac{d!}{(d-k-l-m-n)! k! l! m! n!} \\ & \times \frac{(-1)^{k+m}}{3^k \times 5^l \times 7^m \times 9^n} \\ & \times \frac{1}{(d-k-l-m-n) + 3^2 k + 5^2 l + 7^2 m + 9^2 n}. \end{aligned} \quad (21)$$

The combinatorial factor indicates the fact that summed elements do not depend on the order of indices but on their values. In equation (21), the second term is due to the product of $\sin[i\pi/2]/i$ terms, while the last term is $(i_1^2 + \dots + i_d^2)^{-1}$. The mean first passage time is proportional to L^2/σ^2 for all dimensions d . For $d = 1$, the standard formula is recovered, see equation (7). For the hypercube with $\alpha = 2$, the MFPT calculated according to equation (21) nicely agrees with simulations for $d < 30$, see Figure 1. Formula (21) provides a good, fast and controllable estimation of the MFPT, which significantly outperforms equation (20). Theoretical values of the MFPT are depicted by solid lines which have the interpretation for integer d only and the rest of curves is drawn to guide the eye only. In analogy to the Gaussian case, for $\alpha < 2$, the MFPT is proportional to L^α/σ^α . This particular scaling of the MFPT implies from the dimensional analysis.

Figures 2 and 3 present the mean first passage time for the escape from the d -dimensional hypercube as a function of the dimension d . The additional Figure 4 presents the rescaled mean first passage times, i.e. $\langle \tau \rangle/(L/\sigma)^\alpha$. Top panels of Figures 2–4 show results for spherical Lévy flights, while bottom panels for Cartesian Lévy flights. Various curves correspond to different values of the stability index α . Noise induced escapes from the hypercube are slower than escapes from the hypersphere (of the same

Fig. 2. The mean first passage times $\langle \tau \rangle$ for the escape from the d -dimensional hypercube for spherical (top panel) and Cartesian (bottom panel) Lévy flights. The hypercube's edge is $2L = 10$. Various curves correspond to different values of the stability index α . Error bars are smaller than the symbol size. Lines are drawn to guide the eye only.

dimension d with $L = R$) [32] because the average distance to the absorbing boundary is larger for the hypercube than for the hypersphere. For the hypersphere the MFPT decreases like a power-law with increasing value of the dimension d both for Cartesian and spherical Lévy flights, see [32]. For the hypercube, the dependence is not of the power-law type and follow more complex pattern. Figures 2–4 are drawn in the log-log scale, consequently the mean first passage time scaling of $\mathcal{T} \propto d^{\kappa(\alpha)}$ type would be manifested by straight lines, which is not the case. Nevertheless, for Cartesian Lévy flights and small values of the stability index α apparent power-law decay of the MFPT is visible, see bottom panel of Figures 2–4. The origin of the power-law dependence of the MFPT for small values of the stability index α can be explained with the help of the survival probability $S(t)$, which is the probability that at time t a particle is still in the domain of motion, see the next subsection.

2.3 Survival probability

Examination of Figure 5 and reference [41] provides explanation of the mean first passage time scaling on the

Fig. 3. The same like in Figure 2 for $2L = 20$.

dimension d for Cartesian Lévy flights with small values of the stability index α . Figure 5 presents survival probabilities for spherical (left column) and Cartesian (right column) Lévy flights. For $\alpha = 2$ (bottom panel), spherical and Cartesian Lévy flights are equivalent and correspond to the 2D Brownian motion. Therefore, left and right figures in the bottom panel are the same. The survival probabilities have exponential asymptotics determined by the smallest eigenvalue of the fractional Laplacian. Nevertheless, for small t survival probabilities significantly differ from the exponential decay. With the decreasing value of the stability index α the exponential approximation to the survival probability becomes better. Finally, in 1D for low α , the survival probability can be approximated (not only asymptotically) by $\exp[-\lambda_1 t]$, where λ_1 is the smallest eigenvalue [41]. For Cartesian Lévy flights, due to independence of components, in d dimensions the full survival probability is a product of the d 1D survival probabilities. Consequently, the full survival probability can be approximated by $\exp[-d\lambda_1 t]$ leading to the following scaling of the mean first passage time

$$\langle \tau \rangle \propto \frac{1}{\lambda_1} \times \frac{1}{d}. \quad (22)$$

The approximation (22) works well when the decay of the survival probability is fast, i.e. when $\alpha \approx 0$ and the

Fig. 4. The rescaled mean first passage times $\langle \tau \rangle / (L/\sigma)^\alpha$ for the escape from the d -dimensional hypercube for spherical (top panel) and Cartesian (bottom panel) Lévy flights. Various curves correspond to different values of the stability index α . Error bars are smaller than the symbol size. Lines are drawn to guide the eye only.

effective (smallest) eigenvalue is large (see [41], Fig. 3). Indeed, such a scaling is visible for Cartesian Lévy flights with $\alpha < 1$, see the bottom panels of Figures 2–4. For larger values of the stability index α , the survival probability cannot be approximated by a single eigenfunction, consequently, the $1/d$ scaling is not visible for Cartesian Lévy flights with larger values of α . For spherical Lévy flights, the situation is more complex because components of spherical Lévy flights are no longer independent. The lack of independence of noise components is responsible for slower escape kinetics in comparison to Cartesian Lévy flights. Therefore, for spherical Lévy flights, the exponential asymptotics of the survival probability is characterized by a smaller exponent than for Cartesian Lévy flights, compare left and right columns of Figure 5. The exponential asymptotics of the survival probability is the general property of Markovian escape problems.

In general, except discrete spectral measures concentrated on the intersections of axes with the unit hypersphere, components of multi-variate α -stable random variables are dependent [19]. For $\alpha < 2$, the standard

Fig. 5. Survival probabilities $S(t)$ for various values of the stability index α : $\alpha = 1.0$ (top row), $\alpha = 1.5$ (middle row) and $\alpha = 2$ (bottom row). Different columns present results for spherical (left column) and Cartesian (right column) Lévy flights. Solid lines in the bottom row represent the first term of equation (19). Various curves correspond to different values of the dimension d . The hypercube edge is $2L = 10$.

correlation cannot be used because it diverges. Dependence of components can be characterized by covariation or codifference [19,42]. For $\alpha = 2$, the dependence can be characterized by the covariance matrix. In the current studies, due to symmetry, the covariation matrix is proportional to the identity matrix resulting in the independence of components. For $\alpha < 2$, the components of α -stable random variables are independent if and only if the spectral measure is discrete and concentrated on the intersections of axes with the unit hypersphere [19]. Such a very special spectral measures produce already mentioned Cartesian Lévy flights [40] for which independent jumps are generated along axes.

Escape from the hypercube can be used to verify if increments of the $\mathbf{L}_\alpha(t)$ process along various directions are independent and identically distributed. If axes are perpendicular to the walls of the hypercube and jumps along axes are independent the escape from the hypercube is equivalent to the first order statistics of the d $L_\alpha(t)$ processes. More precisely, in d dimensions, if all components of $\mathbf{L}_\alpha(t)$ are independent, one can calculate d first passage times $\tau_1, \dots, \tau_d$, where τ_i represents the first exit time from the interval $[-L, L]$ of the i th component of $\mathbf{L}_\alpha(t)$. Since the hypercube is the Cartesian product of d $[-L, L]$ intervals, i.e. $[-L, L] \times \dots \times [-L, L]$, and components of $\mathbf{L}_\alpha(t)$ are independent, the first exit from the hypercube

takes place for

$$\tau_{(1)} = \min\{\tau_1, \dots, \tau_d\}, \quad (23)$$

which is the minimal first exit time, i.e. the first order statistics [43,44]. For Cartesian Lévy flights, escape events are independent and the cumulative distribution $\mathcal{F}_{(1)}(t)$ of the first order statistics, see [43,44], is:

$$\mathcal{F}_{(1)}(t) = 1 - \prod_{i=1}^d [1 - \mathcal{F}_i(t)], \quad (24)$$

where $\mathcal{F}_i(t)$ is the cumulative density associated with τ_i , i.e. escapes of the i -th component of $\mathbf{L}_\alpha(t)$. For the survival probability, one gets

$$S_d^{\min}(t) = \prod_{i=1}^d S_i(t), \quad (25)$$

because $S_i(t) = 1 - \mathcal{F}_i(t)$. $S_d^{\min}(t)$ is not only the survival probability of the first order statistics but it is also the survival probability $S(t)$ for the d -dimensional hypercube. The factorization of $S(t)$ is the immediate consequence of the fact that for Cartesian Lévy flights escapes of components of $\mathbf{L}_\alpha(t)$ from intervals $[-L, L]$ are independent. Additionally, if jumps' components are identically distributed, the survival probability is:

$$S(t) = S_d^{\min}(t) = [S_1(t)]^d, \quad (26)$$

where $S_1(t)$ is the survival probability for the escape from the $[-L, L]$ interval. Consequently, if components of $\mathbf{L}_\alpha(t)$ are independent, identically distributed the $[S(t)]^{1/d}$ curves with various d should coincide. Indeed, that is the case for spherical Lévy flights with $\alpha = 2$, while for $\alpha < 2$ they differ, see left column of Figure 6. For $\alpha = 2$, if the covariance matrix is proportional to the identity matrix, the $\mathbf{L}_2(t)$ process has independent components, which is not necessary the case for $\alpha < 2$. The disagreement of $[S(t)]^{1/d}$ curves for $\alpha < 2$, see the left column of Figure 6, is due to the fact that $\mathbf{L}_\alpha(t)$ represents here the α -stable random vector with the uniform continuous spectral measure. In such a case, components of the vector are no longer independent, see [19], and the escape from the hypercube cannot be seen as the escape from the Cartesian product of d $[-L, L]$ intervals. On the contrary to spherical Lévy flights, Cartesian Lévy flights have independent components. Moreover, if the spectral measure is symmetric, discrete and concentrated on the intersections of axes with the unit hypersphere components of Cartesian Lévy flights are not only independent but also identically distributed. In such a case, the property $[S(t)]^{1/d} = S_1(t)$ perfectly holds, see the right column of Figure 6, which gives a possibility to check if increments are independent and identically distributed.

3 Summary and conclusions

The multi-variate α -stable noises are characterized by the spectral measure $\Lambda(\cdot)$ which is responsible for statistical properties of α -stable motions. Typically, one assumes

Fig. 6. The rescaled survival probabilities $[S(t)]^{1/d}$ for various values of the stability index α : $\alpha = 1.0$ (top row), $\alpha = 1.5$ (middle row) and $\alpha = 2$ (bottom row). Different columns present results for spherical (left column) and Cartesian (right column) Lévy flights. Various lines correspond to different values of the dimension d . The hypercube edge is $2L = 10$.

that spectral measure is continuous and uniform or discrete and uniform concentrated on the intersections of the unit hypersphere with axes. Such common choices correspond to spherical or Cartesian Lévy flights. The main difference between these two scenarios is manifested by displacements of α -stable motions which are isotropic (spherical Lévy flight) or anisotropic with the preference to be along axes (Cartesian Lévy flights). Moreover, components of Cartesian Lévy flights are independent, while components of spherical Lévy flights are in general not independent, except for the $\alpha = 2$ case, i.e. 2D Brownian motions.

There is no analytic formula for the mean first passage time from the d -dimensional hypercubes except limiting $\alpha = 2$ (Gaussian) case. The formula for $\alpha = 2$ is very complicated although it can be very well approximated by combinatorial methods. In the non-Gaussian regime, i.e. for $\alpha < 2$, the mean first passage time can be estimated numerically. Properties of escape processes driven by multi-variate α -stable noises are determined by the choice of the spectral measure. Cartesian and spherical Lévy flights result in different scaling of the mean first passage time with the increasing dimension d . For Cartesian Lévy flights with small value of the stability index α , the mean first passage time attains power-law dependence on the hypercube dimension d , i.e. $\langle \tau \rangle \propto 1/d$. At the same time, for spherical Lévy flights, the dependence of the MFPT on d follow more complex pattern. Finally,

the escape process from the d -dimensional hypercube can be used to verify if components of multi-variate α -stable motions are independent and identically distributed.

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