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On the Existence of Global Solutions for the Vlasov–Poisson System in a Half-space and Plasma Confinement

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Abstract—We consider the Vlasov–Poisson system with external magnetic field in a half-space with the Neumann boundary condition for the electric potential and specular reflection on a boundary. For arbitrary compactly supported initial density distribution functions, we obtain sufficient conditions for external magnetic field, which provide global existence of density distribution functions with compact supports lying at some distance from a boundary.

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1. INTRODUCTION

For the first time, the Vlasov equations were considered in [1] These equations describe many important physical processes: kinetics of high temperature plasma, Landau damping effect, distribution of gravitating particles and etc. The relevance of investigation to mixed problems for the Vlasov–Poisson system is closely connected with the creation of controlled thermonuclear fusion. If sufficiently large number of charged particles reach the wall of the fusion reactor, it can lead to a destruction of the wall. In order to confine charged particles at some distance from the reactor wall an external magnetic field is used [2]. This means that a solution of mixed problem for the Vlasov–Poisson system has a support lying at some distance from the boundary of a domain because of the influence of an external magnetic field. Theorem 5.2 holds the main result of this paper. This theorem guarantees an existence of classical solution to a mixed problem for the Vlasov–Poisson system with external magnetic field in the half-space $\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : x_1 > 0\}$ with Neumann boundary condition for the electric potential $\varphi(x,t)$, $x \in \overline{\mathbb{R}^3_+}$, $0 \leqslant t \leqslant T$ on the boundary $\partial\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : x_1 = 0\}$ and the specular reflection on $\partial\mathbb{R}^3_+$ such that the supports of the density distribution functions $f^\beta(x,v,t)$, $x \in \overline{\mathbb{R}^3_+}$, $v \in \mathbb{R}^3$, $0 \leqslant t \leqslant T$, $\beta \pm 1$, with respect to the space variables x belong to the half-space $\mathbb{R}^3_\delta := \{x \in \mathbb{R}^3 : x_1 > \delta\}$, $\delta > 0$.

Classical solutions to the Cauchy problem for the Vlasov–Poisson system were studied in [3–7] and others. The so-called "velocity lemma" played the important role in these investigations. For the first time, this result was formulated by J. Batt, see [3], and was proved there for the spherically symmetrical initial data. For arbitrary initial data this lemma was proved independently in [4, 5]. From [3–5] it follows the theorem on the existence of global classical solution to the Cauchy problem for the Vlasov–Poisson system with arbitrary smooth initial density distribution functions with compact supports.

The mixed problems for the Vlasov–Poisson system in a half-space were studied in [8, 9] and in convex bounded domain in [10].

For the first time, conditions for existence of classical solutions to mixed problems for the Vlasov–Poisson system with external magnetic field providing confinement of two-components plasma in the cases of half-space, infinite cylinder ("mirror trap") and torus ("tokamak") were obtained in [11–17]. We also note the paper [18].

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Section 2 is devoted to statement of problem and notation. In Section 3, we formulate the auxiliary results concerning a priori estimates of solutions to the Poisson equation in $\mathbb{R}^3$ and in $\mathbb{R}^3_+$. In Section 4, we prove the existence of global classical solution to a mixed problem for the Vlasov–Poisson system with external magnetic field in a half-space. This result is a generalization of Theorem 6.1 from [8] to the case of the Vlasov–Poisson system with external magnetic field. Similarly to [8] we assume that initially plasma is neutral, i.e.,

$$\int\limits_{\mathbb{R}^3_+} \rho_0(x)dx = 0,$$

where

$$\rho_0(x) = \int\limits_{\mathbb{R}^3} \sum_{\beta=\pm 1} f^\beta(x, v, 0)dv.$$

Accordingly to physical definition of plasma [2] this assumption is natural. Section 5 is devoted to the proof of existence of global classical solutions with supports of density distribution functions (with respect to space variables x) on the half-space $\mathbb{R}^3_\delta$.

2. STATEMENT OF PROBLEM. NOTATION

We consider the Vlasov–Poisson system

$$-\Delta\varphi(x,t) = 4\pi e \int\limits_{\mathbb{R}^3} \sum_{\beta} \beta f^\beta(x, v, t)\, dv, \quad x \in \mathbb{R}^3_+, \quad 0 < t < T, \quad \beta = \pm 1, \tag{2.1}$$

$$\frac{\partial f^\beta}{\partial t} + \Big(v, \nabla_x f^\beta\Big) + \frac{\beta e}{m_\beta} \left(-\nabla_x \varphi + \frac{1}{c}\,[v, B], \nabla_v f^\beta\right) = 0,$$
$$x \in \mathbb{R}^3_+, \quad v \in \mathbb{R}^3, \quad 0 < t < T, \quad \beta = \pm 1, \tag{2.2}$$

with respect to unknown functions $\varphi(x,t)$ and $f^\beta(x, v, t)$, $\beta = \pm 1$. Here $\varphi = \varphi(x,t)$ is a potential of self-consistent electric field; $f^\beta = f^\beta(x, v, t)$ is a density distribution function of positively charged ions, if $\beta = +1$, and negatively charged electrons, if $\beta = -1$, at the point $x = (x_1, x_2, x_3)$ with velocity $v = (v_1, v_2, v_3)$ at the moment t; ∇_x and ∇_v are gradients with respect to x and v, respectively; m_{+1} and m_{-1} are the masses of ion and electron, respectively; e is the charge of electron; c is the velocity of light; $B = B(x)$ is the induction of external magnetic field; $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^3$; $[\cdot,\cdot]$ is the vector product in $\mathbb{R}^3$.

Let the following initial conditions hold

$$f^\beta(x, v, t)\big|_{t=0} = f^\beta_0(x, v), \quad x \in \overline{\mathbb{R}^3_+}, \quad v \in \mathbb{R}^3, \quad \beta = \pm 1, \tag{2.3}$$

and let the specular reflection boundary conditions on the boundary hold

$$f^\beta(0, x', v_1, v', t) = f^\beta(0, x', -v_1, v', t), \quad x' \in \mathbb{R}^2, \quad v \in \mathbb{R}^3, \quad 0 < t < T, \quad \beta = \pm 1, \tag{2.4}$$

where $x' = (x_2, x_3)$, $v' = (v_2, v_3)$.

We also assume that the potential φ satisfies the Neumann boundary condition

$$\frac{\partial \varphi(x,t)}{\partial x_1}\Big|_{x_1=0} = 0, \quad x' \in \mathbb{R}^2, \quad 0 \leqslant t \leqslant T, \tag{2.5}$$

and the decreasing condition at infinity

$$\lim_{|x|\to\infty} \varphi(x,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{2.6}$$

Denote by $C^s(\mathbb{R}^n)$, $s \geqslant 0$, $n \in \mathbb{N}$, the Hölder space of continuous functions on $\mathbb{R}^n$ having all continuous derivatives on $\mathbb{R}^n$ up to the kth order, $k = [s]$, with the finite norm

$$||u||_s = \max_{|\alpha|\leqslant k} \sup_x |D^\alpha u(x)|, \quad \text{if} \quad s = k \in \mathbb{Z}, \quad 0 \leqslant k,$$

$$||u||_s = ||u||_k + |u|_\sigma, \quad \text{if} \quad s = k + \sigma, \quad 0 \leqslant k \in \mathbb{Z}, \quad 0 < \sigma < 1, \tag{2.7}$$

where

$$|u|_\sigma = \max_{|\alpha|=k} \sup_{x \neq y} |x-y|^{-\sigma} |D^\alpha u(x) - D^\alpha u(y)|,$$

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = \alpha_1 + \dots + \alpha_n. \tag{2.8}$$

Similarly, we denote by $C^s\left(\overline{\mathbb{R}^n_+}\right)$, $s \geqslant 0$, $n \in \mathbb{N}$, the Hölder space of continuous on $\overline{\mathbb{R}^n_+}$ functions $u(x)$ having all continuous derivatives $D^\alpha u(x)$, $|\alpha| \leqslant k$, on $\mathbb{R}^n_+$ admitting continuous bounded extensions on $\overline{\mathbb{R}^n_+}$ with the finite norm given by (2.7) and (2.8).

Let $C(\mathbb{R}^n) = C^0(\mathbb{R}^n)$, and let $C(\overline{\mathbb{R}^n_+}) = C^0(\overline{\mathbb{R}^n_+})$.

As above, we can introduce the space $C^1(\overline{\mathbb{R}^3_+} \times \mathbb{R}^3 \times [0,T])$ consisting of continuous bounded functions on $\overline{\mathbb{R}^3_+} \times \mathbb{R}^3 \times [0,T]$ having continuous bounded derivatives of the first order on $\mathbb{R}^3_+ \times \mathbb{R}^3 \times [0,T]$ and their continuous bounded extensions on $\overline{\mathbb{R}^3_+} \times \mathbb{R}^3 \times [0,T]$.

Let $\mathring{C}^k(\mathbb{R}^n)$, $k, n \in \mathbb{N}$, be a set of k times continuously differentiable functions on $\mathbb{R}^n$ with compact supports.

We introduce the Banach space $C([0,T], C^s(\overline{\mathbb{R}^3_+}))$, $s > 0$, consisting of continuous functions $[0,T] \ni t \mapsto \varphi(\cdot, t) \in C^s(\overline{\mathbb{R}^3_+})$ with the norm

$$||\varphi||_{s,T} = \sup_{0 \leqslant t \leqslant T} ||\varphi(\cdot, t)||_s. \tag{2.9}$$

Similarly, we can define the space $C([0,T], C^s_\Omega(\overline{\mathbb{R}^3_+}))$, where $C^s_\Omega(\overline{\mathbb{R}^3_+}) = \{w \in C^s(\overline{\mathbb{R}^3_+}) : supp\ w \subset \overline{\Omega}\}$, Ω is a bounded domain in $\mathbb{R}^3_+$, $\overline{\Omega} \subset \mathbb{R}^3_+$.

Let $B_\rho(x^0) = \{x \in \mathbb{R}^3 : |x - x^0| < \rho\}$, and let $B_\rho = B_\rho(0)$. Denote by $|B_\rho| = 4\pi\rho^3/3$ the volume of the ball B_ρ.

Further, we assume that k_i, c_j are positive constants in inequalities.

3. BOUNDARY VALUE PROBLEMS FOR THE POISSON EQUATION ON $\mathbb{R}^3$ AND $\mathbb{R}^3_+$

For a proof of theorem on the behavior of characteristics of the Vlasov–Poisson system we need some auxiliary results.

Let $C_0(\mathbb{R}^3) = \{g \in C(\mathbb{R}^3) : g(x) \to 0 \text{ as } |x| \to \infty\}$, and let $C_0(\overline{\mathbb{R}^3_+}) = \{w \in C(\overline{\mathbb{R}^3_+}) : w(x) \to 0 \text{ as } |x| \to \infty\}$.

We consider the equation

$$-\Delta u(x) = F(x), \quad x \in \mathbb{R}^3, \tag{3.1}$$

with decreasing condition at infinity

$$u(x) \to 0 \quad \text{as } |x| \to \infty. \tag{3.2}$$

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Then, for any function $F \in C^\sigma_\Omega(\mathbb{R}^3)$, $0 < \sigma < 1$, there exists a unique solution of equation (3.1) with condition (3.2) $u \in C^{2+\sigma}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$. Moreover,*

$$||\,|\nabla u|\,||_0 \leqslant c_1 ||F||_0, \tag{3.3}$$

where

$$0 < c_1 = \frac{1}{4\pi} \sup_{x \in \mathbb{R}^3} \left\{ \sum_{i=1}^3 \left(\int_\Omega \frac{|x_i - y_i|}{|x-y|^3} dy \right)^2 \right\}^{1/2} < \infty.$$

Proof. We consider the Newtonian potential

$$w(x) = \int\limits_{\Omega} \Gamma(x-y)F(y)dy, \tag{3.4}$$

where $\Gamma(x) = \frac{1}{4\pi|x|}$.

By virtue of Lemma 2.2 from [19, Chap. 3, Sect. 2] and Lemma 4.2 from [20, Chap. 4, Sect. 4.2], the function w belongs to $C^{2+\sigma}(\mathbb{R}^3)$ and satisfies equation (3.1).

From the boundedness of Ω it follows that

$$|w(x)| = k_1(1+|x|)^{-1}, \quad x \in \mathbb{R}^3, \tag{3.5}$$

where $k_1 = k_1(F) > 0$ does not depend on x, i.e., $w \in C_0(\mathbb{R}^3)$. Therefore, problem (3.1) and (3.2) has the classical solution $u = w \in C^{2+\sigma}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$.

Differentiating right-hand side of (3.4), we obtain the expression for constant c_1.

In order to prove a uniqueness of solution to the problem (3.1) and (3.2), we put $F(x) = 0$. Then, by virtue of the Liouville theorem, $u(x)$ is a polynomial. However, a nontrivial polynomial does not satisfy the decreasing condition (3.2). □

We now consider the Poisson equation in a half-space

$$-\Delta u(x) = F(x), \quad x \in \mathbb{R}^3_+, \tag{3.6}$$

with the Neumann condition

$$\frac{\partial u(x)}{\partial x_1}\Big|_{x_1=0} = 0, \quad x' \in \mathbb{R}^2, \tag{3.7}$$

and the decreasing condition at infinity

$$u(x) \to 0 \quad \text{as } x \in \mathbb{R}^3_+, \quad |x| \to \infty, \tag{3.8}$$

where $x = (x_1, x')$.

Lemma 3.2. *Let $\Omega_+ \subset \mathbb{R}^3$ be a bounded domain, and let $\overline{\Omega}_+ \subset \mathbb{R}^3_+$. Then, for every function $F \in C^\sigma_{\Omega_+}(\overline{\mathbb{R}^3_+})$ there exists a unique solution of problem (3.6)–(3.8) $u \in C^{2+\sigma}(\overline{\mathbb{R}^3_+}) \cap C_0(\overline{\mathbb{R}^3_+})$. Moreover,*

$$||\, |\nabla u|\, ||_0 \leqslant c_1 ||F||_0, \tag{3.9}$$

where $c_1 > 0$ is the constant from Lemma 3.1.

Proof. Denote by $F_e(x)$ an even extension of the function $F(x)$ onto $\mathbb{R}^3$. Clearly,

$$F_e \in C^\sigma_{B_R}(\mathbb{R}^3) = \{w \in C^\sigma(\mathbb{R}^3) : \operatorname{supp} w \subset \overline{B_R}\},$$

where $R > 0$ is such that $\Omega_+ \cup \Omega_- \subset B_R$, $\Omega_- = \{x \in \mathbb{R}^3 : (-x_1, x') \in \Omega_+, x_1 < 0\}$.

Let $u_e \in C^{2+\sigma}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$ be a solution of problem (3.1) and (3.2) if the right-hand side of (3.1) is equal to $F_e(x)$. Since the function $F_e(x)$ is even with respect to x_1, the function $u_e(-x_1, x')$ is also a solution of problem (3.1) and (3.2). By virtue of uniqueness of solution to problem (3.1) and (3.2), see Lemma 3.1, we have $u_e(-x_1, x') = u_e(x_1, x')$ for every $x \in \mathbb{R}^3$. Therefore, the function $u_e(x)$ is even with respect to x_1. Hence,

$$\frac{\partial u_e(x)}{\partial x_1}\Big|_{x_1=0+0} = \frac{\partial u_e(x)}{\partial x_1}\Big|_{x_1=0-0} = 0.$$

Thus, the function $u_e|_{\overline{\mathbb{R}^3_+}}$ is a solution of problem (3.6)–(3.8). By virtue of Lemma 6.2 from [12], this solution is unique. A constant in inequality (3.9) is defined as following:

$$||\, |\nabla u|\, ||_0 = ||\, |\nabla u_e|_{\overline{\mathbb{R}^3_+}}|\, ||_0 = ||\, |\nabla u_e|\, ||_0 \leqslant c_1 ||F_e||_0 = c_1 ||F||_0.$$

□

4. THE EXISTENCE OF CLASSICAL SOLUTION TO THE SECOND MIXED PROBLEM OF THE VLASOV–POISSON SYSTEM WITH EXTERNAL MAGNETIC FIELD

Assume that the following conditions hold.

Condition 4.1. Let $B(x) = (0, 0, b(x_1))$, where

$$b(x_1) = \xi(x_1)h, \quad h > 0, \tag{4.1}$$

$\xi \in C^\infty(\overline{\mathbb{R}_+}), 0 \leqslant \xi(x_1) \leqslant 1, x_1 \in \overline{\mathbb{R}_+}, \xi(x_1) = 1, 0 < \delta/2 \leqslant x_1, \xi(x_1) = 0, 0 \leqslant x_1 < \delta/4, \mathbb{R}_+ = \{x_1 \in \mathbb{R} : x_1 > 0\}$.

Condition 4.2. Let $f_0^\beta \in \mathring{C}^1(\mathbb{R}^6)$, and let $\operatorname{supp} f_0^\beta \subset D_0 := (\mathbb{R}^3_{2\delta} \cap B_\lambda) \times B_\rho$, where $\delta, \lambda, \rho > 0$, $D_0 \neq \varnothing$.

The following result is a generalization of Theorem 6.1 from [8] to the case of the second mixed problem for the Vlasov–Poisson system with external magnetic field.

Theorem 4.1. *Let Conditions 4.1, 4.2 hold. Then, there exists a solution of problem (2.1)–(2.6) $\{\varphi, f^\beta\}$ such that $\varphi \in C([0,T], C^{2+\sigma}(\overline{\mathbb{R}^3_+}))$ and $f^\beta \in C^1\left(\overline{\mathbb{R}^3_+} \times \mathbb{R}^3 \times [0,T]\right)$. Moreover, f^β have compact supports.*

Proof. We prove the existence of solution. Denote by $\overline{f}{}_0^\beta(x, v)$, $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, and $\overline{B}(x)$, $x \in \mathbb{R}^3$, the extensions of functions $f_0^\beta(x, v)$ and $B(x)$ onto the space $\mathbb{R}^3 \times \mathbb{R}^3$ and $\mathbb{R}^3$, respectively, such that

$$\overline{B}(x) = (0, 0, \overline{b}(x_1)), \quad \overline{b}(x_1) = -b(-x_1), \quad x_1 \leqslant 0, \tag{4.2}$$

$$\overline{f}{}_0^\beta(x, v) = f_0^\beta(-x_1, x', -v_1, v'), \qquad x_1 \leqslant 0. \tag{4.3}$$

We consider the Cauchy problem for the Vlasov–Poisson system with respect to unknown functions $\overline{\varphi}(x, t)$ and $\overline{f}(x, v, t)$

$$-\Delta\overline{\varphi}(x,t) = 4\pi e \int\limits_{\mathbb{R}^3} \sum_\beta \beta \overline{f}^\beta(x, v, t)\, dv, \quad x \in \mathbb{R}^3, \quad 0 < t < T, \quad \beta = \pm 1, \tag{4.4}$$

$$\frac{\partial \overline{f}^\beta}{\partial t} + \left(v, \nabla_x \overline{f}^\beta\right) + \frac{\beta e}{m_\beta}\left(-\nabla_x\overline{\varphi} + \frac{1}{c}\,[v, \overline{B}], \nabla_v \overline{f}^\beta\right) = 0,$$
$$x \in \mathbb{R}^3, \quad v \in \mathbb{R}^3, \quad 0 < t < T, \quad \beta = \pm 1, \tag{4.5}$$

$$\overline{f}^\beta(x, v, t)|_{t=0} = \overline{f}{}_0^\beta(x, v), \quad x \in \mathbb{R}^3, \quad v \in \mathbb{R}^3, \quad \beta = \pm 1, \tag{4.6}$$

$$\lim_{|x|\to\infty} \overline{\varphi}(x, t) = 0, \quad 0 \leqslant t \leqslant T, \tag{4.7}$$

where the induction of external magnetic field $\overline{B}$ and initial density distribution functions $\overline{f}{}_0^\beta$ are given by the formulas (4.2) and (4.3), respectively.

From Conditions 4.1 and 4.2 and the formulas (4.2) and (4.3) it follows that

$$\overline{b} \in C^\infty(\mathbb{R}), \tag{4.8}$$

$$\overline{f}{}_0^\beta \in \mathring{C}^1(\mathbb{R}^3) \quad \text{and} \quad \mathit{supp}\, \overline{f}{}_0^\beta \in B_{R_0}, \quad \text{where} \quad R_0 = (\lambda^2 + \rho^2)^{\frac{1}{2}}. \tag{4.9}$$

By virtue of Theorem 1 from [3] and Theorem from [5], there exists a unique solution of the problem (4.4)–(4.7) $\overline{\varphi} \in C([0,T], C^{2+\sigma}(\mathbb{R}^3))$, $\overline{f}^\beta \in C^1\left(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T]\right)$, $\beta = \pm 1$. Moreover,

$$R(T) := \{1 + \max_\beta \sup_{v \in \mathbb{R}^3} |v| : \text{ there exist } x \in \mathbb{R}^3 \text{ and } t \in [0,T] \text{ such that } f^\beta(x, v, t) \neq 0\} < \infty. \tag{4.10}$$

Here the functions $(0,T) \ni t \mapsto \overline{\varphi}(\cdot, t) \in C^{2+\sigma}(\mathbb{R}^3)$, $(0,T) \ni t \mapsto \overline{f}^\beta(\cdot,\cdot,t) \in C^1(\mathbb{R}^3 \times \mathbb{R}^3)$, and $\dfrac{\partial f^\beta(\cdot,\cdot,t)}{\partial t} \in C(\mathbb{R}^3 \times \mathbb{R}^3)$ are extended by continuity onto the points $t = 0$ and $t = T$.

We consider the vector-function $\{\hat{\varphi}, \hat{f}^\beta\}$ given by the formulas

$$\hat{\varphi}(x,t) = \overline{\varphi}(-x_1, x', t), \quad x \in \mathbb{R}^3, \quad t \in [0,T], \tag{4.11}$$

$$\hat{f}^\beta(x,v,t) = \overline{f}(-x_1, x', -v_1, v', t), \quad x \in \mathbb{R}^3, \quad v \in \mathbb{R}^3, \quad t \in [0,T]. \tag{4.12}$$

Substituting $\{\hat{\varphi}, \hat{f}^\beta\}$ into (4.4)–(4.7) instead of $\{\overline{\varphi}, \overline{f}^\beta\}$ and using differentiation of composite function, we verify that $\{\hat{\varphi}, \hat{f}^\beta\}$ is a solution of problem (4.4)–(4.7). Moreover, $supp\, \hat{f}_0^\beta \subset B_{R_0}$.

From the uniqueness of solution to the problem (4.4)–(4.7) it follows that

$$\overline{\varphi}(-x_1, x', t) = \overline{\varphi}(x,t), \quad x \in \mathbb{R}^3, \quad t \in [0,T], \tag{4.13}$$

$$\overline{f}^\beta(-x_1, x', -v_1, v', t) = \overline{f}^\beta(x,v,t), \qquad x \in \mathbb{R}^3, \quad v \in \mathbb{R}^3, \quad t \in [0,T], \quad \beta = \pm 1. \tag{4.14}$$

We set

$$\varphi(x,t) := \overline{\varphi}(x,t), \quad x \in \overline{\mathbb{R}^3_+}, \quad t \in [0,T], \tag{4.15}$$

$$f^\beta(x,v,t) := \overline{f}^\beta(x,v,t), \quad x \in \overline{\mathbb{R}^3_+}, \quad v \in \mathbb{R}^3, \quad t \in [0,T]. \tag{4.16}$$

By definition, the vector-function $\{\varphi, f^\beta\}$ satisfies the equations (2.1), (2.2) and the conditions (2.3), (2.6).

We also prove that $\{\varphi, f^\beta\}$ satisfies the conditions (2.4) and (2.5).

Since the function $\overline{\varphi}(x,t)$ is continuously differentiable with respect to x in $\mathbb{R}^3$, from (4.13) we obtain

$$\frac{\partial \overline{\varphi}(x,t)}{\partial x_1}\Big|_{x_1=0+0} = -\frac{\partial \overline{\varphi}(x,t)}{\partial x_1}\Big|_{x_1=0-0} = 0,$$

i.e., the function φ given by (4.15) satisfies condition (2.5).

On the other hand, by virtue of continuity of $\overline{f}^\beta(x,v,t)$ and equality (4.14), it follows that

$$\begin{aligned} f^\beta(0,x',v_1,v',t) &= \overline{f}^\beta(0+0,x',v_1,v',t) = \overline{f}^\beta(0-0,x',-v_1,v',t) \\ &= \overline{f}^\beta(0+0,x',-v_1,v',t) = f^\beta(0,x',-v_1,v',t). \end{aligned}$$

Therefore, the functions f^β given by (4.16) satisfy condition (2.4). Thus, we have proved the existence of classical solution to the problem (2.1)–(2.6) $\{\varphi, f^\beta\}$ such that $\varphi \in C([0,T], C^{2+\sigma}(\overline{\mathbb{R}^3_+}))$ and $f^\beta \in C^1\left(\overline{\mathbb{R}^3_+} \times \mathbb{R}^3 \times [0,T]\right)$, $\beta = \pm 1$. Moreover, f^β have compact supports. $\square$

5. TRAJECTORIES OF CHARGED PARTICLES IN EXTERNAL MAGNETIC FIELD

As above, we shall assume that the conditions 4.1 and 4.2 hold.

Let $\{\overline{\varphi}, \overline{f}^\beta\}$, $\overline{\varphi} \in C([0,T], C^{2+\sigma}(\mathbb{R}^3))$, $\overline{f}^\beta \in C^1\left(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T]\right)$, $\beta = \pm 1$, be a solution of problem (4.4)–(4.7). The existence and uniqueness of this solution was stated in the proof of Theorem 4.1.

For the present function $\overline{\varphi} \in C([0,T], C^{2+\sigma}(\mathbb{R}^3))$, the Vlasov equations (2.2) with the initial conditions (2.3) can be solved with the help of characteristics method. For this, we consider the following system of ordinary differential equations

$$\frac{dX^\beta_{\overline{\varphi}}(\tau)}{d\tau} = V^\beta_{\overline{\varphi}}(\tau), \quad 0 < \tau < T, \quad \beta = \pm 1, \tag{5.1}$$

$$\frac{dV^\beta_{\overline{\varphi}}(\tau)}{d\tau} = -\frac{\beta e}{m_\beta} \nabla_x \overline{\varphi}\big(X^\beta_{\overline{\varphi}}(\tau), \tau\big) + \frac{\beta e}{m_\beta c}\big[V^\beta_{\overline{\varphi}}(\tau), B(X^\beta_{\overline{\varphi}}(\tau))\big], \quad 0 < \tau < T, \quad \beta = \pm 1, \tag{5.2}$$

with initial conditions

$$X^{\beta}_{\varphi}(\tau)|_{\tau=t} = x, \qquad \beta = \pm 1, \tag{5.3}$$

$$V^{\beta}_{\varphi}(\tau)|_{\tau=t} = v, \qquad \beta = \pm 1. \tag{5.4}$$

From Theorem in [5] it follows that for any $(x, v) \in B_{R(T)}$ there exists a unique solution of the problem (5.1)–(5.4) $(X^{\beta}_{\varphi}(\tau), V^{\beta}_{\varphi}(\tau))$ on the closed interval $[0, T]$. We denote this solution by $(X^{\beta}_{\varphi}(\tau, x, v, t), V^{\beta}_{\varphi}(\tau, x, v, t)) := S^{\beta}_{\varphi}(\tau, x, v, t)$.

The following result gives an a priori estimate of norm for strength of self-consistent electric field to the problem (4.4)–(4.7) by the norms of initial density distribution functions for the charged particles $\overline{f}^{\beta}_0$.

Theorem 5.1. *Let the vector-function $\overline{B}(x)$ and the functions $\overline{f}^{\beta}_0(x, v)$ are defined by the formulas (4.2) and (4.3), respectively, and let the vector-functions $B(x)$ and $f^{\beta}_0(x, v)$ satisfy the Conditions 4.1 and 4.2. Then, the following estimate holds*

$$||\, |\nabla \overline{\varphi}|\, ||_{0,T} \leqslant c_1 |B_{R(T)}| 4\pi e \max_{\beta} ||\overline{f}^{\beta}_0||_0, \tag{5.5}$$

where the number $R(T)$, $0 < R(T) < \infty$, is given by (4.10), and $c_1 > 0$ is a constant from inequality (3.3).

Proof. From formula (4.10), well-known equality

$$\overline{f}^{\beta}(x, v, t) = f^{\beta}_0(S^{\beta}_{\varphi}(0, x, v, t)), \quad x \in \mathbb{R}^3, \quad v \in \mathbb{R}^3, \quad t \in (0, T), \tag{5.6}$$

and Condition 4.2 we obtain

$$R(T) = \{1 + \max_{\beta} \sup_{v \in \mathbb{R}^3} \{|v| : \text{ there exist } x \in \mathbb{R}^3 \text{ and } t \in [0, T] \text{ such that } S^{\beta}_{\varphi}(0, x, v, t) \subset D_0\}. \tag{5.7}$$

From Lemma 3.1 and the equalities (5.6) and (5.7) we have

$$||\, |\nabla \overline{\varphi}|\, ||_{0,T} \leqslant c_1 4\pi e \sup_{x,t} \int\limits_{\mathbb{R}^3} \Big| \sum_{\beta} \beta \overline{f}^{\beta}(x, v, t) \Big| dv \leqslant c_1 4\pi e \sup_{x,t} \int\limits_{\mathbb{R}^3} \max_{\beta} |\overline{f}^{\beta}(x, v, t)| dv$$

$$= c_1 4\pi e \sup_{x,t} \int\limits_{\mathbb{R}^3} \max_{\beta} |\overline{f}^{\beta}_0(S^{\beta}_{\varphi}(0, x, v, t))| dv \leqslant c_1 |B_{R(T)}| 4\pi e \max_{\beta} \sup_{(y,w) \in B_{R_0}} |\overline{f}^{\beta}_0(y, w)|, \quad t \in [0, T].$$

From this it follows inequality (5.5). □

Lemma 5.1. Let the Conditions 4.1 and 4.2 hold, and let the vector-function $\overline{B}(x)$ and functions $\overline{f}^{\beta}_0(x, v)$ be given by the formulas (4.2) and (4.3), respectively. Then, for every $(x, v) \in D_0$ and $0 \leqslant t \leqslant T$ the following estimates take place

$$|V^{\beta}_{\varphi}(t, x, v, 0)| \leqslant \rho + \frac{eT}{m_{\beta}} c_1 |B_{R(T)}| 4\pi e \max_{\beta} ||\overline{f}^{\beta}_0||_0, \tag{5.8}$$

$$|X^{\beta}_{\varphi}(t, x, v, 0)| \leqslant \left(\rho + \frac{eT}{m_{\beta}} c_1 |B_{R(T)}| 4\pi e \max_{\beta} ||\overline{f}^{\beta}_0||_0 \right) T + \lambda. \tag{5.9}$$

Proof. Multiplying the left and the right sides of equation (5.2) by V^{β}_{φ}, we have

$$\frac{1}{2} \frac{d}{d\tau} |V^{\beta}_{\varphi}(\tau, x, v, 0)|^2 = -\frac{\beta e}{m_{\beta}} \left(\nabla_x \overline{\varphi}(X^{\beta}_{\varphi}(\tau, x, v, 0)), V^{\beta}_{\varphi}(\tau, x, v, 0) \right), \quad 0 \leqslant \tau \leqslant T.$$

From this and from the Cauchy–Bunyakovskii inequality we obtain

$$\frac{1}{2} \frac{d}{d\tau} |V^{\beta}_{\varphi}(\tau, x, v, 0)|^2 \leqslant \frac{e}{m_{\beta}} |\nabla_x \overline{\varphi}(X^{\beta}_{\varphi}(\tau, x, v, 0))| \cdot |V^{\beta}_{\varphi}(\tau, x, v, 0)|, \quad 0 \leqslant \tau \leqslant T.$$

Hence,

$$\frac{d}{d\tau}|V^{\beta}_{\overline{\varphi}}(\tau,x,v,0)| \leqslant \frac{e}{m_\beta}|\nabla_x\overline{\varphi}(X^{\beta}_{\overline{\varphi}}(\tau,x,v,0))|, \quad 0\leqslant\tau\leqslant T. \tag{5.10}$$

Integrating equality (5.10) by τ from 0 to t, by virtue of Theorem 5.1 and Condition 4.2, we have

$$|V^{\beta}_{\overline{\varphi}}(\tau,x,v,0)| \leqslant |v| + \frac{e}{m_\beta}\int\limits_0^t |\nabla_x\overline{\varphi}(X^{\beta}_{\overline{\varphi}}(\tau,x,v,0))|d\tau \leqslant \rho + \frac{eT}{m_\beta}c_1|B_{R(T)}|4\pi e\max_\beta ||\overline{f}^{\beta}_0||_0. \tag{5.11}$$

Integrating equation (5.1) by τ from 0 to t and using inequality (5.11), we obtain (5.9). □

Assume the following condition holds.

Condition 5.1. Let a constant h in formula (4.1) satisfy the inequality

$$\frac{4c}{e\delta}\left(\rho m_{+1} + eTc_1|B_{R(T)}|4\pi e\max_\beta ||f^{\beta}_0||_0\right) < h, \tag{5.12}$$

where $c_1 > 0$ is a constant from Lemma 3.1.

We introduce the matrix

$$R(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \quad \theta\in\mathbb{R}.$$

Multiplication by the matrix $R(\theta)$ corresponds to rotation by the angle θ on the plane. The next statement allows us to apply properties of this operator to the investigation of trajectories of charged particles, if equation (5.2) contains magnetic field.

Lemma 5.2.

(a) $R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2)$, $\theta_1,\theta_2\in\mathbb{R}$.

(b) $R(\theta)^m = R(m\theta)$, $\quad \theta\in\mathbb{R}, m\in\mathbb{Z}$.

(c) $\dfrac{d}{d\theta}R(\theta) = R(\pi/2)R(\theta) = R(\theta+\pi/2)$, $\theta\in\mathbb{R}$.

(d) $|R(\theta)\,x| = |x|$, $\theta\in\mathbb{R}$, $x\in\mathbb{R}^2$.

(e) $\exp(tR(\theta)) = \exp(t\cos\theta)R(t\sin\theta)$.

For a proof, see [15].

Let $\hat{x} := (x_1,x_2)$, and let $\hat{X}^{\beta}_{\overline{\varphi}}(\tau,x,v,0) := \left(X^{\beta}_{\overline{\varphi},1}(\tau,x,v,0), X^{\beta}_{\overline{\varphi},2}(\tau,x,v,0)\right)$.

The following result is a generalization of Lemma 3.3 from [15]. For the convenience of reader we present here a complete proof.

Lemma 5.3. *Let Conditions 4.1, 4.2, and 5.1 hold, and let the vector-function $\overline{B}(x)$ and the functions $\overline{f}^{\beta}_0(x,v)$ be given by the formulas (4.2) and (4.3), respectively. Then, the solution of the problem (5.1)–(5.4) given by $\left(X^{\beta}_{\overline{\varphi}}(\tau,x,v,0), V^{\beta}_{\overline{\varphi}}(\tau,x,v,0)\right) := S^{\beta}_{\overline{\varphi}}(\tau,x,v,0)$ have the following properties*

$$|\hat{X}^{\beta}_{\overline{\varphi}}(\tau,x,v,0) - \hat{x}| < \frac{\delta}{2}, \quad V^{\beta}_{\overline{\varphi}}(\tau,x,v,0)\in B_{\rho_1}$$

for all $(x,v)\in D_0$ and $\tau\in[0,T]$, where

$$\rho_1 = \rho + \frac{eT}{m_{-1}}c_1|B_{R(T)}|4\pi e\max_\beta ||\overline{f}^{\beta}_0||_0.$$

Proof. We prove that

$$|\hat{X}^{\beta}_{\overline{\varphi}}(\tau,x,v,0)-\hat{x}|<\frac{\delta}{2}\ \text{for all}\ \tau\in[0,T]. \tag{5.13}$$

Assume to the contrary that there exists $\tau_0\in[0,T]$ such that

$$|\hat{X}^{\beta}_{\overline{\varphi}}(\tau_0,x,v,0)-\hat{x}|\geqslant\frac{\delta}{2}.$$

Since $\hat{X}^{\beta}_{\overline{\varphi}}(0,x,v,0)=\hat{x}$, then for some τ_1, $0<\tau_1\leqslant\tau_0\leqslant T$, we have

$$|\hat{X}^{\beta}_{\overline{\varphi}}(\tau_1,x,v,0)-\hat{x}|=\frac{\delta}{2}, \tag{5.14}$$

$$|\hat{X}^{\beta}_{\overline{\varphi}}(\tau,x,v,0)-\hat{x}|<\frac{\delta}{2},\qquad \tau\in[0,\tau_1). \tag{5.15}$$

By virtue of Condition 4.1, we can rewrite characteristic equation (5.2) in the form

$$\frac{dV^{\beta}_{\overline{\varphi}}(\tau)}{d\tau}=-\frac{\beta e}{m_\beta}\nabla_x\overline{\varphi}(X^{\beta}_{\overline{\varphi}},\tau)+\frac{\beta e}{m_\beta c}\begin{pmatrix}0&h&0\\-h&0&0\\0&0&0\end{pmatrix}V^{\beta}_{\overline{\varphi}}(\tau),\quad \tau\in(0,\tau_1).$$

Hence,

$$\frac{d}{d\tau}\begin{pmatrix}V^{\beta}_{\overline{\varphi},1}(\tau)\\V^{\beta}_{\overline{\varphi},2}(\tau)\end{pmatrix}+\frac{\beta eh}{m_\beta c}R\Big(\frac{\pi}{2}\Big)\begin{pmatrix}V^{\beta}_{\overline{\varphi},1}(\tau)\\V^{\beta}_{\overline{\varphi},2}(\tau)\end{pmatrix}=-\frac{\beta e}{m_\beta}\nabla_{(x_1,x_2)}\overline{\varphi}(X^{\beta}_{\overline{\varphi}},\tau),\quad \tau\in(0,\tau_1).$$

Multiplying the last equation by $\exp\Big(\tau\,\frac{\beta eh}{m_\beta c}R\Big(\frac{\pi}{2}\Big)\Big)$, we obtain

$$\frac{d}{d\tau}\left[\exp\left(\tau\frac{\beta eh}{m_\beta c}R\left(\frac{\pi}{2}\right)\right)\begin{pmatrix}V^{\beta}_{\overline{\varphi},1}(\tau)\\V^{\beta}_{\overline{\varphi},2}(\tau)\end{pmatrix}\right]=-\frac{\beta e}{m_\beta}\exp\left(\tau\frac{\beta eh}{m_\beta c}R\left(\frac{\pi}{2}\right)\right)\nabla_{(x_1,x_2)}\overline{\varphi}\big(X^{\beta}_{\overline{\varphi}},\tau\big),$$
$$\tau\in(0,\tau_1).$$

Integrating the achieved equation from 0 to t, $t\in(0,\tau_1]$, we have

$$\exp\left(t\frac{\beta eh}{m_\beta c}R\left(\frac{\pi}{2}\right)\right)\begin{pmatrix}V^{\beta}_{\overline{\varphi},1}(t)\\V^{\beta}_{\overline{\varphi},2}(t)\end{pmatrix}-\begin{pmatrix}v_1\\v_2\end{pmatrix}=-\frac{\beta e}{m_\beta}\int\limits_0^t\exp\left(\tau\frac{\beta eh}{m_\beta c}R\left(\frac{\pi}{2}\right)\right)\nabla_{(x_1,x_2)}\overline{\varphi}\big(X^{\beta}_{\overline{\varphi}},\tau\big)\,d\tau. \tag{5.16}$$

By virtue of Lemma (5.2) (e), we see that

$$\exp\left(t\frac{\beta eh}{m_\beta c}R\left(\frac{\pi}{2}\right)\right)=\exp\left(t\frac{\beta eh}{m_\beta c}\cos\frac{\pi}{2}\right)R\left(t\frac{\beta eh}{m_\beta c}\sin\frac{\pi}{2}\right)=R\left(t\frac{\beta eh}{m_\beta c}\right).$$

Therefore, multiplying (5.16) by $R\left(-t\frac{\beta eh}{m_\beta c}\right)$, we obtain

$$\begin{pmatrix}V^{\beta}_{\overline{\varphi},1}(t)\\V^{\beta}_{\overline{\varphi},2}(t)\end{pmatrix}=R\left(-t\,\frac{\beta eh}{m_\beta c}\right)\begin{pmatrix}v_1\\v_2\end{pmatrix}-\frac{\beta e}{m_\beta}\int\limits_0^t R\left((\tau-t)\,\frac{\beta eh}{m_\beta c}\right)\nabla_{(x_1,x_2)}\overline{\varphi}\big(X^{\beta}_{\overline{\varphi}},\tau\big)\,d\tau,\qquad t\in(0,\tau_1].$$

From the last equation and from (5.1) it follows that

$$\begin{pmatrix}X^{\beta}_{\overline{\varphi},1}(\tau_1)\\X^{\beta}_{\overline{\varphi},2}(\tau_1)\end{pmatrix}=\begin{pmatrix}x_1\\x_2\end{pmatrix}+I_1+I_2, \tag{5.17}$$

where

$$I_1 = \int_0^{\tau_1} R\left(-t\,\frac{\beta eh}{m_\beta c}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dt,$$

$$I_2 = -\frac{\beta e}{m_\beta}\int_0^{\tau_1}\int_0^{t} R\left((\tau - t)\,\frac{\beta eh}{m_\beta c}\right)\nabla_{(x_1,x_2)}\overline{\varphi}\big(X^{\beta}_{\overline{\varphi}}, \tau\big)\,d\tau dt.$$

We now calculate I_1 and I_2.

By virtue of Lemma 5.2 (c), we have

$$R\left(-t\,\frac{\beta eh}{m_\beta c}\right) = -\frac{m_\beta c}{\beta eh}\frac{d}{dt}\left(R\left(-t\,\frac{\beta eh}{m_\beta c} - \frac{\pi}{2}\right)\right).$$

Hence,

$$I_1 = \frac{m_\beta c}{\beta eh}\left[-R\left(-t\,\frac{\beta eh}{m_\beta c} - \frac{\pi}{2}\right)\right]\Bigg|_{t=0}^{t=\tau_1}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{m_\beta c}{\beta eh}\left\{-R\left(-\tau_1\,\frac{\beta eh}{m_\beta c} - \frac{\pi}{2}\right) + R\left(-\frac{\pi}{2}\right)\right\}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \frac{m_\beta c}{\beta eh}\left\{-R\left(-\tau_1\,\frac{\beta eh}{m_\beta c}\right) + E\right\} R\Big(-\frac{\pi}{2}\Big)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \frac{m_\beta c}{\beta eh}\begin{pmatrix} 1 - \cos\left(\tau_1\,\frac{\beta eh}{m_\beta c}\right) & -\sin\left(\tau_1\,\frac{\beta eh}{m_\beta c}\right) \\ \sin\left(\tau_1\,\frac{\beta eh}{m_\beta c}\right) & 1 - \cos\left(\tau_1\,\frac{\beta eh}{m_\beta c}\right) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \frac{m_\beta c}{\beta eh}\begin{pmatrix} \sin\left(\tau_1\frac{\beta eh}{m_\beta c}\right) & 1 - \cos\left(\tau_1\,\frac{\beta eh}{m_\beta c}\right) \\ -\left(1 - \cos\left(\tau_1\,\frac{\beta eh}{m_\beta c}\right)\right) & \sin\left(\tau_1\,\frac{\beta eh}{m_\beta c}\right) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \frac{m_\beta c}{\beta eh}\begin{pmatrix} \beta\sin\left(\tau_1\frac{eh}{m_\beta c}\right) v_1 + \left(1 - \cos\left(\tau_1\,\frac{eh}{m_\beta c}\right)\right) v_2 \\ -\left(1 - \cos\left(\tau_1\,\frac{eh}{m_\beta c}\right)\right) v_1 + \beta\sin\left(\tau_1\,\frac{eh}{m_\beta c}\right) v_2 \end{pmatrix},$$

where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, we have

$$|I_1| = \frac{m_\beta c}{eh}\Bigg\{\left(\beta\sin\left(\tau_1\frac{eh}{m_\beta c}\right) v_1 + \left(1 - \cos\left(\tau_1\,\frac{eh}{m_\beta c}\right)\right) v_2\right)^2$$
$$+ \left(-\left(1 - \cos\left(\tau_1\,\frac{eh}{m_\beta c}\right)\right) v_1 + \beta\sin\left(\tau_1\,\frac{eh}{m_\beta c}\right) v_2\right)^2\Bigg\}^{1/2}$$
$$= \frac{m_\beta c}{eh}\left\{\left(v_1^2 + v_2^2\right)\left(\left(1 - \cos\left(\tau_1\,\frac{eh}{m_\beta c}\right)\right)^2 + \sin^2\left(\tau_1\,\frac{eh}{m_\beta c}\right)\right)\right\}^{1/2}$$
$$= \frac{m_\beta c}{eh}|v|\sqrt{2}\sqrt{1 - \cos\left(\tau_1\,\frac{eh}{m_\beta c}\right)} \leqslant \frac{2c}{eh} m_{+1}|v|. \tag{5.18}$$

On the other hand, using Lemma 5.2 (c), we conclude that

$$I_2 = -\frac{\beta e}{m_\beta}\int\limits_0^{\tau_1}\left\{\int\limits_\tau^{\tau_1} R\left((\tau-t)\,\frac{\beta eh}{m_\beta c}\right)dt\right\}\nabla_{(x_1,x_2)}\overline{\varphi}\big(X^\beta_{\overline{\varphi}},\tau\big)\,d\tau$$

$$= \frac{c}{h}\int\limits_0^{\tau_1}\left\{R\left((\tau-\tau_1)\,\frac{\beta eh}{m_\beta c}\right)-E\right\}R\Big(-\frac{\pi}{2}\Big)\nabla_{(x_1,x_2)}\overline{\varphi}\big(X^\beta_{\overline{\varphi}},\tau\big)\,d\tau$$

$$= \frac{c}{h}\int\limits_0^{\tau_1}\begin{pmatrix}\cos\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right)-1 & -\beta\sin\left((\tau-\tau_1)\frac{eh}{m_\beta c}\right)\\ \beta\sin\left((\tau-\tau_1)\frac{eh}{m_\beta c}\right) & \cos\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right)-1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\frac{\partial\overline{\varphi}\left(X^\beta_{\overline{\varphi}},\tau\right)}{\partial x_1}\\ \frac{\partial\overline{\varphi}\left(X^\beta_{\overline{\varphi}},\tau\right)}{\partial x_2}\end{pmatrix}d\tau$$

$$= \frac{c}{h}\int\limits_0^{\tau_1}\begin{pmatrix}\beta\sin\left((\tau-\tau_1)\frac{eh}{m_\beta c}\right) & \cos\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right)-1\\ 1-\cos\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right) & \beta\sin\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right)\end{pmatrix}\begin{pmatrix}\frac{\partial\overline{\varphi}\left(X^\beta_{\overline{\varphi}},\tau\right)}{\partial x_1}\\ \frac{\partial\overline{\varphi}\left(X^\beta_{\overline{\varphi}},\tau\right)}{\partial x_2}\end{pmatrix}d\tau$$

$$= \frac{c}{h}\int\limits_0^{\tau_1}\begin{pmatrix}\beta\sin\left((\tau-\tau_1)\frac{eh}{m_\beta c}\right)\frac{\partial\overline{\varphi}\left(X^\beta_{\overline{\varphi}},\tau\right)}{\partial x_1}+\left(\cos\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right)-1\right)\frac{\partial\overline{\varphi}\left(X^\beta_{\overline{\varphi}},\tau\right)}{\partial x_2}\\ \left(1-\cos\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right)\right)\frac{\partial\overline{\varphi}\left(X^\beta_{\overline{\varphi}},\tau\right)}{\partial x_1}+\beta\sin\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right)\frac{\partial\overline{\varphi}\left(X^\beta_{\overline{\varphi}},\tau\right)}{\partial x_2}\end{pmatrix}d\tau.$$

Therefore, we obtain

$$|I_2| = \frac{c}{h}\int\limits_0^{\tau_1}\left\{\left(\left(1-\cos\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right)\right)^2+\sin^2\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right)\right)\right.$$
$$\left.\times\left(\left(\frac{\partial\overline{\varphi}\big(X^\beta_{\overline{\varphi}},\tau\big)}{\partial x_1}\right)^2+\left(\frac{\partial\overline{\varphi}\big(X^\beta_{\overline{\varphi}},\tau\big)}{\partial x_2}\right)^2\right)\right\}^{1/2}d\tau$$

$$= \frac{c}{h}\int\limits_0^{\tau_1}\left(\left(1-\cos\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right)\right)^2+\sin^2\left((\tau-\tau_1)\,\frac{eh}{m_\beta c}\right)\right)^{\frac{1}{2}}\left|\nabla_{(x_1,x_2)}\overline{\varphi}\big(X^\beta_{\overline{\varphi}},\tau\big)\right|d\tau$$

$$\leqslant \frac{2c}{h}\int\limits_0^{\tau_1}\left|\nabla_{(x_1,x_2)}\overline{\varphi}\big(X^\beta_{\overline{\varphi}},\tau\big)\right|d\tau \leqslant \frac{2c}{h}T||\,|\nabla\overline{\varphi}|\,||_{0,T}. \tag{5.19}$$

From (5.14), (5.17)–(5.19) and Theorem 5.1 it follows that

$$\frac{\delta}{2} = \left|\hat{X}^\beta_{\overline{\varphi}}(\tau_1,x,v,0)-\hat{x}\right| \leqslant |I_1|+|I_2| \leqslant \frac{2c}{eh}\,\left(\rho m_{+1}+eT||\,|\nabla\overline{\varphi}|\,||_{0,T}\right)$$
$$\leqslant \frac{2c}{eh}\left(\rho m_{+1}+eTc_1|B_{R(T)}|4\pi e\max_\beta||\overline{f}^\beta_0||_0\right). \tag{5.20}$$

By virtue of (4.3), $||\overline{f}^\beta_0||_0 = ||f^\beta_0||_0$. Therefore, (5.12) implies that

$$\frac{2c}{eh}\left(\rho m_{+1}+eTc_1|B_{R(T)}|4\pi e\max_\beta||\overline{f}^\beta_0||\right) < \frac{\delta}{2}.$$

This inequality contradicts to (5.20). Thus, we have proved that $|\hat{X}^\beta_{\overline{\varphi}}(\tau,x,v,0)-\hat{x}| < \frac{\delta}{2}$ for all $\tau\in[0,T]$.

From Lemma (5.1) it follows that $V^\beta_{\overline{\varphi}}(\tau,x,v,0)\in B_{\rho_1}$ for all $(x,v)\in D_0$ and $\tau\in[0,T]$. □

Similarly to Lemma 5.3 one can prove the following statement.

Lemma 5.4. *Let the assumptions of Lemma 5.3 hold. Then, the solution of problem (5.1)–(5.4)* $S^{\beta}_{\varphi}(\tau,y,w,t) := \Big(X^{\beta}_{\varphi}(\tau,y,w,t), V^{\beta}_{\varphi}(\tau,y,w,t)\Big)$ *have the following properties*

$$|\hat{X}^{\beta}_{\varphi}(\tau,y,w,t)-\hat{x}|<\frac{\delta}{2},\quad V^{\beta}_{\varphi}(\tau,y,w,t)\in B_{\rho_2}$$

for all $(y,w)\in D_1 := \left(\mathbb{R}^3_{\frac{3\delta}{2}}\cap B_{\lambda_1}\right)\times B_{\rho_1}$ *and* $0\leqslant\tau\leqslant t\leqslant T$, *where* $\lambda_1=\lambda+T\rho_1$,

$$\rho_2=\rho+\frac{2eT}{m_{-1}}c_1|B_{R(T)}|4\pi e\max_{\beta}||f^{\beta}_0||_0.$$

Lemma 5.5. *Let the assumptions of Lemma 5.3 hold. Then, supp* $f^{\beta}_0\left(S^{\beta}_{\varphi}(0,\cdot,\cdot,t)\right)\subset D_1$, $0\leqslant t\leqslant T$.

Proof. By virtue of Condition 4.2, it is sufficient to prove that $S^{\beta}_{\varphi}(t,x,v,0)\in D_1$ for $(x,v)\in D_0$, $0\leqslant t\leqslant T$. Let $(x,v)\in D_0$. Then, from Lemma 5.3 it follows that $S^{\beta}_{\varphi}(t,x,v,0)\in\mathbb{R}^3_{3\delta/2}$. By Condition 4.2, $x\in B_{\lambda}$. Therefore, since $|V^{\beta}_{\varphi}(t,x,v,0)|<\rho_1$, $0\leqslant t\leqslant T$, from the equality $\lambda_1=\lambda+T\rho_1$ we obtain

$$|X^{\beta}_{\varphi}(t,x,v,0)|\leqslant|x|+\int\limits_0^t|V^{\beta}_{\varphi}(\tau,x,v,0)|d\tau\leqslant\lambda_1.$$

□

We define the function $f^{\beta}(x,v,t)$ by the formula

$$f^{\beta}(x,v,t)=\begin{cases}f^{\beta}_0(S^{\beta}_{\varphi}(0,y,w,t)), & (y,w)\in D_1,\quad 0\leqslant t\leqslant T,\\ 0, \quad (y,w)\in\overline{\mathbb{R}^3_+}\times\mathbb{R}^3\setminus D_1, & 0\leqslant t\leqslant T.\end{cases}\tag{5.21}$$

By virtue of Lemma 5.5, $\operatorname{supp} f^{\beta}_0\left(S^{\beta}_{\varphi}(0,\cdot,\cdot,t)\right)\subset D_1$. Therefore, using the characteristics method, continuous differentiability of functions $S^{\beta}_{\varphi}(0,y,w,t)$ with respect to y, w, t, and Theorem 4.1, we see that there exists a solution of problem (2.1)–(2.6) $\varphi\in C([0,T],C^{2+\sigma}(\overline{\mathbb{R}^3_+}))$, $f^{\beta}\in C^1\left(\overline{\mathbb{R}^3_+}\times\mathbb{R}^3\times[0,T]\right)$, $\beta\pm1$, given by the formulas (4.15), (4.16), where $\{\overline{\varphi},\overline{f}^{\beta}\}$ is a solution of problem (4.4)–(4.7). Moreover, *supp* $f^{\beta}\subset D_1$, $0\leqslant t\leqslant T$. Thus, we have proved the following statement on existence of global classical solutions to the problem (2.1)–(2.6), which correspond to plasma confinement in a half-space.

Theorem 5.2 *Let the Conditions 4.1, 4.2, and 5.1 hold. Then, there exists a solution of problem (2.1)–(2.6)* $\{\varphi,f^{\beta}\}$ *such that* $\varphi\in C([0,T],C^{2+\sigma}(\overline{\mathbb{R}^3_+}))$ *and* $f^{\beta}\in C^1\left(\overline{\mathbb{R}^3_+}\times\mathbb{R}^3\times[0,T]\right)$. *Moreover, supp* $f^{\beta}(\cdot,\cdot,t)\subset D_1$, $0\leqslant t\leqslant T$.

Remark 5.1. From Theorem 5.2 it follows the existence of global weak solution of problem (2.1)–(2.6) with supports of functions f^{β} with respect to x on $\overline{\mathbb{R}^3_{3\delta/2}}$ in a sense of integral identity, see [21].

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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