Unique Solvability of a Boundary Value Problem for a Loaded Fractional Parabolic-Hyperbolic Equation with Nonlinear Terms

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Abstract—This work is devoted to study the existence and uniqueness of solution of an analogue of the Gellerstedt problem with nonlocal assumptions on the boundary and integral gluing conditions for the parabolic-hyperbolic type equation with nonlinear terms and Gerasimov–Caputo operator of differentiation. Using the method of integral energy, the uniqueness of solution have been proved. Existence of solution was proved by the method of successive approximations of factorial law for Volterra type nonlinear integral equations.

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1. INTRODUCTION

One of the most important areas of mathematical analysis is the theory of fractional order integro-differential operators. Today, the theory and applications of operators of fractional differentiation and integration have become a powerful direction of theoretical and applied research in different sciences and technologies. In particular, the operators of fractional differentiation and integration are used in the study of problems associated with the study of the coronavirus COVID–19 (see, for example [1–3]). In [4], the mathematical problems of an Ebola epidemic model by fractional order equations are considered. In [5], fractional models of the dynamics of tuberculosis infection are considered.

On May 29, 1947 A. N. Gerasimov made report at the Institute of Mechanics of the USSR Academy of Sciences (see [6]). There, he has introduced a concept of fractional derivative, which today we know as a Gerasimov–Caputo fractional derivative. This Gerasimov’s report was published in [7]. Fractional order differentiation operators of Riemann–Liouville and Gerasimov–Caputo describe diffusion processes [3, vol. 1, 47–85]. A physical and engineering interpretations of the generalized fractional operators are given in [3, vol. 4–8], [8–13]. In [9], in particular, by the aid of operational calculus of Mikusinski type were studied the problems of existence and representation of solution of initial value problem for the general ordinary linear fractional differential equation with generalized Riemann–Liouville fractional derivatives and constant coefficients.

In [14, 15], the fractional integro-differentiation operators are applied in studying the dielectric relaxation in glass-forming liquids with different chemical compositions. For this a classical Debye-type model was used. This model describes exponential relaxation and it was determined by a first-order differential equation. The ubiquitous feature of the dynamics of supercooled liquids and amorphous polymers is just non-exponential relaxation, which is the result of slow relaxation. To successfully

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describe the relaxation dynamics of glassy materials, the author in [15] proposed a new model of dielectric relaxation containing derivatives and integrals of the noninteger order, which are a natural generalization of the Debye equation.

The theory of boundary value problems for equations of mixed type of fractional order is also one of the intensively developing directions in the general theory of partial differential equations. It should be noted that local and nonlocal boundary value problems for equations of parabolic-hyperbolic and elliptic-hyperbolic types, including various integro-differential operators of fractional order, have been studied by many authors (see works [16–21]).

There are works [22, 23], in which were investigated local and nonlocal boundary value problems for parabolic-hyperbolic equations with Gerasimov–Caputo operator without a loaded part. Similar problems were considered in [24–26] for loaded equations of parabolic type, whose solutions trace was included in various fractional integro-differential operators of Riemann–Liouville, Erdaley–Kober and others types.

When modelling the problems of optimal control of the agroeconomic system for regulating ground-water marks and soil moisture, it became necessary to study boundary value problems for loaded partial differential equations [27, 28]. A large number of publications, in particular, [29–33], have been devoted to study of boundary value problems of various kinds for loaded equations of parabolic, parabolic-hyperbolic and elliptic-hyperbolic and other types.

In the present paper we study the existence and uniqueness of solution of the Gellerstedt type problem with nonlocal boundary and integral gluing conditions for the parabolic-hyperbolic type equation with nonlinear terms and Gerasimov–Caputo operator of differentiation.

2. FORMULATION OF THE PROBLEM

We consider a loaded fractional order parabolic-hyperbolic equation with nonlinear terms:

$$0 = \begin{cases} u_{xx} - {}_CD_{oy}^\alpha u + a_1(x, y)u^{p_1}(x, y) + f_1(x, y; u(x, 0)), & \text{for } y > 0, \\ u_{xx} - u_{yy} + a_2(x, y)u^{p_2}(x, y) + f_2(x, y; u(x, 0)), & \text{for } y < 0, \end{cases} \quad (1)$$

where

$${}_CD_{oy}^\alpha f(y) = \frac{1}{\Gamma(1-\alpha)} \int_0^y (y-t)^{-\alpha} f'(t) dt, \quad 0 < \alpha < 1, \quad (2)$$

$a_i(x, y)$, $f_i(x, y; u(x, 0))$ are given functions, p_i , α are constants and $p_i > 0$, $0 < \alpha < 1$, $i = 1, 2$.

The main goal of this work is to study the unique solvability of a boundary value problem with an integral gluing condition for the equation (1). Let Ω be region, bounded by intervals: $A_1A_2 = \{(x, y) : x = 1, 0 < y < h\}$, $B_1B_2 = \{(x, y) : x = 0, 0 < y < h\}$, $B_2A_2 = \{(x, y) : y = h, 0 < x < 1\}$ for $y > 0$, and by characteristics: $A_1C : x - y = 1$, $B_1C : x + y = 0$ of the equation (1) for $y < 0$, where $A_1 = (1; 0)$, $A_2 = (1; h)$, $B_1 = (0; 0)$, $B_2 = (0; h)$ and $C = (\frac{1}{2}; \frac{-1}{2})$.

We introduce the designations (see Fig. 1):

$$\Omega_1 = \Omega \cap (y > 0), \quad \Omega_2 = \Omega \cap (y < 0), \quad I_1 = \left\{x : 0 < x < \frac{1}{2}\right\}, \quad I_2 = \left\{x : \frac{1}{2} < x < 1\right\}.$$

Problem BV. It is required to find a solution $u(x, y)$ of the equation (1) from the class of functions:

$$W = \{u(x, y) : u(x, y) \in C(\overline{\Omega}) \cap C^2(\Omega_2); u_{xx}, {}_CD_{oy}^\alpha u \in C(\Omega_1); u_x \in C^1(\overline{\Omega}_1 \setminus A_2B_2)\}, \quad (3)$$

satisfying boundary value conditions:

$$u_x(x, y)|_{A_1A_2} = \varphi_1(y), \quad u_x(x, y)|_{B_1B_2} = \varphi_2(y), \quad 0 \leq y < h; \quad (4)$$

$$u(x, y)|_{B_1C} = \psi(x), \quad 0 < x < \frac{1}{2} \quad (5)$$

Fig. 1.

and integral gluing condition:

$$\lim_{y \rightarrow +0} y^{1-\alpha} u_y(x, y) = \lambda_1(x) u_y(x, -0) + \lambda_2(x) u_x(x, -0) + \lambda_3(x) \int_0^x r(t) u(t, 0) dt + \lambda_4(x) u(x, 0) + \lambda_5(x), \quad 0 < x < 1, \quad (6)$$

where $\psi(x)$, $\varphi_i(y)$ ($i = 1, 2$) and $\lambda_k(x)$ ($k = \overline{1, 5}$) are given continuous functions, $\phi_2(0) = \psi'(0)$ and $\sum_{k=1}^4 \lambda_k^2(x) \neq 0$.

Remark 1. As we know, the same above problems for the equation (1) for the integer order $\alpha = 1$ have not been investigated, too. On the another hand, we would like to note, that fundamental solution of equation (1) for $y > 0$, $\alpha = 1$, completely coincides with the fundamental solution of the heat equation $u_{xx} - u_t = 0$. Therefore, all results in this work remain valid in case of integer order $\alpha = 1$, too.

3. FUNCTIONAL RELATIONS FOR SOLUTION

We note that solution of the Cauchy–Goursat problem with condition (5) and $u(x, 0) = \tau(x)$ for equation (1) in Ω_2 has the form

$$u(x, y) = \tau(x + y) - \frac{1}{4} \int_{x+y}^{x-y} d\eta \int_{x+y}^0 a_2 \left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2} \right) u^{p_2} \left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2} \right) d\xi - \frac{1}{4} \int_{x+y}^{x-y} d\eta \int_{x+y}^0 f_2 \left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}; \tau \left(\frac{\xi + \eta}{2} \right) \right) d\xi + \psi \left(\frac{x - y}{2} \right) - \psi \left(\frac{x + y}{2} \right). \quad (7)$$

Further, from the equation (1) as $y \rightarrow +0$ taking into account (2), (6) and

$$\lim_{y \rightarrow 0} D_{0y}^{\alpha-1} f(y) = \Gamma(\alpha) \lim_{y \rightarrow 0} y^{1-\alpha} f(y),$$

we derive

$$\tau''(x) - \Gamma(\alpha) \lambda_1(x) u_y(x, -0) - \Gamma(\alpha) \lambda_2(x) \tau'(x) - \Gamma(\alpha) \lambda_3(x) \int_0^x r(t) \tau(t) dt - \Gamma(\alpha) \lambda_4(x) \tau(x) - \Gamma(\alpha) \lambda_5(x) + a_1(x, 0) \tau^{p_1}(x) + f_1(x, 0; \tau(x)) = 0, \quad 0 < x < 1.$$

Hence, taking into account (see the equation (7)) the following relation

$$u_y(x, -0) = \tau'(x) - \psi' \left(\frac{x}{2} \right) + \frac{1}{2} \int_0^x a_2 \left(\frac{\xi + x}{2}, \frac{\xi - x}{2} \right) u^{p_2} \left(\frac{\xi + x}{2}, \frac{\xi - x}{2} \right) d\xi$$

$$+ \frac{1}{2} \int_0^x f_2 \left(\frac{\xi+x}{2}, \frac{\xi-x}{2}; \tau \left(\frac{\xi+x}{2} \right) \right) d\xi,$$

we obtain

$$\begin{aligned} & \tau''(x) - \Gamma(\alpha)(\lambda_1(x) + \lambda_2(x))\tau'(x) - \Gamma(\alpha)\lambda_4(x)\tau(x) - \Gamma(\alpha)\lambda_3(x) \int_0^x r(t)\tau(t)dt \\ & - \frac{\Gamma(\alpha)}{2}\lambda_1(x) \int_0^x f_2 \left(\frac{\xi+x}{2}, \frac{\xi-x}{2}; \tau \left(\frac{\xi+x}{2} \right) \right) d\xi + a_1(x, 0)\tau^{p_1}(x) + f_1(x, 0; \tau(x)) \\ & = \frac{\Gamma(\alpha)}{2}\lambda_1(x) \int_0^x a_2 \left(\frac{\xi+x}{2}, \frac{\xi-x}{2} \right) u^{p_2} \left(\frac{\xi+x}{2}, \frac{\xi-x}{2} \right) d\xi \\ & + \Gamma(\alpha)\lambda_5(x) - \Gamma(\alpha)\lambda_1(x)\psi' \left(\frac{x}{2} \right), \quad 0 < x < 1. \end{aligned} \quad (8)$$

By virtue of properties of functions from the class (3) for solution of the problem *BV*, we have

$$\tau(0) = \psi(0), \quad \tau'(0) = \varphi_2(0). \quad (9)$$

By integration the equation (8) with initial value conditions (9) we derive

$$\begin{aligned} & \tau(x) - \Gamma(\alpha) \int_0^x K_1(x, t)\tau(t)dt - \frac{\Gamma(\alpha)}{2} \int_0^x (x-t)\lambda_1(t)dt \int_0^t f_2 \left(\frac{\xi+t}{2}, \frac{\xi-t}{2}; \tau \left(\frac{\xi+t}{2} \right) \right) d\xi \\ & + \int_0^x (x-t)a_1(t, 0)\tau^{p_1}(t)dt + \int_0^x (x-t)f_1(t, 0; \tau(t))dt \\ & = \frac{\Gamma(\alpha)}{2} \int_0^x (x-t)\lambda_1(t)dt \int_0^t a_2 \left(\frac{\xi+t}{2}, \frac{\xi-t}{2} \right) u^{p_2} \left(\frac{\xi+t}{2}, \frac{\xi-t}{2} \right) d\xi + g_1(x), \quad 0 < x < 1, \end{aligned} \quad (10)$$

where

$$K_1(x, t) = [(x-t)(\lambda_1(t) + \lambda_2(t))] - (x-t)\lambda_4(t) - r(t) \int_t^x \lambda_3(s)(x-s)ds, \quad (11)$$

$$g_1(x) = \Gamma(\alpha) \int_0^x (x-t)\lambda_5(t)dt - \Gamma(\alpha) \int_0^x (x-t)\lambda_1(t)\psi' \left(\frac{t}{2} \right) dt + \varphi_2(0)x + \psi(0). \quad (12)$$

4. EXISTENCE AND UNIQUENESS OF SOLUTION OF THE PROBLEM *BV*

Theorem 1. *We suppose that $p_i = \text{const} > 1$ and the following conditions are fulfilled:*

$$a_i(x, y) \in C(\overline{\Omega}_i) \cap C^1(\Omega_i), \quad \varphi_i(y) \in C[0, h] \cap C^1(0, h), \quad \psi(x) \in C\left[0, \frac{1}{2}\right] \cap C^2\left(0, \frac{1}{2}\right), \quad (13)$$

$$f_i(x, y, u(x, 0)) \in C(\overline{\Omega}_i) \cap C^1(\Omega_i), \quad \lambda_k(x) \in C[0, 1] \cap C^1(0, 1) \quad (k = 1, 2, \dots, 5), \quad (14)$$

$$|f_i(x, (i-1)y; u_2(x, 0)) - f_i(x, (i-1)y; u_1(x, 0))| \leq L_i |u_2(x, 0) - u_1(x, 0)|, \quad (x, y) \in \Omega_i, \quad (15)$$

where $L_i = \text{const} > 0 (i = 1, 2)$. Then the problem has a unique solution.

Proof. By virtue of properties (13) and (14), taking designations (11) and (12) into account we have estimates:

$$\|K_1(x, t)\|_C \leq M_1, \quad \|g_1(x, t)\|_C \leq g_{10}, \quad \|f_i(x, (i-1)y; u(x, 0))\|_C \leq f_{i0}, \quad (16)$$

where $M_1, g_{10}, f_{i0} = \text{const} > 0$ ($i = 1, 2$). The equations (10) and (7) we consider as a system of nonlinear second kind integral equations of Volterra type with respect to unknown functions $\tau(x)$ and $u(x, y)$ for $y \leq 0$:

$$\left\{ \begin{aligned} u(x, y) &= \mathbb{A}(x, y, u; \tau) \equiv -\frac{1}{4} \int_{x+y}^{x-y} d\eta \int_{x+y}^0 a_2 \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2} \right) u^{p_2} \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2} \right) d\xi \\ &+ \tau(x+y) - \frac{1}{4} \int_{x+y}^{x-y} d\eta \int_{x+y}^0 f_2 \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}; \tau \left(\frac{\xi+\eta}{2} \right) \right) d\xi + \psi \left(\frac{x-y}{2} \right) - \psi \left(\frac{x+y}{2} \right); \\ \tau(x) &= \mathbb{B}(x, y, u; \tau) \equiv \Gamma(\alpha) \int_0^x K_1(x, t) \tau(t) dt - \int_0^x (x-t) a_1(t, 0) \tau^{p_1}(t) dt \\ &+ \frac{\Gamma(\alpha)}{2} \int_0^x (x-t) \lambda_1(t) dt \int_0^t f_2 \left(\frac{\xi+t}{2}, \frac{\xi-t}{2}; \tau \left(\frac{\xi+t}{2} \right) \right) d\xi - \int_0^x (x-t) f_1(t, 0; \tau(t)) dt \\ &+ \frac{\Gamma(\alpha)}{2} \int_0^x (x-t) \lambda_1(t) dt \int_0^t a_2 \left(\frac{\xi+t}{2}, \frac{\xi-t}{2} \right) u^{p_2} \left(\frac{\xi+t}{2}, \frac{\xi-t}{2} \right) d\xi + g_1(x), \quad 0 < x < 1. \end{aligned} \right. \quad (17)$$

We define a sequence of functions $\tau_n(x)$ and $u_n(x, y)$ ($n = 0, 1, \dots$) from the following system of recurrent equations:

$$\left\{ \begin{aligned} u_0(x, y) &= g_1(x) + \psi \left(\frac{x-y}{2} \right) - \psi \left(\frac{x+y}{2} \right), \quad u_n(x, y) = \mathbb{A}(x, y, u_{n-1}; \tau_{n-1}); \\ \tau_0(x) &= g_1(x), \quad \tau_n(x) = \mathbb{B}(x, y, u_{n-1}; \tau_{n-1}). \end{aligned} \right. \quad (18)$$

By virtue of properties (13) and estimates (16), we have following estimates:

$$\|a_i(x, y)\|_C \leq m_i, \quad \|\lambda_1(x)\|_C \leq \lambda_{10}, \quad \|\tau_0(x)\|_C \leq c_{11}, \quad \|u_0(x, y)\|_C \leq c_{21},$$

where $m_i, \lambda_{10}, c_{i1} = \text{const} > 0$ ($i = 1, 2$). Further, taking these estimates into account, from the iteration process (18) we derive

$$\left\{ \begin{aligned} \|\tau_1(x) - \tau_0(x)\|_C &\leq \Gamma(\alpha) M_1 c_{11} x + (m_1 c_{11}^{p_1} + f_{10}) x^2 + \frac{\Gamma(\alpha)}{2} \lambda_{10} (f_{20} + m_2 c_{21}^{p_2}) x^3 \leq \beta x, \\ \|u_1(x, y) - u_0(x, y)\|_C &\leq \frac{1}{2} (m_2 c_{21}^{p_2} + f_{20}) |x+y| \leq \frac{\gamma}{2} |x+y|, \end{aligned} \right. \quad (19)$$

where

$$\beta = \max \left\{ \Gamma(\alpha) M_1 c_{11}, \quad m_1 c_{11}^{p_1} + f_{10}, \quad \frac{\Gamma(\alpha)}{2} \lambda_{10} (f_{20} + m_2 c_{21}^{p_2}) \right\}, \quad \gamma = m_2 c_{21}^{p_2} + f_{20}.$$

To continue the proof of the Theorem 1 we need on following lemma.

Lemma 1. *If the Lipschitz condition (15) is fulfilled for $j = 2$ and*

$$\|\tau_2(x) - \tau_1(x)\|_C \leq \beta^* \frac{x^k}{k!} \quad (20)$$

are true, then there holds

$$\left\| \int_{x+y}^{x-y} d\eta \int_0^0 \left| f_2 \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}; \tau_2 \left(\frac{\xi+\eta}{2} \right) \right) - f_2 \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}; \tau_1 \left(\frac{\xi+\eta}{2} \right) \right) \right| d\xi \right\|_C \leq \frac{L_2 \beta^*}{k!} x^k, \quad (x, y) \in \overline{\Omega}_2, \quad (21)$$

where $\beta^* = \text{const} > 0, k$ is a fixed natural number.

Proof. By virtue of Lipschitz condition (15) and estimate (20), we have

$$\begin{aligned} & \left\| \int_{x+y}^{x-y} d\eta \int_0^{x+y} f_2 \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}; \tau_2 \left(\frac{\xi+\eta}{2} \right) \right) d\xi - \int_{x+y}^{x-y} d\eta \int_0^{x+y} f_2 \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}; \tau_1 \left(\frac{\xi+\eta}{2} \right) \right) d\xi \right\|_C \\ & \leq L_2 \left\| \int_{x+y}^{x-y} d\eta \int_0^{x+y} \left| \tau_2 \left(\frac{\xi+\eta}{2} \right) - \tau_1 \left(\frac{\xi+\eta}{2} \right) \right| d\xi \right\|_C \\ & \leq \frac{L_2 \beta^*}{k!} \left\| \int_{x+y}^{x-y} d\eta \int_0^{x+y} \left| \frac{\xi+\eta}{2} \right|^k d\xi \right\|_C \leq \frac{L_2 \beta^*}{k!} x^k, \quad 0 \leq x \leq 1. \end{aligned}$$

So, the Lemma is proved. $\square$

Further, we take into account that for a functions $F(\tau) = \tau^p(x)$ and $G(u) = u^p(x, y)$ in the case $p > 1$ Lipschitz conditions hold:

$$\begin{aligned} \|\tau_2^{p_1}(x) - \tau_1^{p_1}(x)\|_C &\leq c_{12} \|\tau_2(x) - \tau_1(x)\|_C, \\ \|u_2^{p_2}(x, y) - u_1^{p_2}(x, y)\|_C &\leq c_{22} \|u_2(x, y) - u_1(x, y)\|_C. \end{aligned}$$

Therefore, taking estimates (19) into account, from successive approximations (18) we obtain

$$\|\tau_2(x) - \tau_1(x)\|_C \leq \beta \Gamma(\alpha) M \frac{x^2}{2!} + (\beta m_1 c_{12} + \beta L_1) \frac{x^3}{3!} + \frac{\Gamma(\alpha)}{2} \lambda_{10} (\beta L_2 + \gamma m_2 c_{22}) \frac{x^4}{4!},$$

where $c_{12}, c_{22} = \text{const} > 0$. We put $L_2 \lambda_{10} \Gamma(\alpha) = \max\{m_2 c_{22}; m_1 c_{12} + L_1\}$ and $\beta = \Gamma(\alpha) \lambda_{10} (f_{20} + m_2 c_{21}^{p_2})$. Then assuming that $M_1 < L_2 \lambda_{10}$ (you can always get it by imposing a condition on the function $\lambda_1(x)$) from last estimate, we have

$$\|\tau_2(x) - \tau_1(x)\|_C \leq 3 \cdot 4 \beta \Gamma(\alpha) L_2 \lambda_{10} \frac{x^2}{2!}, \quad 0 \leq x \leq 1. \quad (22)$$

Taking the estimate (21) into account, from successive approximations (18) yields

$$\|u_2(x, y) - u_1(x, y)\|_C \leq 3\beta|x+y| + \frac{1}{4} m_2 c_{22} \gamma \frac{(x+y)^2}{2} + \frac{3}{4} L_2 \beta |x| \leq \beta(L_2 + 3)|x|, \quad (x, y) \in \overline{\Omega}_2.$$

Taking also the estimate (22) and last inequality into account, from iteration process (18) we obtain

$$\begin{aligned} \|\tau_3(x) - \tau_2(x)\|_C &\leq \text{const} \cdot 4^2 \beta \Gamma^2(\alpha) \lambda_{10}^2 L_2^2 \frac{x^3}{3!}, \quad 0 \leq x \leq 1, \\ \|u_3(x, y) - u_2(x, y)\|_C &\leq \text{const} \cdot 4 \beta \Gamma(\alpha) \lambda_{10} L_2^2 \frac{x^2}{2!}, \quad (x, y) \in \overline{\Omega}_2. \end{aligned}$$

Continuing the above reasoning for arbitrary n , we have

$$\begin{aligned} \|\tau_n(x) - \tau_{n-1}(x)\|_C &\leq \text{const} \cdot 4^{n-1} \beta \Gamma^{n-1}(\alpha) \lambda_{10}^{n-1} L_2^{n-1} \frac{x^n}{n!}, \quad 0 \leq x \leq 1, \\ \|u_n(x, y) - u_{n-1}(x, y)\|_C &\leq \text{const} \cdot \beta 4^{n-2} \Gamma^{n-2}(\alpha) \lambda_{10}^{n-2} L_2^{n-2} \frac{x^{n-1}}{(n-1)!}, \quad (x, y) \in \overline{\Omega}_2. \end{aligned}$$

By virtue of the obtained estimates, we conclude that the functional sequences of functions $\{\tau_n(x)\}_{n=1}^\infty$ and $\{u_n(x, y)\}_{n=1}^\infty$ has a unique limit functions $\tau(x)$ and $u(x, y)$:

$$\lim_{n \rightarrow \infty} \tau_n(x) = \tau(x), \quad \lim_{n \rightarrow \infty} u_n(x, y) = u(x, y).$$

Thus, the existence of a solution of the system (17) has been proved.

Now we prove the uniqueness of the solution of the system (17). We suppose that this system (17) have two solutions $(u_1(x, y); \tau_1(x))$ and $(u_2(x, y); \tau_2(x))$. Then introducing the following designations $u(x, y) = u_1(x, y) - u_2(x, y)$ and $\tau(x) = \tau_1(x) - \tau_2(x)$, from this system (17) we obtain

$$\left\{ \begin{aligned} & u(x, y) + \frac{1}{4} \int_{x+y}^{x-y} d\eta \int_{x+y}^0 a_2 \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2} \right) \left| u_1^{p_2} \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2} \right) - u_2^{p_2} \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2} \right) \right| d\xi \\ & = \tau(x+y) - \frac{1}{4} \int_{x+y}^{x-y} d\eta \int_{x+y}^0 \left| f_2 \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}; \tau_1 \left(\frac{\xi+\eta}{2} \right) \right) - f_2 \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}; \tau_2 \left(\frac{\xi+\eta}{2} \right) \right) \right| d\xi; \\ & \tau(x) - \Gamma(\alpha) \int_0^x K_1(x, t) \tau(t) dt + \int_0^x (x-t) a_1(t, 0) \left| \tau_1^{p_1}(t) - \tau_2^{p_1}(t) \right| dt \\ & - \frac{\Gamma(\alpha)}{2} \int_0^x (x-t) \lambda_1(t) dt \int_0^t \left| f_2 \left(\frac{\xi+t}{2}, \frac{\xi-t}{2}; \tau_1 \left(\frac{\xi+t}{2} \right) \right) - f_2 \left(\frac{\xi+t}{2}, \frac{\xi-t}{2}; \tau_2 \left(\frac{\xi+t}{2} \right) \right) \right| d\xi \\ & + \int_0^x (x-t) \left| f_1(t, 0; \tau_1(t)) - f_1(t, 0; \tau_2(t)) \right| dt \\ & = \frac{\Gamma(\alpha)}{2} \int_0^x (x-t) \lambda_1(t) dt \int_0^t a_2 \left(\frac{\xi+t}{2}, \frac{\xi-t}{2} \right) \left| u_1^{p_2} \left(\frac{\xi+t}{2}, \frac{\xi-t}{2} \right) - u_2^{p_2} \left(\frac{\xi+t}{2}, \frac{\xi-t}{2} \right) \right| d\xi, \quad 0 < x < 1. \end{aligned} \right. \quad (23)$$

Taking into account the class of given functions of the system (23), we have

$$\begin{aligned} |u(x, y)| & \leq \frac{m_2 c_{22}}{4} \left| \int_{x+y}^{x-y} d\eta \int_{x+y}^0 \left| u \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2} \right) \right| d\xi \right| \\ & + |\tau(x+y)| + \frac{L_2}{4} \left| \int_{x+y}^{x-y} d\eta \int_{x+y}^0 \left| \tau \left(\frac{\xi+\eta}{2} \right) \right| d\xi \right|, \\ |\tau(x)| & \leq \Gamma(\alpha) M_1 \left| \int_0^x \tau(t) dt \right| + (m_1 c_{12} + L_1) \left| \int_0^x (x-t) \tau(t) dt \right| \\ & + \frac{\Gamma(\alpha) L_2}{2} \lambda_{10} \left| \int_0^x (x-t) dt \int_0^t \left| \tau \left(\frac{\xi+t}{2} \right) \right| d\xi \right| \\ & + \frac{\Gamma(\alpha) c_{22} m_2}{2} \lambda_{10} \left| \int_0^x (x-t) dt \int_0^t \left| u \left(\frac{\xi+t}{2}, \frac{\xi-t}{2} \right) \right| d\xi \right|. \end{aligned}$$

We assume that $0 < \max \left\{ \frac{m_2 c_{22}}{4}, \frac{L_2}{4} \right\} = \delta$ is small number. Then from the last inequalities we obtain estimates

$$|u(x, y)| \leq \delta |I_1(u, \tau)| + |\tau(x+y)|, \quad |\tau(x)| \leq 2\delta \lambda_{10} \Gamma(\alpha) |I_2(u, \tau)|, \quad (24)$$

where

$$\begin{aligned} I_1(u, \tau) & = \int_{x+y}^{x-y} d\eta \int_{x+y}^0 \left| \tau \left(\frac{\xi+\eta}{2} \right) + u \left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2} \right) \right| d\xi, \\ I_2(u, \tau) & = \int_0^x (x-t) dt \int_0^t \left| \tau \left(\frac{\xi+t}{2} \right) + u \left(\frac{\xi+t}{2}, \frac{\xi-t}{2} \right) \right| d\xi. \end{aligned}$$

Strengthening the first estimate of (24), taking into account the second one of (24), we have

$$\begin{cases} |u(x, y)| \leq \delta |I_1(u, \tau)| + 2\delta \lambda_{10} \Gamma(\alpha) |I_3(u, \tau)|, \\ |\tau(x)| \leq 2\delta \lambda_{10} \Gamma(\alpha) |I_2(u, \tau)|, \end{cases} \quad (25)$$

where

$$I_3(u, \tau) = \int_0^{x+y} (x+y-t) dt \int_0^t \left| \tau\left(\frac{\xi+t}{2}\right) + u\left(\frac{\xi+t}{2}, \frac{\xi-t}{2}\right) \right| d\xi.$$

Introducing the designation $|v(x, y)| = |u(x, y)| + |\tau(x)|$, from (25) we obtain the following estimate

$$|v(x, y)| = |u(x, y)| + |\tau(x)| \leq \delta |I_1(v)| + 2\delta\lambda_{10}\Gamma(\alpha) |I_3(v)| + 2\delta\lambda_{10}\Gamma(\alpha) |I_2(v)|.$$

We take into account that $|I_3(v)| \leq |I_2(v)|$ and

$$\begin{aligned} |I_1(v)| &\leq \left| \int_0^{x-y} dt \int_0^{x+y} \left| v\left(\frac{\xi+t}{2}, \frac{\xi-t}{2}\right) \right| d\xi \right| = \left| \int_0^{x-y} d\xi \int_\xi^{x-y} \left| v\left(\frac{\xi+t}{2}, \frac{\xi-t}{2}\right) \right| dt \right|, \\ |I_2(v)| &\leq \left| \int_0^x dt \int_0^t \left| v\left(\frac{\xi+t}{2}, \frac{\xi-t}{2}\right) \right| d\xi \right| \leq \left| \int_0^{x-y} dt \int_0^t \left| v\left(\frac{\xi+t}{2}, \frac{\xi-t}{2}\right) \right| d\xi \right| \\ &= \left| \int_0^{x-y} d\xi \int_\xi^{x-y} \left| v\left(\frac{\xi+t}{2}, \frac{\xi-t}{2}\right) \right| dt \right| \equiv I(v). \end{aligned}$$

From the last estimate we have

$$|v(x, y)| \leq \delta_0 |I_1(v)| + \delta_0 |I_2(v)| \leq 2\delta_0 |I(v)|, \quad (26)$$

where $\delta_0 = \max\{\delta, 4\delta\lambda_{10}\Gamma(\alpha)\}$. Taking into account that $I(v)$ is a linear operator with respect to function $v(x, y)$, from the estimate (26) we obtain

$$|v(x, y)| \leq (2\delta_0)^n |I^n(v)|, \quad (x, y) \in \Omega^-. \quad (27)$$

For the estimation $|I^2(v)|$ we have

$$\begin{aligned} |I^2(v)| &\leq \left| \int_0^{x-y} dt \int_0^t d\xi \int_0^t ds \int_s^t \left| v\left(\frac{s+z}{2}, \frac{s-z}{2}\right) \right| dz \right| \\ &\leq \left| \int_0^{x-y} t dt \int_0^t ds \int_s^t \left| v\left(\frac{s+z}{2}, \frac{s-z}{2}\right) \right| dz \right| \leq \text{const} \cdot \frac{|x-y|^2}{2} \left| \int_0^{x-y} ds \int_s^{x-y} \left| v\left(\frac{s+z}{2}, \frac{s-z}{2}\right) \right| dz \right| \\ &\leq \text{const} \cdot \frac{|x-y|^2}{2} I(v). \end{aligned}$$

It is easy to check that

$$|I^n(v)| \leq \text{const} \cdot \frac{|x-y|^{2n-2}}{2^{\sigma_{n-1}}} \left| \int_0^{x-y} d\xi \int_\xi^{x-y} \left| v\left(\frac{\xi+t}{2}, \frac{\xi-t}{2}\right) \right| dt \right|,$$

where $\sigma_n = 2\sigma_{n-1} + n$ and $\sigma_0 = 0$.

As $n \rightarrow \infty$ from (27) implies that $|v(x, y)| = 0$. Consequently, from the designation $|v(x, y)| = |u(x, y)| + |\tau(x)| = 0$, we come to identities $|u(x, y)| \equiv 0$ and $|\tau(x)| \equiv 0$. Hence, $u_1(x, y) = u_2(x, y)$ and $\tau_1(x) = \tau_2(x)$. Consequently, the solution of the system (17) is unique.

After determination $\tau(x)$ we restore the unique solution of the considering problem BV in the domain Ω_2 as a solution of the Cauchy–Goursat problem (see the equation (7)). Further, we take the existence

of function $\tau(x)$ into account and use the solution of second boundary value problem for the equation (1) in domain Ω_1 [34]:

$$u(x, y) = \int_0^y G_\xi(x, y, 0, \eta) \varphi_2(\eta) d\eta - \int_0^y G_\xi(x, y, 1, \eta) \varphi_1(\eta) d\eta + \int_0^1 G_0(x - \xi, y) \tau(\xi) d\xi \\ - \int_0^y \int_0^1 G(x, y, \xi, \eta) f_1(\xi, \eta; \tau(\xi)) d\xi d\eta - \int_0^y \int_0^1 G(x, y, \xi, \eta) a_1(\xi, \eta) u^{p_1}(\xi, \eta) d\xi d\eta.$$

Then we consider the existence of the problem BV for equation (1) in domain Ω_1 , where

$$G_0(x - \xi, y) = \frac{1}{\Gamma(1 - \alpha)} \int_0^y (y - \eta)^{-\alpha} G(x, \eta, \xi, 0) d\eta, \\ G(x, y, \xi, \eta) = \frac{(y - \eta)^{\alpha/2 - 1}}{2} \sum_{n=-\infty}^{\infty} \left[e_{1, \alpha/2}^{1, \alpha/2} \left(-\frac{|x - \xi + 2n|}{(y - \eta)^{\alpha/2}} \right) - e_{1, \alpha/2}^{1, \alpha/2} \left(-\frac{|x + \xi + 2n|}{(y - \eta)^{\alpha/2}} \right) \right]$$

is Green's function of the second boundary value problem for the equation (1) in the domain Ω_1 [18, 34],

$$e_{1, \delta}^{1, \delta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\delta - \delta n)}$$

is Wright type function.

Remark 2. Notice that the Wright type function $e_{1, \delta}^{1, \delta}(z)$ is partial case of the generalized (Fox-) Wright function ${}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{matrix} (a_i, \alpha_i)_{1, p} \\ (b_i, \beta_i)_{1, q} \end{matrix} \middle| z \right]$, when $p = 0, q = 2, b_1 = 0, \beta_1 = 1$ (see[34]).

It is investigated the last equation as a nonlinear Volterra type integral equation of the second kind

$$u(x, y) + \int_0^y \int_0^1 K_2(\xi, \eta) u^{p_1}(\xi, \eta) d\xi d\eta = F(x, y)$$

by well known methods from the work [35], where

$$F(x, y) = \int_0^y G_\xi(x, y, 0, \eta) \varphi_2(\eta) d\eta - \int_0^y G_\xi(x, y, 1, \eta) \varphi_1(\eta) d\eta + \int_0^1 G_0(x - \xi, y) \tau(\xi) d\xi \\ - \int_0^y \int_0^1 G(x, y, \xi, \eta) f_1(\xi, \eta; \tau(\xi)) d\xi d\eta.$$

The Theorem 1 is proved. □

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