Initial-boundary Value Problem for a Time-fractional Subdiffusion Equation with an Arbitrary Elliptic Differential Operator

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Abstract—An initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary order elliptic differential operator is considered. Uniqueness and existence of the classical solution of the posed problem are proved by the classical Fourier method. Sufficient conditions for the initial function and for the right-hand side of the equation are indicated, under which the corresponding Fourier series converge absolutely and uniformly. In the case of an initial-boundary value problem on N -dimensional torus, one can easily see that these conditions are not only sufficient, but also necessary.

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1. MAIN RESULTS

The theory of differential equations with fractional derivatives has gained considerable popularity and importance in the past few decades, mainly due to its applications in numerous seemingly distant fields of science and technology (see, for example, [1–8]). As the most important applications of this theory one may consider the recent investigations on modeling of COVID-19 outbreak [2, 3]. The data [9] presented by Johns Hopkins University on the outbreak from different countries seem to show fractional order dynamical processes.

In turn, the well-deserved popularity of the theory attracted the attention of specialists, causing a large number of investigations on mathematical aspects of fractional differential equations and methods to solve them (see, for example, [1] and references therein, [10–13]). Interesting results were obtained in [14–16]. The Fourier method is widely used in modern studies of partial differential equations and optimal control (see, for example [17–21]).

In this paper we will investigate by the Fourier method the solvability (in the classical sense) of initial-boundary value problems for a time-fractional subdiffusion equation with elliptic differential operators of any order, defined on an arbitrary N -dimensional bounded domain Ω with smooth boundary $\partial\Omega$. The fractional part of our equation will be the Riemann–Liouville fractional derivative ∂_t^ρ of order $0 < \rho \leq 1$.

Many special cases of this problem have been considered by a number of authors using different methods. It has been mainly considered the case of one spatial variable $x \in \mathbb{R}$ and subdiffusion equation with “elliptical part” u_{xx} (see, for example, handbook [1], book of A.A. Kilbas et al. [6] and monograph of A.V. Pskhu [11], and references in these works). In multidimensional case ($x \in \mathbb{R}^N$) instead of the differential expression u_{xx} it has been considered either the Laplace operator [6, 22], or the elliptic differential or pseudo-differential operator in the whole space $\mathbb{R}^N$ with constant coefficients [10]. In both cases the authors investigated the Cauchy type problems applying either the Laplace transform or the Fourier transform. In his recent paper [23] Pskhu considered an initial-boundary value problem for

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subdiffusion equation with the Laplace operator and domain Ω - a multidimensional rectangular region. The author succeeded to construct the Green's function. Naturally, based on the physical meaning of the problem, it is of interest to consider similar problems for arbitrary bounded spatial domains Ω .

In an arbitrary domain Ω initial-boundary value problems for subdiffusion equations (the fractional part of the equation is a multi-term and initial conditions are non-local) with the Caputo derivatives has been investigated by M. Ruzhansky et al. [24]. The authors proved the existence and uniqueness of the generalized solution to the problem.

Let $A(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ be an arbitrary positive formally selfadjoint (symmetric) elliptic differential operator of order $m = 2l$ with sufficiently smooth coefficients $a_\alpha(x)$ in Ω , where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ - multi-index and $D = (D_1, D_2, \dots, D_N)$, $D_j = \frac{\partial}{\partial x_j}$. Recall, an operator $A(x, D)$ is elliptic in Ω , if for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$ one has

$$\sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha > 0 \quad \xi \neq 0.$$

The fractional integration of order $\rho < 0$ of a function f defined on $[0, \infty)$ in the Riemann–Liouville sense is defined by the formula

$$\partial_t^\rho f(t) = \frac{1}{\Gamma(-\rho)} \int_0^t \frac{f(\xi)}{(t-\xi)^{\rho+1}} d\xi, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\rho)$ is Euler's gamma function. Using this definition one can define the Riemann–Liouville fractional derivative of order ρ , $k-1 < \rho \leq k$, $k \in \mathbb{N}$, as (see, for example, [11, p. 14])

$$\partial_t^\rho f(t) = \frac{d^k}{dt^k} \partial_t^{\rho-k} f(t).$$

Note if $\rho = k$, then fractional derivative coincides with the ordinary classical derivation: $\partial_t^k f(t) = \frac{d^k}{dt^k} f(t)$.

Let $\rho \in (0, 1]$ be a constant number. Consider the differential equation

$$\partial_t^\rho u(x, t) + A(x, D)u(x, t) = f(x, t), \quad x \in \Omega, \quad t > 0; \quad (1)$$

with initial

$$\lim_{t \rightarrow 0} \partial_t^{\rho-1} u(x, t) = \varphi(x), \quad x \in \Omega; \quad (2)$$

and boundary

$$B_j u(x, t) = \sum_{|\alpha| \leq m_j} b_{\alpha,j}(x) D^\alpha u(x, t) = 0, \quad 0 \leq m_j \leq m-1, \quad j = 1, 2, \dots, l; \quad x \in \partial\Omega; \quad (3)$$

conditions, where $f(x, t)$, $\varphi(x)$ and coefficients $b_{\alpha,j}(x)$ are given functions.

Definition 1. A function $u(x, t)$ with the properties $\partial_t^\rho u(x, t), A(x, D)u(x, t) \in C(\bar{\Omega} \times (0, \infty))$, $\partial_t^{\rho-1} u(x, t) \in C(\bar{\Omega} \times [0, \infty))$ and satisfying all the conditions of problem (1)–(3) in the classical sense is called **the regular solution** of initial-boundary value problem (1)–(3).

We draw attention to the fact, that in this definition the requirement of continuity in the closed domain of all derivatives of a solution, included in equation (1), is not caused by the merits. However, on the one hand, the uniqueness of just such a solution is proved quite simply, and on the other, the solution found by the Fourier method satisfies the above conditions.

We will also call the regular solution simply the solution to the boundary value problem.

Application of the Fourier method to problem (1)–(3) leads us to consider the following spectral problem

$$A(x, D)v(x) = \lambda v(x) \quad x \in \Omega; \quad (4)$$

$$B_j v(x) = 0, \quad j = 1, 2, \dots, l; \quad x \in \partial\Omega. \quad (5)$$

S. Agmon [25] found the necessary conditions for boundary $\partial\Omega$ of the domain Ω and for coefficients of operators A and B_j , which guarantee compactness of the inverse operator, i.e. the existence of a complete in $L_2(\Omega)$ system of orthonormal eigenfunctions $\{v_k(x)\}$ and a countable set of positive eigenvalues λ_k of spectral problem (4)–(5). We will call these conditions as *condition (A)*.

In accordance with the Fourier method, we will look for a solution to problem (1)–(3) in the form of a series

$$u(x, t) = \sum_{j=1}^{\infty} T_j(t) v_j(x),$$

where functions $T_j(t)$ are solutions of the Cauchy type problem

$$\partial_t^\rho T_j + \lambda_j T_j = f_j(t), \quad \lim_{t \rightarrow 0} \partial_t^{\rho-1} T_j(t) = \varphi_j. \quad (6)$$

Here we denoted by $f_j(t)$ and φ_j the Fourier coefficients of functions $f(x, t)$ and $\varphi(x)$ with respect to the system of eigenfunctions $\{v_k(x)\}$ correspondingly, defined as a scalar product on $L_2(\Omega)$, i.e. for example, $\varphi_j = (\varphi, v_j)$. The unique solution of problem (6) has the form (see, for example, [11, p. 16])

$$T_j(t) = \varphi_j t^{\rho-1} E_{\rho, \rho}(-\lambda_j t^\rho) + \int_0^t f_j(t-\xi) \xi^{\rho-1} E_{\rho, \rho}(-\lambda_j \xi^\rho) d\xi, \quad (7)$$

where $E_{\rho, \mu}$ is the Mittag-Leffler function of the form

$$E_{\rho, \mu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k + \mu)}.$$

A uniqueness result for the solution of problem (1)–(3) can be formulated as

Theorem 1. *Let condition (A) be satisfied and $f(x, t) \in C(\bar{\Omega} \times (0, \infty))$ and $\varphi \in C(\bar{\Omega})$. Then problem (1)–(3) may have only one regular solution.*

Throughout what follows, we assume that the condition (A) is satisfied.

To formulate the existence theorem we need to introduce for any real number τ an operator $\hat{A}^\tau$, acting in $L_2(\Omega)$ in the following way

$$\hat{A}^\tau g(x) = \sum_{k=1}^{\infty} \lambda_k^\tau g_k v_k(x), \quad g_k = (g, v_k).$$

Obviously, operator $\hat{A}^\tau$ with the domain of definition

$$D(\hat{A}^\tau) = \left\{ g \in L_2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |g_k|^2 < \infty \right\}$$

is selfadjoint. If we denote by A the operator in $L_2(\Omega)$, acting as $Ag(x) = A(x, D)g(x)$ and with the domain of definition $D(A) = \{g \in C^m(\bar{\Omega}) : B_j g(x) = 0, \quad j = 1, \dots, l, \quad x \in \partial\Omega\}$, then it is not hard to show, that operator $\hat{A} \equiv \hat{A}^1$ is a selfadjoint extension in $L_2(\Omega)$ of operator A .

Theorem 2. *Let $\tau > \frac{N}{2m}$ and $\varphi \in D(\hat{A}^\tau)$. Moreover, let $f(x, t) \in D(\hat{A}^\tau)$ for all $t \in [0, T]$, and the function $F(t) = \hat{A}^\tau f(x, t)$ be continuous in the norm of space $L_2(\Omega)$ for all $t \in [0, T]$. Then there exists a solution of initial-boundary value problem (1)–(3) and it has the form of series*

$$u(x, t) = \sum_{j=1}^{\infty} \left[\varphi_j t^{\rho-1} E_{\rho, \rho}(-\lambda_j t^\rho) + \int_0^t f_j(t-\xi) \xi^{\rho-1} E_{\rho, \rho}(-\lambda_j \xi^\rho) d\xi \right] v_j(x), \quad (8)$$

which absolutely and uniformly converges on $x \in \bar{\Omega}$ and for each $t \in (0, T]$.

Note, when $\rho = 1$ this theorem states existence of the unique solution of the problem, when the equation and initial condition have the form

$$u_t(x, t) + A(x, D)u(x, t) = f(x, t), \quad x \in \Omega, \quad t > 0; \quad u(x, 0) = \varphi(x), \quad x \in \Omega,$$

and the boundary condition is the same as (3).

It should be noted that Sh. A. Alimov in his paper [27] presents sufficient conditions for a given function to belong to the domain of definition of operator $\hat{A}^\tau$ in terms of various classes of differentiable functions. At the end of this paper we consider initial-boundary value problem (1)–(3) in torus $\mathbb{T}^N$ when operator $A(x, D)$ has constant coefficients. We will see that in this case the domain of definition $D(\hat{A}^\tau)$ coincides with the corresponding Sobolev spaces.

2. UNIQUENESS

In this section we prove Theorem 1. Suppose that all the conditions of this theorem are satisfied and let initial-boundary value problem (1)–(3) have two regular solutions $u_1(x, t)$ and $u_2(x, t)$. Our aim is to prove that $u(x, t) = u_1(x, t) - u_2(x, t) \equiv 0$. Since the problem is linear, then we have the following homogenous problem for $u(x, t)$:

$$\partial_t^\rho u(x, t) + A(x, D)u(x, t) = 0, \quad x \in \Omega, \quad t > 0; \quad (9)$$

$$\lim_{t \rightarrow 0} \partial_t^{\rho-1} u(x, t) = 0, \quad x \in \Omega; \quad (10)$$

$$B_j u(x, t) = \sum_{|\alpha| \leq m_j} b_{\alpha,j}(x) D^\alpha u(x, t) = 0, \quad 0 \leq m_j \leq m-1, \quad j = 1, 2, \dots, l; \quad x \in \partial\Omega; \quad (11)$$

Let $u(x, t)$ be a regular solution of problem (9)–(11) and v_k be an arbitrary eigenfunction of the problem (4)–(5) with the corresponding eigenvalue λ_k . Consider the function

$$w_k(t) = \int_{\Omega} u(x, t) v_k(x) dx. \quad (12)$$

By definition of the regular solution we may write

$$\partial_t^\rho w_k(t) = \int_{\Omega} \partial_t^\rho u(x, t) v_k(x) dx = - \int_{\Omega} A(x, D)u(x, t) v_k(x) dx, \quad t > 0,$$

or, integrating by parts,

$$\partial_t^\rho w_k(t) = - \int_{\Omega} u(x, t) A(x, D) v_k(x) dx = -\lambda_k \int_{\Omega} u(x, t) v_k(x) dx = -\lambda_k w_k(t), \quad t > 0.$$

Since condition (2) can be rewritten as (see, for example, [11, p. 104])

$$\lim_{t \rightarrow 0} t^{1-\rho} u(x, t) = \frac{\varphi(x)}{\Gamma(\rho)}, \quad (13)$$

then, using in (12) the homogenous initial condition (10) in this form, we have the following Cauchy problem for $w_k(t)$:

$$\partial_t^\rho w_k(t) + \lambda_k w_k(t) = 0, \quad t > 0; \quad \lim_{t \rightarrow 0} \partial_t^{\rho-1} w_k(t) = 0.$$

This problem has the unique solution; therefore, the function defined by (12), is identically zero: $w_k(t) \equiv 0$ (see (7)). From completeness in $L_2(\Omega)$ of the system of eigenfunctions $\{v_k(x)\}$, we have $u(x, t) = 0$ for all $x \in \bar{\Omega}$ and $t > 0$. Hence Theorem 1 is proved.

3. EXISTENCE

Here we have borrowed some original ideas from the method developed in the work of M. A. Krasnoselski et al. [26]. The following lemma plays a key role in this method [26, p. 453].

Lemma . *Let $\sigma > 1 + \frac{N}{2m}$. Then for any $|\alpha| \leq m$ operator $D^\alpha \hat{A}^{-\sigma}$ (completely) continuously maps from $L_2(\Omega)$ into $C(\bar{\Omega})$ and moreover the following estimate holds true*

$$\|D^\alpha \hat{A}^{-\sigma} g\|_{C(\Omega)} \leq C \|g\|_{L_2(\Omega)}. \quad (14)$$

Using this lemma, we prove that one can validly apply the operators D^α with $|\alpha| \leq m$ and ∂_t^ρ to the series (8) term-by-term.

For the Mittag-Leffler function with a negative argument we have an estimate $|E_{\rho,\mu}(-t)| \leq \frac{C}{1+t}$ (see, for example, [11], p. 13). Therefore, since all eigenvalues λ_j are positive,

$$|t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho)| \leq \frac{C t^{\rho-1}}{1 + \lambda_j t^\rho} \leq \frac{C}{\lambda_j t} (t^\rho \lambda_j)^{\varepsilon/\rho}, \quad t > 0, \quad (15)$$

where $0 < \varepsilon < \rho$. Indeed, let $t^\rho \lambda_j < 1$, then

$$\frac{1}{\lambda_j t} (t^\rho \lambda_j)^{\varepsilon/\rho} > \frac{1}{\lambda_j t} t^\rho \lambda_j > t^{\rho-1},$$

and if $t^\rho \lambda_j > 1$, then

$$\frac{1}{\lambda_j t} (t^\rho \lambda_j)^{\varepsilon/\rho} > \frac{1}{\lambda_j t}.$$

Note the series (8) is in fact the sum of two series. Consider the first series:

$$S_k^1(x, t) = \sum_{j=1}^k v_j(x) \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho), \quad (16)$$

and suppose that function φ satisfies the condition of Theorem 2, i.e. for some $\tau > \frac{N}{2m}$

$$\sum_1^\infty \lambda_j^{2\tau} |\varphi_j|^2 \leq C_\varphi < \infty.$$

We choose a small $\varepsilon > 0$ in such a way, that $\tau + 1 - \varepsilon/\rho > 1 + \frac{N}{2m}$. Since $\hat{A}^{-\tau-1+\varepsilon/\rho} v_j(x) = \lambda_j^{-\tau-1+\varepsilon/\rho} v_j(x)$, we may rewrite the sum (16) as

$$S_k^1(x, t) = \hat{A}^{-\tau-1+\varepsilon/\rho} \sum_{j=1}^k v_j(x) \lambda_j^{\tau+1-\varepsilon/\rho} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho).$$

Therefore by virtue of Lemma 1 one has

$$\begin{aligned} \|D^\alpha S_k^1\|_{C(\Omega)} &= \left\| D^\alpha \hat{A}^{-\tau-1+\varepsilon/\rho} \sum_{j=1}^k v_j(x) \lambda_j^{\tau+1-\varepsilon/\rho} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) \right\|_{C(\Omega)} \\ &\leq C \left\| \sum_{j=1}^k v_j(x) \lambda_j^{\tau+1-\varepsilon/\rho} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) \right\|_{L_2(\Omega)}. \end{aligned} \quad (17)$$

Using the orthonormality of the system $\{v_j\}$, we will have

$$\|D^\alpha S_k^1\|_{C(\Omega)}^2 \leq C \sum_{j=1}^k |\lambda_j^{\tau+1-\varepsilon/\rho} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho)|^2. \quad (18)$$

Application of inequality (15) gives

$$\sum_{j=1}^k |\lambda_j^{\tau+1-\varepsilon/\rho} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho)|^2 \leq C t^{2(\varepsilon-1)} \sum_{j=1}^k \lambda_j^{2\tau} |\varphi_j|^2 \leq C t^{2(\varepsilon-1)} C_\varphi.$$

Therefore we can rewrite the estimate (18) as

$$\|D^\alpha S_k^1\|_{C(\Omega)}^2 \leq C t^{2(\varepsilon-1)} C_\varphi.$$

This implies uniformly on $x \in \bar{\Omega}$ convergence of the differentiated sum (16) with respect to the variables x_j for each $t \in (0, T]$. On the other hand, the sum (17) converges for any permutation of its members as well, since these terms are mutually orthogonal. This implies the absolute convergence of the differentiated sum (16) on the same interval $t \in (0, T]$.

Now we consider the second part of the series (8):

$$S_k^2(x, t) = \sum_{j=1}^k v_j(x) \int_0^t f_j(t - \xi) \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \quad (19)$$

and suppose that function $f(x, t)$ satisfies all the conditions of Theorem 2, i.e. the following series converges uniformly on $t \in [0, T]$ for some $\tau > \frac{N}{2m}$:

$$\sum_1^\infty \lambda_j^{2\tau} |f_j(t)|^2 \leq C_f < \infty.$$

We have

$$S_k^2(x, t) = \hat{A}^{-\tau-1+\varepsilon/\rho} \sum_{j=1}^k v_j(x) \int_0^t \lambda_j^{\tau+1-\varepsilon/\rho} f_j(t - \xi) \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi.$$

Then by virtue of Lemma 1 one has

$$\begin{aligned} \|D^\alpha S_k^2\|_{C(\Omega)} &= \left\| D^\alpha \hat{A}^{-\tau-1+\varepsilon/\rho} \sum_{j=1}^k v_j(x) \int_0^t \lambda_j^{\tau+1-\varepsilon/\rho} f_j(t - \xi) \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \right\|_{C(\Omega)} \\ &\leq C \left\| \sum_{j=1}^k v_j(x) \int_0^t \lambda_j^{\tau+1-\varepsilon/\rho} f_j(t - \xi) \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \right\|_{L_2(\Omega)}. \end{aligned} \quad (20)$$

Using the orthonormality of the system $\{v_j\}$, we will have

$$\|D^\alpha S_k^2\|_{C(\Omega)}^2 \leq C \sum_{j=1}^k \left| \int_0^t \lambda_j^{\tau+1-\varepsilon/\rho} f_j(t - \xi) \xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi \right|^2.$$

Now we use estimate (15) and apply the generalized Minkowski inequality. Then

$$\|D^\alpha S_k^2\|_{C(\Omega)}^2 \leq C \left(\int_0^t \xi^{\varepsilon-1} \left(\sum_{j=1}^k \lambda_j^{2\tau} |f_j(t - \xi)|^2 \right)^{1/2} d\xi \right)^2 \leq C C_f \left(\frac{T^\varepsilon}{\varepsilon} \right)^2.$$

Hence, using the same argument as above, we see that the differentiated sum (19) with respect to the variables x_j converges absolutely and uniformly on $(x, t) \in \bar{\Omega} \times [0, T]$.

It is not hard to see that

$$\partial_t^\rho \sum_1^k T_j(t) v_j(x) = - \sum_1^k \lambda_j T_j(t) v_j(x) + \sum_1^k f_j(t) v_j(x)$$

$$= -A(x, D)\hat{A}^{-\tau-1+\varepsilon/\rho} \sum_1^k \lambda_j^{\tau+1-\varepsilon/\rho} T_j(t) v_j(x) + \hat{A}^{-\tau-1+\varepsilon/\rho} \sum_1^k \lambda_j^{\tau+1-\varepsilon/\rho} f_j(t) v_j(x).$$

Absolutely and uniformly convergence of the latter series can be proved as above.

Obviously, function (8) satisfies the boundary conditions (3). Considering the initial condition as in (13), it is not hard to verify, that this condition is also satisfied.

Thus Theorem 2 is completely proved.

Observe, taking $\alpha = 0$ in the above estimations, one may obtain an estimate for $\|u(x, t)\|_{C(\Omega)}$, which gives the stability of the solution to problem (1)–(3).

4. THE INITIAL-BOUNDARY VALUE PROBLEM ON $\mathbb{T}^N$

In the proof of Theorem 2 we only use the fact that elliptic operator A has a complete in $L_2(\Omega)$ set of orthonormal eigenfunctions and Lemma 1. This lemma reduces the study of uniform convergence to the study of convergence in L_2 , where the Parseval equality gives a solution to the problem immediately. Therefore, similar to Theorem 2 statement holds true for any operator with these properties. As an example we may consider differential operator with involution and the Bessel operator (see the paper of M. Ruzhansky et al. [24]), or the initial-boundary value problem in N -dimensional torus: $\mathbb{T}^N = (\pi, \pi]^N$.

Let us consider the latter case. So let $A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ be a homogeneous symmetric positive elliptic differential operator with constant coefficients. Let the differential equation and initial condition have the form ($0 < \rho \leq 1$)

$$\partial_t^\rho u(x, t) + A(D)u(x, t) = f(x, t), \quad x \in \mathbb{T}^N, \quad t > 0; \quad (21)$$

$$\lim_{t \rightarrow 0} \partial_t^{\rho-1} u(x, t) = \varphi(x), \quad x \in \mathbb{T}^N. \quad (22)$$

Instead of boundary conditions (3) we consider the 2π -periodic in each argument x_j functions and suppose that $\varphi(x)$ and $f(x, t)$ are also 2π -periodic functions in x_j .

Let A stands for the operator $A(D)$, defined on 2π -periodic functions from $C^m(\mathbb{R}^N)$. The closure $\hat{A}$ of this operator in $L_2(\mathbb{T}^N)$ is selfadjoint. It is not hard to see that operator $\hat{A}$ has a complete (in $L_2(\mathbb{T}^N)$) set of eigenfunctions $\{(2\pi)^{-N/2} e^{inx}\}$, and corresponding eigenvalues $A(n)$. Therefore, by virtue of J. von Neumann theorem, for any $\tau > 0$ operator $\hat{A}^\tau$ acts as $\hat{A}^\tau g(x) = \sum_{A(n) < \lambda} A^\tau(n) g_n e^{inx}$, where g_n is

Fourier coefficients of $g \in L_2(\mathbb{T}^N)$. The domain of definition of this operator is defined from the condition $\hat{A}^\tau g(x) \in L_2(\mathbb{T}^N)$ and has the form

$$D(\hat{A}^\tau) = \left\{ g \in L_2(\mathbb{T}^N) : \sum_{n \in \mathbb{Z}^N} A^{2\tau}(n) |g_n|^2 < \infty \right\}.$$

In order to define the domain of definition of operator $\hat{A}^\tau$ in terms of the Sobolev spaces, we remind the definition of these spaces (see, for example, [28]): we say that function $g \in L_2(\mathbb{T}^N)$ belongs to the Sobolev space $L_2^a(\mathbb{T}^N)$ with the real number $a > 0$, if the norm

$$\|g\|_{L_2^a(\mathbb{T}^N)}^2 = \left\| \sum_{n \in \mathbb{Z}^N} (1 + |n|^2)^{\frac{a}{2}} g_n e^{inx} \right\|_{L_2(\mathbb{T}^N)}^2 = \sum_{n \in \mathbb{Z}^N} (1 + |n|^2)^a |g_n|^2$$

is bounded. When a is not integer, then this space is also called the Liouville space.

It is not hard to show the existence of constants c_1 and c_2 such that

$$c_1(1 + |n|^2)^{\tau m} \leq 1 + A^{2\tau}(n) \leq c_2(1 + |n|^2)^{\tau m}.$$

Therefore, $D(\hat{A}^\tau) = L_2^{\tau m}(\mathbb{T}^N)$.

Repeating verbatim the proof of Theorem 2, the following statement is proved.

Theorem 3. Let $\tau > \frac{N}{2}$ and $\varphi \in L_2^\tau(\mathbb{T}^N)$. Moreover, let $f(x, t) \in L_2^\tau(\mathbb{T}^N)$ for any $t \in [0, T]$ and the function $F(t) = \hat{A}^{\tau/m} f(x, t)$ be continuous in the norm of space $L_2(\mathbb{T}^N)$ for all $t \in [0, T]$. Then there exists a solution of initial-boundary value problem (21)–(22) and it has the form

$$u(x, t) = \sum_{n \in \mathbb{Z}^N} \left[\varphi_n t^{\rho-1} E_{\rho, \rho}(-A(n)t^\rho) + \int_0^t f_n(t-\xi) \xi^{\rho-1} E_{\rho, \rho}(-A(n)\xi^\rho) d\xi \right] e^{inx},$$

which absolutely and uniformly converges on $x \in \mathbb{T}^N$ and for each $t \in (0, T]$, where φ_n and $f_n(t)$ are corresponding Fourier coefficients.

Note, when $\tau > \frac{N}{2}$, according to the Sobolev embedding theorem, all functions in $L_2^\tau(\mathbb{T}^N)$ are 2π -periodic continuous functions. The fulfillment of the inverse inequality $\tau \leq \frac{N}{2}$, admits the existence of unbounded functions in $L_2^\tau(\mathbb{T}^N)$ (see, for example, [28]). Therefore, condition $\tau > \frac{N}{2}$ of this theorem is not only sufficient for the statement to be hold, but it is also necessary.

A 2π -periodical continuous function $\varphi(x)$, defined on $\mathbb{T}^N$, is said to be a Hölder continuous with exponent $\beta \in (0, 1]$, if for some constant C ,

$$|\varphi(x) - \varphi(y)| \leq C|x - y|^\beta, \quad \text{for all } x, y \in \mathbb{T}^N.$$

We define $C^a(\mathbb{T}^N)$ to be the Hölder space of 2π -periodical functions in $C^{[a]}(\mathbb{T}^N)$, $[a]$ is the integer part of a , all derivatives $D^\alpha \varphi$, $|\alpha| = [a]$, of which are Hölder continuous with exponent $a - [a]$. It follows from the embedding theorem $C^a(\mathbb{T}^N) \rightarrow L_2^{a-\varepsilon}(\mathbb{T}^N)$, $\varepsilon > 0$, that the conditions of the Theorem 3 can be formulated in terms of the Hölder spaces, replacing the classes $L_2^\tau(\mathbb{T}^N)$ by the classes $C^\tau(\mathbb{T}^N)$. Again, condition $\tau > \frac{N}{2}$ is precise: if $\tau = \frac{N}{2}$, then there exists a function in $C^\tau(\mathbb{T}^N)$, such that the Fourier series of which diverges at some point (see, for example, [28]).

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