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MATHEMATICS

Mathematical Modeling of the Consumer Loan Market in Russia under Sanctions

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Abstract—The article develops and investigates a new model for the formation of interest rates on consumer loans based on an analysis of commercial interests and the logic of behavior of commercial banks. The model assumes that the borrowers' incomes are described by a geometric Brownian motion. Commercial banks assess the default risk of borrowers. According to the Feynman–Kac formula, the assessment is reduced to solving a boundary value problem for partial differential equations. An analytical solution to this problem is constructed. The model is used to analyze the problem of maintaining consumer credit under the current conditions as a mechanism for social adaptation of households.

Keywords: Ramsey model, consumer loan, Feynman–Kac formula, mathematical modeling

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1. INTRODUCTION

The consumer loan market in Russia has considerably grown over the last 20 years. The debt of households on consumer credit has reached 12.5 trillions of rubles, which is more than 12% of the GDP of the Russian Federation and more than 9% of the consolidated balance assets of Russian Federation commercial banks. Since the ratio of consumer loan interest rates to deposits varied from 2.5 to 3.5 in the last decade, consumer credit was one of the most attractive assets for commercial banks. On the one hand, the increase in demand for consumer loans has reflected the formation of a "middle layer" in the Russian Federation population over the last 20 years. Households from the middle layer manage their expenditures using consumer loans so that their demand for loans is commensurate with the loan interest rates. On the other hand, consumer loans in Russia work as a mechanism of social adaptation and more than half of the total amount of loans fall on low-income households, who take loans in difficult life circumstances regardless of the interest rate value. This structure of borrowers increases the probability of borrower default and generates risks for the banking system, which was a matter of concern for the Ministry of Economic Development as early as the summer of 2019. The COVID-19 pandemic affected the economic state of households and aggravated the situation in the consumer loan market.

The sanctions imposed on Russia in 2022 are aimed at lowering the standard of living of households due to the crisis of the banking system and inflation. The goals of the sanctions are to aggravate social tension, to reduce the management efficiency at the federal level, and to cause power disintegration. The countermeasures of the government include compensation for losses of households in the real disposable income. There are two mechanisms of support and social adaptation of households: direct transfers from the budget and consumer loans. The question arises as to how the consumer loan market will change under these circumstances and whether it will be able to work as a mechanism of social adaptation in combination with direct subsidies from the budget. In the new conditions, commercial banks have changed their approaches to the delivery of consumer loans. Accordingly, to analyze the possible strategies of the Russian Federation government and answer the above question, we need to use mathematical models in which the behavior, in the consumer loan market, of commercial banks and households from various social groups is described taking into account their commercial interests and the logic of behavior under varying conditions. Models of consumer loan demand among

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households from various social groups were developed in [1, 2], and they were used to analyze the state of the consumer loan market of Russia in 2019–2021. In this analysis, it was supposed that commercial banks, having a large liquidity buffer, exhibit inertial behave that can be described by econometric models. Under the new conditions of liquidity deficit, the behavior of major commercial banks in the consumer loan market has changed and it has to be modeled as based on the analysis of their commercial interests and the logic of behavior. The modeling if the interaction between commercial banks and households in the consumer loan market is based on the concept of Stackelberg equilibrium. It is assumed that commercial banks set the interest rate on consumer credit relying on their commercial interests, assessing the response behavior of households.

Section 2 describes the results of [2] concerning the modeling of consumer credit demand based on the concept of a rational representative household. By analyzing the optimal control problem modeling the economic behavior of a rational representative household, we identify four types of borrowers corresponding to various social groups of the population. The problem of setting an interest rate on consumer credit in a commercial bank is set up and studied in Section 3. The model assumes that the incomes of borrowers are described by a geometric Brownian motion, and commercial banks set an interest rate that maximizes the expectation of the net present value, taking into account the risks of the borrowers. With the use of the Feynman–Kac formula, taking into account the risk of borrower default is reduced to solving a boundary value problem for a partial differential equation. After establishing its relationship with the Abel equation, the boundary value problem is reduced to the Cauchy problem for the heat equation with an external source and the default risk is estimated in analytical form. In Section 4, the constructed model is identified using Rosstat household budget survey data and the influence of shocks exerted by the sanctions on the consumer loan market in Russia is analyzed.

2. MODEL OF HOUSEHOLD BEHAVIOR

In modeling the economic behavior of households, we rely on the concept of a rational representative economic agent going back to Ramsey's work [3]. Specifically, we use a model of the behavior of a representative household in the form of an optimal control problem [2, 4, 5]. In this model, the dynamics of household liquid assets $M_0(t)$ is described by the equation

$$\frac{dM_0(t)}{dt} = S(t) - p(t)C(t) + H_L(t) - H_D(t). \tag{1}$$

Here, $S(t)$ is the household income at the time t, which is assumed to grow at the rate γ, i.e., $S(t) = S_0e^{\gamma t}$; $C(t)$ is the household consumption; $p(t)$ is the consumer price index, which is assumed to grow at the rate j, i.e., $p(t) = p_0e^{jt}$; $H_L(t)$ is consumer credit; and $H_D(t)$ is savings in the form of deposits in a commercial bank. Note that $H_L > 0$ if the household receives a loan of size H_L and $H_L < 0$ in the case of credit repayment. Similarly, putting an amount of money of H_D in a deposit account leads to a decrease in the money supply M_0; therefore, $H_D > 0$; if the money is withdrawn from a deposit account, then $H_D < 0$. The consumer debt $L(t) \ge 0$ is governed by the equation

$$\frac{dL(t)}{dt} = H_L(t) + r_L L(t), \tag{2}$$

where r_L is the interest rate on the loan credit. Savings in the form of deposits $D(t) \ge 0$ with an interest rate r_D are described by the equation

$$\frac{dD(t)}{dt} = H_D(t) + r_D D(t). \tag{3}$$

The absence of arbitration assumes that the credit interest rate is higher than the deposit interest rate, i.e., $r_L > r_D$. According to Fisher's equation of exchange, the money supply $M_0(t)$ is related to the household consumption $C(t)$ via the coefficient $\theta > 0$:

$$M_0(t) = \theta p(t)C(t),$$

where $\frac{1}{\theta} > 0$ is the velocity of money.

For the subsequent presentation, it is convenient to introduce a variable $X(t)$ defined as

$$X(t) = M_0(t) - L(t) + D(t),$$

which will be referred to as the financial state of the household. The rationality of household behavior implies that consumer credit is not taken simultaneously with making savings, so

$$L(t) = (M_0(t) - X(t))_+,$$

$$D(t) = (X(t) - M_0(t))_+.$$

From Eqs. (1)–(3), it follows that the dynamics of the financial state is described by the dynamical system

$$\frac{dX}{dt} = S - \frac{1}{\theta}M_0 - r_L(M_0 - X)_+ + r_D(X - M_0)_+, \tag{4}$$

$$X(0) = x_0. \tag{5}$$

The household seeks to maximize the discounted consumption with a discount factor $\delta_0 > 0$ and a constant risk aversion $0 < 1 - \alpha < 1$ on the time interval $[0, T]$ by controlling the money supply $M_0 \ge 0$, i.e.,

$$\int_0^T (M_0(t))^\alpha e^{-\delta_0 t} dt \to \max_{M_0 \ge 0}. \tag{6}$$

We say that a financial state $X(t)$ is liquid if there exists a control $M_0(t)$ ensuring that $X(T) \geq 0$. In other words, the household has to repay its credit by a finite time. Suppose that $x(t) = X(t)e^{-\gamma t}$, $M(t) = M_0(t)e^{-\gamma t}$, $\delta = \delta_0 + \alpha j$, and $\hat{H}_L(t) = H_L(t)e^{-\gamma t}$. The optimal control problem at a finite time horizon is stated as follows:

$$\int_0^T M^{\alpha} e^{-(\delta-\alpha\gamma)t} dt \to \max_M, \tag{7}$$

$$\frac{dx}{dt} = S - \gamma x - \frac{1}{\theta}M - r_L\left(M - x\right)_+ + r_D\left(x - M\right)_+, \tag{8}$$

$$x(0) = x_0, \quad x(T) \geq 0, \tag{9}$$

$$M(t) \geq 0. \tag{10}$$

To ensure that $x(T) \geq 0$, it is necessary that $X \geq -\dfrac{S}{r_L - \gamma}$ (if this inequality is violated, then the household cannot repay its credit and a financial pyramid forms). If $T \to \infty$, then this condition is not only necessary, but also sufficient.

Given an optimal trajectory $M(t)$ that maximizes the discounted consumption (7), it is possible to find the financial state $x(t)$ from the Cauchy problem (7)–(10) and to determine the dynamics of the consumer debt $L(t)$ and the dynamics of deposit savings $D(t)$. It was shown in [2] that an optimal trajectory $M(t)$ exists. Since the right-hand side of differential equation (8) is nonsmooth, the Pontryagin maximum principle is applied in the Clarke form. As a result, three modes of household economic behavior can be distinguished: getting a loan, taking no credit from banks, and saving in the form of deposits depending on the financial state and the market conditions (for more details, see [2]).

If $T \to \infty$ in the optimal control problem (7)–(10), then an optimal control synthesis can be constructed in a nearly analytical form (see [2]). Given a synthesis, the optimal control can be determined depending on the current value of the state variable x and market condition parameters. The synthesis depends on the relations between the quantities r_L, r_D, $\delta - \dfrac{1-\alpha}{\theta}$, and $\delta + (1-\alpha)\gamma$ and determines the social group to which the household belongs. Searching all possible admissible combinations of r_L, r_D, $\delta - \dfrac{1-\alpha}{\theta}$, $\delta + (1-\alpha)\gamma$, we can obtain various types of social groups (for a detailed description, see [2]). Note that modes of household behavior are determined for each social group.

We are interested in the problem of consumer debt growth. Let us show how the model described can be used to compute the consumer credit demand $\hat{H}_L$. Then, from the balance equation for the borrower's liquid assets (1), taking into account the dynamics of the household's financial state (4) and the introduced renormalizations, we conclude that the consumer credit demand is described by the equation

$$\hat{H}_L = \left(\frac{dM(x)}{dx}\cdot\frac{dx}{dt} - S + \gamma M(x) + \frac{M(x)}{\theta}\right)_+$$

or, equivalently, in view of (4),

$$\hat{H}_L = \left(\frac{dM(x)}{dx}\left(S - \gamma x - \frac{M(x)}{\theta} - r_L(M(x) - x)_+\right) - S + \gamma M(x) + \frac{M(x)}{\theta}\right)_+, \tag{11}$$

where $M(x) = M\left(x; r_L, \gamma, \theta, \alpha, \delta\right)$ is the optimal control synthesis.

Depending on the relations between r_L, $\delta - \dfrac{1-\alpha}{\theta}$, and $\delta + (1-\alpha)\gamma$, we can distinguish four types of borrowers' behavior corresponding to different social groups.

Type 1 behavior. If $r_L < \delta - \dfrac{1-\alpha}{\theta}$ and $1 + \gamma\theta \geq 0$ or $r_L < \delta + (1-\alpha)\gamma$ and $1 + \gamma\theta < 0$, then the optimal control synthesis is given by the function

$$M\left(x; r_L, \gamma, \theta, \alpha, \delta\right) = \left(\frac{\theta\left(\delta - \alpha r_L\right)}{(1-\alpha)\left(1 + \theta r_L\right)}\right)\left(x + \frac{S}{r_L - \gamma}\right),$$

whence

$$\frac{dM\left(x; r_L, \gamma, \theta, \alpha, \delta\right)}{dx} = \frac{\theta\left(\delta - \alpha r_L\right)}{(1-\alpha)\left(1 + \theta r_L\right)}.$$

The consumer credit demand is determined as

$$\hat{H}_L = \frac{(\delta - \alpha r_L)((r_L - \delta)\theta + (1-\alpha))}{(1-\alpha)^2(1+\theta r_L)}\cdot x + \frac{(1-\alpha)(r_L - \gamma)(\theta(\delta - r_L) - (1-\alpha))S + ((1-\alpha) - \theta(\delta - \alpha r_L))(\delta - \alpha r_L)S}{(1-\alpha)^2(r_L - \gamma)(1+\theta r_L)}.$$

Under behavior of this type, the household takes a consumer loan regardless of its financial state.

Type 2 behavior. If $\delta + (1-\alpha)\gamma < r_L$ and $1 + \theta\gamma < 0$, then the optimal control synthesis is given by the function

$$M(x; r_L, r_D, \gamma, \theta, \alpha, \delta) = \left(\frac{\theta(\delta - \alpha r_L)}{(1-\alpha)(1+\theta r_L)}\right)x,$$

whence

$$\frac{dM}{dx} = \frac{\theta(\delta - \alpha r_L)}{(1-\alpha)(1+\theta r_L)}.$$

The consumer credit demand is determined as

$$\hat{H}_L = \left(\frac{\theta(\delta - \alpha r_L) + (1-\alpha)(1+\theta r_L)}{(1-\alpha)\theta}\right) \times \left(\frac{\theta(\delta - \alpha r_L)}{(1-\alpha)(1+\theta r_L)}\right) \cdot x + \left(\frac{\theta(\delta - r_L) - (1-\alpha)}{(1-\alpha)(1+\theta r_L)}\right) \cdot S.$$

Under behavior of this type, the household also takes a consumer loan regardless of its financial state.

Type 3 behavior. If $\delta - \frac{1-\alpha}{\theta} < r_L < \delta + (1-\alpha)\gamma$ and the household's financial state satisfies $x < \frac{(\delta - \alpha r_L)}{\left(r_L - \delta + \frac{1-\alpha}{\theta}\right)} \frac{S}{(r_L - \gamma)}$, then the optimal control synthesis and the consumer credit demand are the same as in the case of type 1 behavior. Otherwise, if $1 + \theta\gamma \geq 0$ and $x \geq \frac{(\delta - \alpha r_L)}{\left(r_L - \delta + \frac{1-\alpha}{\theta}\right)} \frac{S}{(r_L - \gamma)}$, then no consumer loan is taken.

Type 4 behavior. If $\delta + (1-\alpha)\gamma < r_L$, $1 + \theta\gamma \geq 0$, and the financial state satisfies $x < \frac{S\theta}{1+\gamma\theta}$, then the optimal control synthesis is determined by solving the equation

$$x + \frac{S}{r_L - \gamma} = \frac{S(1+\theta r_L)(\delta + (1-\alpha)\gamma - r_L)}{(\delta - \alpha r_L)(1+\gamma\theta)(r_L - \gamma)} \times \left(\frac{M_2(1+\gamma\theta)}{S\theta}\right)^{\frac{(1-\alpha)(\gamma - r_L)}{(\delta + (1-\alpha)\gamma - r_L)}} + \left(\frac{(1-\alpha)(1+\theta r_L)}{\theta(\delta - \alpha r_L)}\right) M_2, \tag{12}$$

which has two positive solutions for M_2. Let $M_2(x; r_L, \gamma, \theta, \alpha, \delta)$ be the minimum solution of Eq. (12). By the implicit function theorem, $M_2(x; r_L, \gamma, \theta, \alpha, \delta)$ is a differentiable function. Then

$$\frac{dM(x)}{dx} = \left(\frac{\theta(\delta - \alpha r_L)}{(1-\alpha)(1+\theta r_L)}\right) \times \left(1 - \left(\frac{M(1+\gamma\theta)}{S\theta}\right)^{\frac{-\delta + \alpha r_L}{(\delta + (1-\alpha)\gamma - r_L)}}\right)^{-1}.$$

The consumer credit demand is determined as

$$\hat{H}_L = \frac{M}{\theta} - S + \gamma M + \left(\frac{\theta(\delta - \alpha r_L)}{(1-\alpha)(1+\theta r_L)}\right) \times \left(1 - \left(\frac{M(1+\gamma\theta)}{S\theta}\right)^{\frac{-\delta + \alpha r_L}{(\delta + (1-\alpha)\gamma - r_L)}}\right)^{-1} \times \left(S - \gamma x - \frac{M}{\theta} - r_L(M - x)_+\right).$$

If $x \geq \frac{S\theta}{1+\gamma\theta}$, then no consumer loan is taken.

To use the model, we need to specify behavioral characteristics of a representative household typical for a given social group: the velocity of money $\frac{1}{\theta}$, risk aversion $1 - \alpha$, discount factor δ, and market condition parameters, such as the credit interest rate r_L, the income growth rate γ, and inflation j. The model of household economic behavior was calibrated using Rosstat data. Rosstat provides quarterly statistics of a household budget survey (HBS) [6], questioning 50000 households living in 82 regions of Russia. Depending on the level of consumption per capita, these regions were divided into three groups: rich, middle, and poor. Relying on HBS data, we noted that almost half of households from the poor group live in urban areas, and the other half, in rural ones. Since the urban and rural lifestyles differ from each other, the households from the group of poor regions were divided into another two subgroups. In each group of regions, we identified its own population segments with different levels of income and expenditure.

We divided the HBS respondents into borrowers and other households having no loan repayment obligation. The borrowers were divided into two types: ones with low income, which have a high default risk, and ones with high income. An analysis of HBS statistics showed that the income per capita of high-income borrowers was nearly twice as high as that of low-income borrowers. Behavioral characteristics of representative households from a certain population segment were identified by solving inverse problems. A special computer code in the form of a MatLab AppDesigner application [7] was developed for working with HBS statistics, i.e., for distinguishing social groups of householders in different groups of regions, identifying model parameters, verifying the results, and constructing predictions.

3. MODELING OF THE CONSUMER LOAN MARKET

The consumer loan market is an imperfect one in which households show demand for loans and commercial banks issue loans. The formation of an interest rate in the consumer loan market is modeled using the concept of Stackelberg equilibrium. Commercial banks assess the demand for credit depending on the interest rate and set the interest rate on consumer loans at a level that maximizes their profit.

The basic form of payment on a consumer loan is an annuity payment. Suppose, as before, that r_L is the monthly interest rate on a newly issued loan and $\hat{T}$ is the credit period measured in months. The loan debt is equal to $\hat{H}_L(r_L)e^{r_L\hat{T}}$, where $\hat{H}_L(r_L)$ is the current demand for credit. An annuity payment A can be found from the equation

$$\hat{H}_L(r_L)e^{r_L\hat{T}} = A\int_0^{\hat{T}} e^{r_L(\hat{T}-t)}dt,$$

whence

$$A = \frac{r_L}{1-e^{-r_L\hat{T}}}\hat{H}_L(r_L).$$

Commercial banks take into account the risk of borrower default. Let τ denote the time when the borrower fails to repay the annuity payment. From the point of view of commercial banks, the borrowers' income $S(t)$ is not stable. Assume that commercial banks model the borrower's income with the help of a stochastic differential equation for a geometric Brownian motion:

$$dS = S\gamma dt + \sigma S dW_t,$$
$$S(0) = S_0,$$

where γ is the income growth rate, W_t is a Wiener process, and $\sigma > 0$ is the volatility parameter. Assume that the borrower defaults on the loan if its income is not sufficient for the annuity payment A and the consumer expenditure is at the minimum level μ, i.e.,

$$\tau = \min\left(\inf_t\{S(t) < A+\mu\}, \hat{T}\right).$$

Since the borrower's income is a random process, the quantity $\tau \in (0, \hat{T})$ is a random stopping time. Commercial banks set an interest rate on consumer loans to maximize the expectation of the net present value, i.e.,

$$\begin{aligned} NPV(r_L) &= \mathbb{E}_\tau\left(A\int_0^\tau e^{-\lambda t}dt - \hat{H}_L(r_L)\right) \\ &= \mathbb{E}_\tau\left(\left(\frac{r_L(1-e^{-\lambda\tau})}{\lambda(1-e^{-r_L\hat{T}})} - 1\right)\hat{H}_L(r_L)\right) \to \max_{r_L}, \end{aligned} \tag{13}$$

where λ is the discount factor for cash flows, which is equal to the cost of funding expenses for commercial banks.

Define $\alpha_u = \dfrac{\sigma}{\sqrt{2\gamma - \sigma^2}}$.

Theorem. *If* $\gamma > \dfrac{\sigma^2}{2}$, *then*

$$\mathbb{E}_\tau e^{-\lambda\tau} = e^{-\lambda\hat{T}} + \frac{\alpha_u}{\pi}\int_0^{\sqrt{\lambda\hat{T}}} e^{-\left(\frac{\lambda}{4\alpha_u^2\tau^2}\left(\ln\left(\frac{A+\mu}{S_0}\right)\right)^2+\tau^2\right)}\left(\int_0^{\sqrt{\lambda\hat{T}-\tau^2}} e^{-y^2}dy\right)d\tau. \tag{14}$$

Proof. The stochastic differential equation

$$dS = S\gamma dt + \sigma S dW_t,$$
$$S(0) = S_0$$

has strong solution (see [8, p. 364 of the Russian edition]):

$$S(t) = S_0 e^{\left(\gamma - \frac{\sigma^2}{2}\right)t + \sigma W_t}.$$

It follows that

$$\tau = \min\left(\hat{T}, \inf_{t\ge 0}\left\{W_t < \frac{1}{\sigma}\left(\ln\frac{A+\mu}{S_0}\right) - \left(\frac{\gamma}{\sigma} - \frac{\sigma}{2}\right)t\right\}\right). \tag{15}$$

We introduce the following notation:

$$W_t^x = x + W_t,$$
$$\tau_x^\theta = \min\left(\theta, \inf_{t\ge 0}\left\{W_t^x < \frac{1}{\sigma}\left(\ln\frac{A+\mu}{S_0}\right) - \left(\frac{\gamma}{\sigma} - \frac{\sigma}{2}\right)t\right\}\right),$$
$$u(t,x) = \mathbb{E}e^{-\lambda\tau_x^{\hat{T}-t}}.$$

Then $\mathbb{E}_\tau e^{-\lambda\tau} = u(0,0)$. Relying on the Feynman–Kac formula (see Theorem 21.14 in [8, p. 378 of the Russian edition]), we conclude that $u(t,x)$ is the solution of the boundary value problem

$$\left(\gamma - \frac{\sigma^2}{2}\right)\frac{\partial u(t,x)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u(t,x)}{\partial x^2} = 0, \tag{16}$$

$$x \in \left[\ln\left(\frac{A+\mu}{S_0}\right), +\infty\right), \tag{17}$$

$$u(\hat{T}, x) = e^{-\lambda\hat{T}}, \tag{18}$$

$$u\left(t, \ln\left(\frac{A+\mu}{S_0}\right)\right) = e^{-\lambda t}. \tag{19}$$

We introduce $v(t,x) = u(\hat{T}-t, x)$. Then

$$-\left(\gamma - \frac{\sigma^2}{2}\right)\frac{\partial v(t,x)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 v(t,x)}{\partial x^2} = 0,$$
$$v(0,x) = e^{-\lambda\hat{T}},$$
$$x \in \left[\ln\left(\frac{A+\mu}{S_0}\right), +\infty\right),$$
$$v\left(\hat{T}-t, \ln\left(\frac{A+\mu}{S_0}\right)\right) = e^{-\lambda t}.$$

This problem is equivalent to the following Cauchy problem for the heat equation with an external source:

$$-\frac{\partial v(t,x)}{\partial t} + \alpha_u^2 \frac{\partial^2 v(t,x)}{\partial x^2} = \psi(t)\delta\left(x - \ln\left(\frac{A+\mu}{S_0}\right)\right),$$

$$v(0,x) = e^{-\lambda \hat{T}}, \quad x \in \mathbb{R},$$

where $\psi(t)$ is a solution of the Abel equation

$$\frac{\sqrt{\pi}}{2\alpha_u}\int_0^t \frac{\psi(s)}{\sqrt{t-s}} ds = e^{\lambda(t-\hat{T})},$$

i.e.,

$$\psi(t) = \frac{2\alpha_u}{\sqrt{\pi}} e^{-\lambda \hat{T}} \frac{d}{dt}\int_0^t \frac{(e^{\lambda s}-1)}{\sqrt{t-s}} ds,$$

whence

$$\psi(t) = \frac{\alpha_u\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda(\hat{T}-t)} \int_0^{\sqrt{\lambda \hat{T}}} e^{-x^2} dx.$$

Then the Poisson formula yields

$$v(t,x) = e^{-\lambda \hat{T}} + \frac{\alpha_u e^{-\lambda \hat{T}}}{2\pi} \int_0^t \sqrt{\frac{\lambda}{t-z}} e^{-\frac{1}{4\alpha_u^2(t-z)}\left(x - \ln\left(\frac{A+\mu}{S_0}\right)\right)^2} \times e^{\lambda z}\left(\int_0^{\sqrt{\lambda z}} e^{-y^2} dy\right) dz. \tag{20}$$

Taking into account the introduced substitution $v(t,x) = u(\hat{T}-t,x)$, it follows from (20) that

$$u(t,x) = e^{-\lambda \hat{T}} + \frac{\alpha_u e^{-\lambda \hat{T}}}{2\pi} \int_0^{\hat{T}-t} \sqrt{\frac{\lambda}{\hat{T}-t-z}} e^{-\frac{1}{4\alpha_u^2(\hat{T}-t-z)}\left(x - \ln\left(\frac{A+\mu}{S_0}\right)\right)^2} \times e^{\lambda z}\left(\int_0^{\sqrt{\lambda z}} e^{-y^2} dy\right) dz. \tag{21}$$

Substituting $t = 0$ and $x = 0$ into formula (21), we obtain

$$u(0,0) = e^{-\lambda \hat{T}} + \frac{\alpha_u e^{-\lambda \hat{T}}}{2\pi} \int_0^{\hat{T}} \sqrt{\frac{\lambda}{\hat{T}-z}} e^{-\frac{1}{4\alpha_u^2(\hat{T}-z)}\left(\ln\left(\frac{A+\mu}{S_0}\right)\right)^2} e^{\lambda z}\left(\int_0^{\sqrt{\lambda z}} e^{-y^2} dy\right) dz$$

$$= e^{-\lambda \hat{T}} + \frac{\alpha_u}{2\pi}\int_0^{\hat{T}} \sqrt{\frac{\lambda}{\tau}} e^{-\frac{1}{4\alpha_u^2 \tau}\left(\ln\left(\frac{A+\mu}{S_0}\right)\right)^2} e^{-\lambda\tau}\left(\int_0^{\sqrt{\lambda(\hat{T}-\tau)}} e^{-y^2} dy\right) d\tau$$

$$= e^{-\lambda \hat{T}} + \frac{\alpha_u}{\pi}\int_0^{\sqrt{\lambda \hat{T}}} e^{-\left(\frac{\lambda}{4\alpha_u^2\tau^2}\left(\ln\left(\frac{A+\mu}{S_0}\right)\right)^2 + \tau^2\right)}\left(\int_0^{\sqrt{\lambda \hat{T} - \tau^2}} e^{-y^2} dy\right) d\tau,$$

which proves formula (14) and completes the proof of the theorem.

Proposition. *If* $\gamma - \frac{\sigma^2}{2} \le 0$, *then*

$$\mathbb{E}_\tau\left(\frac{r_L(1-e^{-\lambda\tau})}{\lambda(1-e^{-r_L\hat{T}})} - 1\right) \le 0.$$

Proof. Suppose that $\gamma = \frac{\sigma^2}{2}$. Then Eq. (16) on set (17) with boundary conditions (18), (19) has the form

$$\frac{\partial^2 u(t,x)}{\partial x^2} = 0,$$

$$x \in \left[\ln\left(\frac{A+\mu}{S_0}\right), +\infty\right),$$

$$u(\hat{T},x) = e^{-\lambda \hat{T}},$$

$$u\left(t, \ln\left(\frac{A+\mu}{S_0}\right)\right) = e^{-\lambda t},$$

and its unique solution is the function $u(t,x) = e^{-\lambda t}$. Then $u(0,0) = 1$ and

$$\left(\frac{r_L(1-e^{-\lambda\tau})}{\lambda(1-e^{-r_L\hat{T}})} - 1\right) = -1 < 0,$$

i.e., it is unprofitable for the bank to issue a consumer loan to the borrower (recall that a consumer loan issued to the household corresponds to $\hat{H}_L > 0$).

If $\gamma < \frac{\sigma^2}{2}$, then the probability of the household defaulting on the loan grows and the boundary condition (15) is satisfied earlier. Therefore, $u(0,0) > 1$ and

$$\left(\frac{r_L(1-e^{-\lambda\tau})}{\lambda(1-e^{-r_L\hat{T}})} - 1\right) < -1;$$

hence, it is also unprofitable for the bank to issue a consumer loan. The proposition is proved.

Remark. If $\gamma - \frac{\sigma^2}{2} \le 0$, then the delivery of a consumer loan is unprofitable for the bank.

Here, the weighted average interest rate on a consumer loan is used as a parameter of the model, so a representative household is assumed to take a consumer loan with interest rate $\tilde{r}_L$, where

$$\frac{d\tilde{r}_L}{dt} = (r_L - \tilde{r}_L) \cdot \frac{\hat{H}_L(r_L)}{M(x; r_L, r_D, \gamma, \theta, \alpha, \delta)}, \tag{22}$$

and r_L is the maximum net present value (13).

4. NUMERICAL RESULTS

To make predictions of borrower behavior, it is necessary to specify the dynamics of the population

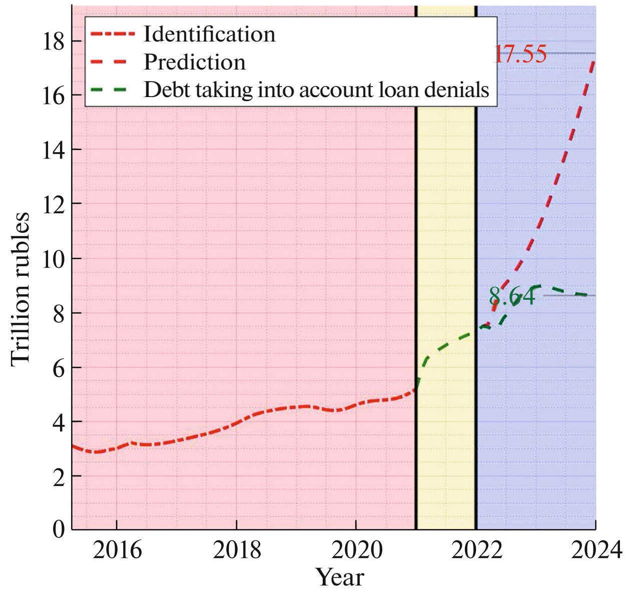

Fig. 1. Total debt of low-income borrowers.

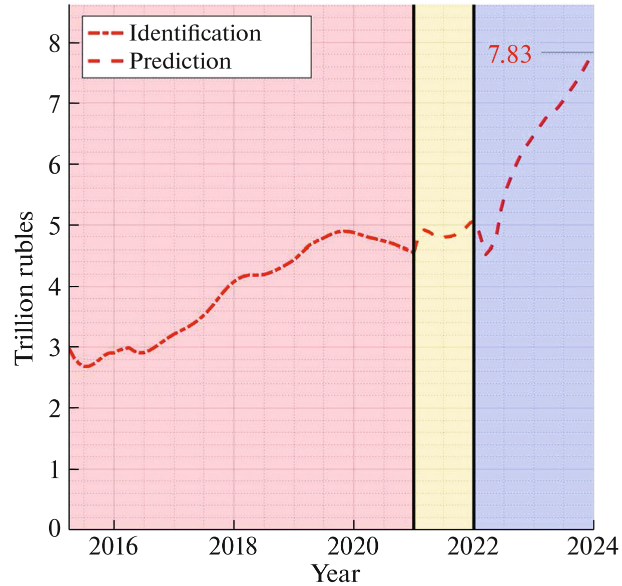

Fig. 2. Total debt of high-income borrowers.

income, the credit interest rate, and the inflation rate. According to Rosstat data, the monthly inflation was 7.8% in March 2022 and reduced sharply afterward. Taking into account Rosstat predictions, we assume that the annual inflation by the end of 2022 will be at 15%.

At the end of February 2022, the Central Bank of Russia key interest rate was sharply raised to 20% per annum, and the number of consumer loan denials grew to 75%. Although the key interest rate fell to 8% later, the interest rate on loans issued to low-income households remains prohibitive and stays at 40% per annum. The interest rate for high-income borrowers can be modeled using the model of the consumer loan market presented in Section 3. According to the numerical results, an increase in the credit interest rate to 25–30% for high-income borrowers is expected by the middle of 2023.

Because large companies pulled out of Russia in the spring of 2022 and some manufacturing plants stopped production, a considerable part of the population lost their income. Low-income population segments suffered the basic loss of income. By the end of 2023, we expect that the nominal income decline in the low-income population will be 15% compared to the early 2022, while the income dynamics in high-income population segments will remain unchanged.

4.1. Scenario 1: Analysis of Financial Position of Borrowers

According to the numerical results, both low- and high-income borrowers in regions of all groups are ruined in the spring of 2022 (the solvency condition $x > -\dfrac{S}{r_L - \gamma}$ is violated). Later, the solvency of borrowers is recovered, but, starting at the late 2022–early 2023, low-income borrowers once again face insolvency, which leads to a noticeable increase in demand for consumer loans among the low-income population. As was noted earlier, the number of loan denials in the low-income population has grown sharply since March 2022. Calculating the total debt over all groups of regions, we took into account this tendency and assumed that, if a low-income household is in the state of insolvency, then a commercial bank rejects their credit application.

Figures 1 and 2 present the total debts over the entire country with differentiation into low- and high-income borrowers. The areas in the plots describe three stages of using the model: identification from March 2015 to December 2020, verification from January 2021 to December 2021, and prediction from January 2022 to December 2023.

Note that, if banks issued consumer loans to low-income households, then the consumption level of the latter would be maintained at the minimum level, but the debt would grow noticeably (to 17.55 trillion rubles by the end of 2023) and the total debt would be 25.38 trillion rubles. Consumer loans denied to the low-income population lead to social tension. By the early 2022, the total debt on consumer credit was 12.6 trillion rubles. In this scenario, taking into account credit denials, the resulting debt by the end of 2023 grows to 16.47 trillion rubles, with the credit denials over the prediction period amounting to 8.91 trillion rubles. Consumer credit ceases to work as a mechanism of social adaptation of households.

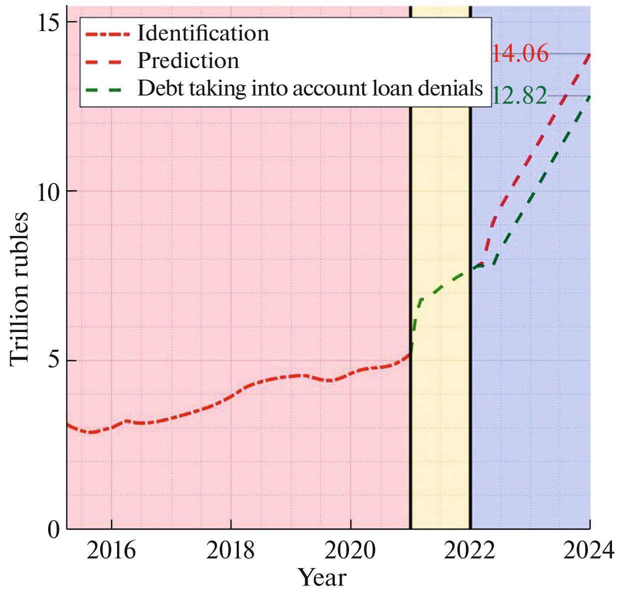

Fig. 3. Total debt of low-income borrowers.

4.2. Scenario 2: Subsidization of Low-Income Borrowers

The main problem of consumer crediting in Russia is the low solvency of low-income borrowers. The given scenario answers the question as to whether the role of consumer credit as a mechanism of social adaptation can be maintained due to targeted financial support of low-income borrowers. It is assumed that the incomes of low-income borrowers grow from August 2022 to the end of 2023 due to their financial support through government institutions. According to the numerical results, over this period, low-income borrowers have to be provided with 58.9 billion rubles in urban areas of the poor group of regions, 70.6 billion rubles in rural areas of the poor group, 594.1 billion rubles in the middle group of regions, and 246 billion rubles in the group of rich regions. These results were obtained assuming that the solvency of low-income borrowers after the spring of 2022 is maintained in all groups of regions. The total debt of low-income borrowers in all groups of regions is presented in Fig. 3.

If subsidies amounting to 969.6 billion rubles are issued to low-income borrowers until the end of 2023, then their solvency is recovered, whereas loan denials would occur during the spring of 2022. The consumer loan denials would amount to 1.24 trillion rubles, and the resulting debt over two years, taking into account the debt of high-income borrowers, would grow by 9.29 trillion rubles.

5. CONCLUSIONS

The economic behavior of a representative household has been described in the form of an optimal control problem of the Ramsey type. Representative households from various groups of regions of Russia were distinguished using Rosstat HBS data. The model was identified using statistical data of HBS from March 2015 to December 2020 and was verified against statistical data over 2021. The solvency of low- and high-income households until the end of 2023 was analyzed using the model. The formation of a credit interest rate for high-income borrowers was modeled by maximizing the net present value for commercial banks. The modeling of the interest rate was based on the Feynman–Kac formula.

It was found that the main problem of consumer crediting is the quick default faced by low-income borrowers, the credit interest rates for whom are currently prohibitive. Denial of loan applications of low-income borrowers leads to social tension. The computations based on the model showed that target subsidies issued to low-income people can recover their solvency and, as a result, can reduce social tension in these population segments. The subsidies to be granted by the end of 2023 amount to less than 1 trillion rubles. The computer code [7] allows the economic states of households in Russia to be analyzed taking into account specific regional features.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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