Covering Planar Sets

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Abstract—Methods for improving upper and lower bounds for various coverings of planar sets are proposed. New bounds for various numbers of partition constituents are presented, and suggestions for the generalization of the presented methods are offered.

Keywords: Borsuk problem, diameter of a set, coverings of planar sets, universal covering systems, chromatic number

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1. INTRODUCTION AND BASIC DEFINITIONS

Let F be an arbitrary bounded planar set and $n \in \mathbb{N}$. Define the quantity

$$d_n(F) = \inf\{x \in \mathbb{R}^+ : \exists F_1, \dots, F_n : F \subseteq F_1 \cup \dots \cup F_n, \\ \forall i \text{ diam}(F_i) \leq x\}.$$

In other words, among all coverings of F by n sets $F_1, \dots, F_n$, we want to choose coverings consisting of sets of as small a diameter as possible.

Note that $d_n(F)$ does not change if all sets of the covering are assumed to be convex and closed. This conclusion follows from the fact that the diameter of the closure of the convex hull of an arbitrary set F coincides with the diameter of F .

Note that, for an arbitrary F , the sequence $d_n(F)$ is nonincreasing, since, in the class of all coverings by $n + 1$ sets, there is a subclass with $F_{n+1} = \emptyset$ which coincides with the class of all coverings by n sets.

Define the quantity $d_n = \sup d_n(F)$, where the supremum is taken over all planar sets F of unit diameter. From the remark made above, it is clear that the sequence d_n is nonincreasing.

The study of d_n is deeply motivated by the classical Borsuk problem of partitioning sets into subsets of smaller diameter (see [1–4]). In different years, d_n for various n was estimated by Lenz [5], Dembinski and Lassak [6], Filimonov [7], Belov and Aleksandrov [8],

and Koval' [9]. We have been able to significantly strengthen many of the previous results. New results are presented in Section 2. Below, we give several definitions important for the subsequent formulations and proofs.

Definition 1. A set $\Omega \subset \mathbb{R}^2$ is called a universal covering set if any planar set F of unit diameter can be completely covered by Ω (i.e., in the plane, there exists a set Ω_0 congruent to Ω such that $F \subset \Omega_0$).

In 1920, Pal proved [10] that the regular hexagon with side length $\frac{1}{\sqrt{3}}$ is a universal covering set. We denote this hexagon by Ω_6 . Let us give another definition.

Definition 2. A system of sets $S = \{\Omega_\alpha\}_{\alpha \in I}$ is called a universal covering system if any planar set F of unit diameter can be completely covered by one of the sets Ω_α . Here, I is an index set (possibly infinite).

Let F be an arbitrary set of unit diameter. We cover it by Ω_6 and draw all six straight lines perpendicular to the segments joining the center of Ω_6 to all its vertices and separated from the center by a distance of $\frac{1}{2}$. These lines cut off six triangles from Ω_6 . Obviously, for each pair of triangles containing opposite vertices, at least one of them does not contain interior points of F (since the distance between these triangles is unity). Such a triangle is called a cutoff triangle. A covering set with three consecutive corners of this type removed (cutoff) is denoted by $\Omega_{6,2}$, while a covering set with three alternating corners removed is denoted by $\Omega_{6,1}$. It is easy to see that $\{\Omega_{6,1}, \Omega_{6,2}\}$ is a universal covering system.

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Fig. 1. Figures used in UCS.

Note that a regular hexagon of side length $\frac{1}{\sqrt{3}}$ (i.e., Ω_6) with two corners at alternating vertices removed as described above is also a universal covering set, because it covers both $\Omega_{6,1}$ and $\Omega_{6,2}$. This universal covering set is denoted by Ω_2 .

Additionally, we consider the quantity

$$d'_n(F) = \inf\{x \in \mathbb{R}^+ : \exists F_1, \dots, F_n : F \subseteq F_1 \cup \dots \cup F_n, \\ \forall i \forall X, Y \in F_i : |XY| \neq x\},$$

where F is an arbitrary bounded planar set and $n \in \mathbb{N}$. In other words, $d'_n(F)$ is the infimum over all x such that F can be properly colored with n colors if x is treated as a forbidden distance. Thus, $d'_n(F)$ is motivated by the classical Hadwiger–Nelson problem of coloring the plane (see [11–17]).

Define $d'_n = \sup d'_n(F)$, where the supremum is taken over all planar sets F of unit diameter. Clearly, the sequence d'_n is nonincreasing. Moreover, $d'_n = 0$ for all positive integers $n \geq 7$ as a consequence of the well-known estimate $\chi(\mathbb{R}^2) \leq 7$ for the chromatic number of the plane (see [11]).

In the next section, we formulate new results, compare them with previously known ones, and briefly discuss the ideas of the proofs.

2. IMPROVED RESULTS

2.1. Formulations

The improved results for elements of the sequence d_n for $1 \leq n \leq 30$ are given in Table 1. The comment column indicates the percentage by which the gap between the upper and lower bounds is reduced due to the improvements proposed in this paper. The last column lists the figures used as a universal covering system in the proof of the upper bound.

Figure 1 shows all the covering sets used to obtain the upper bounds. Partitions for the proofs of the upper bounds are given in [18]. In the proofs of the lower bounds for d_{10} , d_{11} , and d_{54} , the set of diameter 1 is specified as a unit disk.

All constants are given with four significant decimal digits. In additions to the improvements listed in Table 1, we have proved the estimates $\frac{1}{\sqrt{52}} \leq d_{54}$ and $d'_6 \leq 0.5$.

2.2. Ideas behind the Proofs

To prove $d_n \leq \rho$, where ρ is a fixed number, we consider a universal covering system S and partition each of its constituting sets into n subsets, each having a diameter of at most ρ . In Table 1, the column titled as “UCS used” presents the universal covering system used to prove the indicated upper bound on d_n for the corresponding value of n .

Table 1. Results

n	New lower bound	Old lower bound	Old upper bound	New upper bound	Comment	UCS used
1	—	1.0000	1.0000	—	Tight	—
2	—	1.0000	1.0000	—	Tight	—
3	—	0.8660	0.8660	—	Tight	—
4	—	0.7071	0.7071	—	Tight	—
5	—	0.5877	0.6020	0.5953	47%	$\Omega_{6,2}, \Omega_{6,1,1}, \Omega_{6,1,2,1}, \Omega_{6,1,2,2}, \Omega_{6,1,2,3}$
6	—	0.5051	0.5344	—	—	—
7	—	0.5000	0.5000	—	tight	—
8	—	0.4338	0.4456	—	—	—
9	—	0.3826	0.4047	—	—	—
10	0.3667	0.3420	0.4012	—	41%	—
11	0.3420 ¹⁾	0.3333	0.3970	0.3942	19%	Ω_2
12	—	0.3333	0.3660	—	—	—
13	—	0.3333	0.3660	0.3550	33%	Ω_2
14	—	0.3090	0.3324	—	—	—
15	—	0.2928	0.3324	0.3226	24%	Ω_2
16	—	0.2817	0.3324	0.3191	26%	Ω_6
17	—	0.2701	0.3324	0.3010	48%	Ω_2
18	—	0.2588	0.3324	0.2989 ²⁾	46%	Ω_6
19	—	0.2500	0.2857	—	—	—
20	—	0.2500	0.2857	—	—	—
21	—	0.2393	0.2857	0.2723	29%	Ω_2
22	—	0.2323	0.2857	0.2650	39%	Ω_2
23	—	0.2225	0.2857	0.2650	33%	Ω_2
24	—	0.2167	0.2849	0.2610	35%	$\Omega_{6,1}, \Omega_{6,2}$
25	—	0.2079	0.2776	0.2610	24%	$\Omega_{6,1}, \Omega_{6,2}$
26	—	0.2030	0.2709	0.2610	15%	$\Omega_{6,1}, \Omega_{6,2}$
27	—	0.2000	0.2646	0.2610	5%	$\Omega_{6,1}, \Omega_{6,2}$
28	—	0.2000	0.2587	0.2500	15%	Ω_2
29	—	0.1950	0.2531	0.2500	5%	Ω_2
30	—	0.1939	0.2479	—	—	—

¹⁾ The exact value of the lower bound obtained for d_{11} is $\sin(\pi/9)$.

²⁾ The exact value of the upper bound obtained for d_{18} is $\sqrt{\frac{2-\sqrt{3}}{3}}$.

The partitions of elements of the universal covering system into n parts of diameter ρ for all improved bounds are not described in detail in this work, but we consider one of the most elegant partitions that considerably improves the upper bound for d_{18} .

Setting $\rho = \sqrt{\frac{2-\sqrt{3}}{3}} \approx 0.2989$, we prove that $d_{18} \leq \rho$. Consider a universal covering system S consisting of a single set, namely, the universal covering set Ω_6 .

In Fig. 2, the regular hexagon $ABCDEF$ with side length $\frac{1}{\sqrt{3}}$ corresponds to the covering set Ω_6 . The circle ω is inscribed in this hexagon. The segment AO intersects ω at the point N , and the perpendicular to the line AO through the point N intersects the segments AB and AF at the points A_1 and A_2 , respectively. Similarly, we obtain the points $B_1, B_2, \dots, F_1, F_2$. It can be shown that the dodecagon $F_1F_2\dots B_1B_2A_1A_2$ is regular. Then the original hexagon is partitioned into six

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Fig. 2. Partition of Ω_6 into 18 parts.

equal quadrilaterals OA_2AB_2 , OB_2BC_2 , ..., OF_2FA_2 . Consider one of them and show how to partition it into three parts of diameter exactly ρ , which implies that the original hexagon can be divided into 18 parts of diameter ρ . Without loss of generality, consider the quadrilateral OD_2DE_2 , which is inscribed, since $\angle E_2OD_2 = 60^\circ$ and $\angle E_2DD_2 = 120^\circ$. Let γ be the circle circumscribed about OD_2DE_2 and S be the center of γ . It can be shown that the radius of γ is exactly $\rho = \sqrt{\frac{2-\sqrt{3}}{3}}$. The circle of radius ρ centered at the point D_1 intersects the segments OE_2 and OD_2 at the points U and V , respectively. Then the quadrilateral OD_2DE_2 is divided into two quadrilaterals $OUSV$, USD_1E_2 and one pentagon DD_1SVD_2 , each of diameter ρ . In Fig. 2, the lengths of the segments depicted by a thick dotted line are equal to ρ . Thus, we have shown that the hexagon $ABCDEF$ can be partitioned into 18 parts, each of diameter ρ . The parts of this partition are depicted in Fig. 2.

It was shown above how to obtain the upper bound for d_{18} . Importantly, among all bounds obtained in this study, this bound is the only one that is exactly expressed by radicals.

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