Symplectic Geometry of the Koopman Operator

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Abstract—We consider the Koopman operator generated by an invertible transformation of a space with a finite countably additive measure. If the square of this transformation is ergodic, then the orthogonal Koopman operator is a symplectic transformation on the real Hilbert space of square summable functions with zero mean. An infinite set of quadratic invariants of the Koopman operator is specified, which are pairwise in involution with respect to the corresponding symplectic structure. For transformations with a discrete spectrum and a Lebesgue spectrum, these quadratic invariants are functionally independent and form a complete involutive set, which suggests that the Koopman transform is completely integrable.

Keywords: Koopman operator, ergodicity, symplectic structure, quadratic invariants, discrete spectrum, Lebesgue spectrum

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1. KOOPMAN OPERATOR

Let (M, μ) be a space with a finite countably additive measure μ and $T: M \rightarrow M$ be an invertible measure-preserving transformation. Let $L_2(M, \mu)$ be the Hilbert space of real functions on M that are square integrable with respect to μ . The scalar product of functions f and g is defined as usual as

$$(f, g) = \int_M f(x)g(x)d\mu(x).$$

The Koopman operator $U: L_2 \rightarrow L_2$ maps the function $x \mapsto f(x)$ to the function

$$x \mapsto f(T(x)).$$

It is well known that this operator is orthogonal:

$$U^*U = UU^* = I, \quad (1)$$

or, equivalently, $U^{-1} = U^*$. Clearly, $\lambda = 1$ is always its eigenvalue. Nonzero constant functions on M are the corresponding eigenvectors.

An important result for our study is a consequence of (1) stating that the operator U admits a quadratic invariant $F = (f, f)$. In other words,

$$(Uf, Uf) = (f, f) \quad \text{for all } f \in L_2.$$

As an illustration, we consider two examples. Let M be the n -dimensional torus $\mathbb{T}^n = \{x_1, \dots, x_n \bmod 2\pi\}$ and μ be the standard Lebesgue measure on $\mathbb{T}^n$.

Example 1. $T: x \mapsto x + \alpha$, $\alpha \in \mathbb{R}^n$. In the nonresonance case (when the relation $(m, \alpha) = k$, $m \in \mathbb{Z}^n$, $k \in \mathbb{Z}$ implies $m = 0$, $k = 0$) the mapping T is ergodic (Weyl's theorem). The corresponding discrete dynamical system is often called the Kronecker–Weyl cascade.

Example 2. $T: x \mapsto Ax$, where A is a unimodular matrix with integer elements. If none of the eigenvalues of A belong to the unit circle of the complex plane, then T is a fortiori mixing. A classical example is as follows: $n = 2$ and

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

It should be kept in mind that, for the transformations from Example 2, ergodicity is equivalent to mixing.

In Example 1, the functions $\varphi_m = \exp[i(m, x)]$, $m \in \mathbb{Z}^n$, are the eigenfunctions for the corresponding Koopman operator:

$$U\varphi_m = \lambda_m \varphi_m, \quad \lambda_m = e^{i(m, \alpha)}.$$

They make up an orthogonal basis in $L_2(\mathbb{T}^n)$. This simple observation means that we need to consider the space of square integrable functions with complex values. In the real case, the Koopman operator has two-dimensional invariant planes spanned by the vectors $\sin(m, x)$ and $\cos(m, x)$ ($m \neq 0$).

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If the mapping T is a mixing (or even a weak mixing), then the spectrum of the Koopman operator is “continuous” (more precisely, the only eigenvalue is $\lambda = 1$ and this eigenvalue is simple).

The basic facts concerning the spectral theory of the Koopman operator can be found, for example, in [1, 2].

2. SYMPLECTIC STRUCTURE

Consider a more general situation. Let $\mathcal{H}$ be a real Hilbert space (the case $\dim \mathcal{H} < \infty$ is not excluded) with inner product $(\cdot, \cdot)$, and let $U: \mathcal{H} \rightarrow \mathcal{H}$ be an orthogonal operator. Obviously, the mapping $x \mapsto Ux$ admits the quadratic invariant $F(x) = (x, x)$.

Define the operator

$$\Omega = (U^* + I)(U - I). \quad (2)$$

In view of the orthogonality conditions (1),

$$\Omega = U - U^*.$$

Therefore, Ω is a skew-self-adjoint operator. In view of (1), it can also be represented in the form

$$\Omega = (U - I)(U^* + I) = -(U^* - I)(U + I). \quad (3)$$

Operator (2) is associated with the bilinear skew-symmetric form

$$[x, y] = (\Omega x, y) = -(x, \Omega y). \quad (4)$$

Recall that this form is called nondegenerate if $[x, y] = 0$ for all $y \in \mathcal{H}$ implies that $x = 0$. If the 2-form (4) is nondegenerate, it defines a symplectic structure in $\mathcal{H}$ (and it itself is usually called a symplectic structure).

It is well known that the spectrum of an orthogonal transformation lies on the unit circle. It follows from (2) and (3) that the operator Ω is singular if $\lambda = \pm 1$ are eigenvalue of the operator U . Note the following simple result.

Proposition 1. *If the operator U^2 does not have non-zero eigenvectors with a unit eigenvalue, then the 2-form $[\cdot, \cdot]$ is nondegenerate.*

Indeed, suppose that $[\cdot, \cdot]$ is degenerate and $(\Omega x, y) = 0$ for all $y \in \mathcal{H}$. Then $\Omega x = 0$ for at least one $x \neq 0$. This means that $Ux = U^{-1}x$, which is equivalent to the equality $U^2x = x$.

Theorem 1. *The operator U preserves the 2-form $[\cdot, \cdot]$.*

It is necessary to prove that

$$U^*\Omega U = \Omega.$$

In view of (1), the left-hand side is given by

$$U^*(U - I)(U^* + I)U = (I - U^*)(I + U) = \Omega,$$

as required.

Thus, if the 2-form $[\cdot, \cdot]$ is nondegenerate, then $(\mathcal{H}, [\cdot, \cdot])$ is a symplectic space and (by Theorem 1) the

operator U is symplectic. In the finite-dimensional case, this fact was noted in [3]. Specifically, $\dim \mathcal{H}$ is even.

If U is the Koopman operator from Section 1, then the 2-form $[\cdot, \cdot]$ is degenerate: constant functions remain intact under the operator U . To rectify the situation, we need to consider the Hilbert space $\hat{L}_2 \subset L_2$ orthogonal to the one-dimensional subspace consisting of constant functions. In other words, $\hat{L}_2$ consists of square integrable functions with zero mean. Clearly, U maps $\hat{L}_2$ to $\hat{L}_2$.

Specifically, the following result is true.

Proposition 2. *Suppose that the mapping $T^2: M \rightarrow M$ is ergodic. Then the 2-form $[\cdot, \cdot]$ is nondegenerate on $\hat{L}_2$.*

Indeed, $\lambda = 1$ is then a simple eigenvalue of the corresponding Koopman operator $U^2: L_2 \rightarrow L_2$. Therefore, the operator U^2 in $\hat{L}_2$ does not have nonzero eigenvectors with eigenvalue $\lambda = 1$. It remains to use Proposition 1.

Note that the square of an ergodic transformation is not necessarily ergodic. However, this is a fortiori the case in two examples given in Section 1.

Summarizing, in the general case (for cascades with an ergodic square) the Koopman operator $U^2: \hat{L}_2 \rightarrow \hat{L}_2$ is a symplectic operator. However, the invariant symplectic structure depends on this operator.

It is well known that a real orthogonal operator that is a symplectic transformation is a unitary operator. This means that, in the real space $\hat{L}_2$, it is possible to introduce a complex structure such that the action of U is equivalent to the action of a unitary operator in the complex space $\hat{L}_2$. This construction is especially simple for operators with a discrete spectrum. These issues are discussed in [4] for real linear systems of differential equations with a quadratic invariant in a Hilbert space in order to represent them in the form of Schrödinger equations.

3. QUADRATIC INVARIANTS

The quadratic form (Bx, x) , where $B^* = B$, is an invariant of the linear mapping $x \mapsto Ux$ if and only if

$$U^*BU = B.$$

It turns out that, in addition to the original quadratic invariant (when $B = I$), this mapping admits a whole set of quadratic invariants that are pairwise in involution with respect to the symplectic structure from Section 2.

The nondegenerate skew-symmetric 2-form $[\cdot, \cdot]$ makes it possible to introduce the Poisson bracket on the vector space of continuous quadratic forms. Let

$$f = (Fx, x)/2 \quad \text{and} \quad g = (Gx, x)/2$$

be two such forms (F and G are bounded self-adjoint operators). Their Poisson bracket $\{f, g\}$ is the quadratic form

$$h = (Hx, x)/2, \quad \text{where} \quad H = F\Omega G - G\Omega F. \quad (5)$$

The operator H (like F and G) is self-adjoint (symmetric and bounded). In particular, the quadratic form h is also continuous.

The bracket $\{, \}$ is bilinear, skew-symmetric, and satisfies the Jacobi identity. In [5, 6] the Poisson bracket was defined under more general assumptions.

We introduce the self-adjoint operators

$$B_n = [(U + I)(U^* + I)]^n = (U + I)^n(U^* + I)^n, \\ n \geq 0.$$

Theorem 2. *The continuous quadratic forms*

$$F_n = (B_n x, x)/2 \quad (6)$$

are invariants of the mapping $x \mapsto Ux$; moreover,

$$\{F_k, F_l\} = 0 \quad (7)$$

for all $k, l \geq 0$.

To prove the invariance of F_n , we need to check the equality $U^* B_n U = B_n$, or

$$U^* (U + I)^n (U^* + I)^n U = (U + I)^n (U^* + I)^n. \quad (8)$$

Indeed, since U is orthogonal, we have

$$U^* (U + I) = U^* + I, \quad (U^* + I)U = U + I.$$

Therefore, the left-hand side of (8) is

$$(U^* + I)(U + I)^{n-1}(U^* + I)^{n-1}(U + I) \\ = (U + I)^n (U^* + I)^n.$$

To prove the involution of these invariants, we need to check the equality

$$[(U + I)(U^* + I)]^k (U^* + I)(U + I) \\ \times [(U + I)(U^* + I)]^l = [(U + I)(U^* + I)]^l \\ \times (U^* + I)(U + I)[(U + I)(U^* + I)]^k. \quad (9)$$

Since the operators $U^* + I$, $U + I$, and $U - I$ commute with each other, both sides of (9) are symmetric with respect to k and l . This proves equality (7).

The quadratic invariants F_n may be dependent. A simple example is as follows: if $U = I$, then all of them are proportional to (x, x) .

Suppose that $N = \dim \mathcal{H}$ is finite and U is an orthogonal operator having no eigenvalues $\lambda = \pm 1$. Then N is even. Additionally, as was noted in [3], if the operator U has no multiple eigenvalues, then the quadratic forms $F_1, \dots, F_{N/2}$ are functionally independent. Since they are pairwise in involution, their non-empty joint levels

$$\{x \in \mathcal{H}: F_1(x) = c_1, \dots, F_{N/2}(x) = c_{N/2}\}$$

for almost all $c_1, \dots, c_{N/2}$ are $N/2$ -dimensional tori invariant under the action of the operator U . Furthermore, on these tori, we can introduce angular coordinates

$$\varphi_1, \dots, \varphi_{N/2} \pmod{2\pi},$$

such that the action of U is reduced to the Kronecker–Weyl mapping

$$\varphi_j \mapsto \varphi_j + \alpha_j, \quad \alpha_1, \dots, \alpha_{N/2} = \text{const.}$$

Since the operator U is linear, the numbers $\{\alpha_j\}$ do not change from one torus to another.

Note that a linear symplectic mapping is completely integrable without assuming that the spectrum is simple. This result can be derived from the theory of Williamson normal forms [7]. However, in the case of a simple spectrum, a complete set of involution invariants can be presented without preliminarily solving the algebraic eigenvalue and eigenvector problem for a symplectic operator.

In the infinite-dimensional case, the situation is more complicated. We consider two (in a sense, opposite) cases, namely, when the operator U has a simple discrete spectrum and when its spectrum is continuous.

4. DISCRETE SPECTRUM

It is well known that the spectrum of an orthogonal operator lies on the unit circle of the complex plane. The possibility of existence of two real eigenvalues $\lambda = \pm 1$ is ruled out.

Suppose that

$$U(\xi + i\eta) = (\alpha + i\beta)(\xi + i\eta), \quad (10)$$

where ξ and η are vectors from the real Hilbert space $\mathcal{H}$ and $\alpha^2 + \beta^2 = 1$. Equality (10) is equivalent to two real relations

$$U\xi = \alpha\xi - \beta\eta, \quad U\eta = \beta\xi + \alpha\eta.$$

Therefore, the real plane π spanned by the vectors ξ , η is invariant under the action of the orthogonal operator U . Moreover, it is easy to show that

$$(\xi, \xi) = (\eta, \eta) \quad \text{and} \quad (\xi, \eta) = 0,$$

so we can assume that the vectors ξ and η are of unit length; they make up an orthonormal basis in the plane π .

Let

$$x = p\xi + q\eta, \quad p, q \in \mathbb{R}$$

be points of the invariant plane π and $x' = Ux = p'\xi + q'\eta$. Then

$$p' = \alpha p + \beta q, \quad q' = -\beta p + \alpha q \quad (11)$$

define an orthogonal transformation of π ; this is a rotation through the angle

$$\varphi = \arctan \frac{\beta}{\alpha}.$$

Clearly, the linear transformation admits the quadratic invariant

$$p^2 + q^2. \quad (12)$$

The last observation can be made to have an invariant meaning if we introduce the orthoprojector P of the Hilbert space onto the plane π . This operator is bounded and self-adjoint: $P^* = P$ and $P^2 = P$. Invariant (12) can be represented in the form

$$f = (Px, Px), \quad x \in \mathcal{H}.$$

Now we assume that the operator U has infinitely many distinct eigenvalues

$$\alpha_1 + i\beta_1, \alpha_2 + i\beta_2, \dots$$

with eigenvectors

$$\xi_1 + i\eta_1, \xi_2 + i\eta_2, \dots$$

Once again, the two-dimensional planes

$$\pi_n = \{x \in \mathcal{H}: x = p\xi_n + q\eta_n, p, q \in \mathbb{R}\}$$

are invariant under the operator U . It can be shown that the planes π_k and π_l are orthogonal, of course, if $k \neq l$ (see, e.g., [6]).

Thus, in $\mathcal{H}$, there is an orthonormal set of vectors

$$\xi_1, \eta_1, \xi_2, \eta_2, \dots \quad (13)$$

Additionally, the operator U admits infinitely many quadratic invariants

$$f_n = (P_n x, P_n x), \quad n \geq 1, \quad (14)$$

where P_n is the orthoprojector onto π_n .

On the other hand, the planes π_n are also invariant under the action of the skew-self-adjoint operator Ω . In the invariant plane π_n , Ω is represented by the skew symmetric matrix

$$\begin{bmatrix} 0 & 2\beta_n \\ -2\beta_n & 0 \end{bmatrix}.$$

All these matrices are nonsingular, since $\beta_n \neq 0$. Otherwise, the operator U would have the eigenvalues ± 1 .

Theorem 3. Assume that the orthonormal system of vectors (13) is complete. Then the following assertions hold:

(a) Quadratic forms (14) make up a complete set of independent quadratic invariants of the operator U that are pairwise in involution with respect to the symplectic structure in $\mathcal{H}$ defined by the skew-self-adjoint operator Ω .

(b) The joint levels of these invariants

$$\{x \in \mathcal{H}: f_n(x) = c_n, n \geq 1; c_n > 0 \text{ and } \sum c_j^2 < \infty\}$$

are the infinite-dimensional tori

$$\mathbb{T}_c^\infty = \prod_{n=1}^{\infty} \mathbb{T}_n^1.$$

(c) On these tori, it is possible to introduce angular variables $\varphi_1, \varphi_2, \dots \bmod 2\pi$ such that the action of the operator U on $\mathbb{T}_c^\infty$ in these variables is given by the formulas

$$\varphi_n \mapsto \varphi_n + \varphi_n^0, \quad \tan \varphi_n^0 = \frac{\beta_n}{\alpha_n}; \quad n \geq 1. \quad (15)$$

This statement suggests that the mapping $x \mapsto Ux$ is completely integrable (as in the finite-dimensional case, which was mentioned in Section 3). The completeness of the family of involution invariants (14) means that this family cannot be supplemented with another quadratic form that is independent of (14) and is in involution with them. Mapping (15) is an infinite-dimensional variant of the Kronecker–Weyl mapping. Its properties are quite similar to the ergodic properties of continuous Kronecker–Weyl flows on infinite-dimensional tori (they are discussed in [8, 9], where further references can be found).

5. LEBESGUE SPECTRUM

The case of a continuous spectrum is more complicated. We restrict our consideration to cascades with a Lebesgue spectrum. Specifically, they include automorphisms of the torus from Example 2 (see Section 1). Such systems a fortiori have mixing. For simplicity, we consider a simple Lebesgue spectrum.

In this case, $\mathcal{H} = \hat{L}_2$ has a complete orthonormal basis $\{e_j\}$, $j \in \mathbb{Z}$ such that

$$Ue_j = e_{j+1}. \quad (16)$$

In a sense, the general case is reduced to this partial one (see, e.g., [2]).

Let $x = \sum p_j e_j$ and $\sum p_j^2 < \infty$. Then

$$Ux = \sum p_j e_{j+1} = \sum p_{j-1} e_j.$$

Thus, if $\mathcal{H}$ is identified with l_2 (the space of sequences $\{p_j\}$ infinite on both sides with the condition $\sum p_j^2 < \infty$), then the action of U is reduced to a left shift of the elements by unity. Of course, the scalar square $F = \sum p_j^2$ is conserved under this shift. Theorem 2 gives an infinite set of quadratic invariants:

$$F_n = \sum p_{j-n} p_j + \sum p_{j+n} p_j. \quad (17)$$

More precisely, invariants (6) are reduced to finite linear combinations of quadratic forms (17).

Furthermore, $\Omega e_j = e_{j+1} - e_{j-1}$ for $j \in \mathbb{Z}$. Therefore,

$$\Omega x = \sum p_j \Omega e_j = \sum (p_{j-1} - p_{j+1}) e_j. \quad (18)$$

The corresponding 2-form $[x, y] = (\Omega x, y)$ is nondegenerate. Indeed, suppose that $(\Omega x, y) = 0$ for all y . Set $y = e_j$. Then (18) implies that $p_{j-1} = p_{j+1}$. Therefore, the elements of the vector x with even (odd) indices are equal to each other. However, all of them then vanish; otherwise, the series $\sum p_j^2$ diverges. Hence, $x = 0$.

Therefore, the skew-self-adjoint operator Ω defines a symplectic structure in $\mathcal{H} = \hat{L}_2$. Formula (18) implies that this structure is invariant under the transformation U . By Theorem 2, quadratic invariants (17) are in involution with respect to the Poisson bracket generated by this symplectic structure.

It is easy to show that the quadratic forms $F_0, F_1, F_2, \dots$ are independent: their gradients (as vectors from $\mathcal{H}$) are linearly independent (at least at one point of $\mathcal{H}$). Furthermore, it can be shown that any continuous quadratic form on $\mathcal{H}$ that is invariant under the action of U can be represented in the form

$$\sum_{n=0}^{\infty} \alpha_n F_n,$$

where $\alpha_n, n \geq 0$, are constants. This suggests that the Koopman operator is completely integrable for systems with a Lebesgue spectrum. However, the structure of joint levels of quadratic invariants (17) remains an open question. Can the action of the Koopman operator on these infinite-dimensional manifolds be reduced to the Kronecker–Weyl mapping?

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