CONTROL THEORY

Sub-Riemannian (2, 3, 5, 6)-Structures

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Abstract—We describe all Carnot algebras with growth vector (2, 3, 5, 6), their normal forms, an invariant that separates them, and a change of basis that transforms such an algebra into a normal form. For each normal form, Casimir functions and symplectic foliations on the Lie coalgebra are computed. An invariant and normal forms of left-invariant (2, 3, 5, 6)-distributions are described. A classification, up to isometries, of all left-invariant sub-Riemannian structures on (2, 3, 5, 6)-Carnot groups is obtained.

Keywords: sub-Riemannian geometry, Carnot algebras, Carnot groups, left-invariant sub-Riemannian structures

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Sub-Riemannian structures [1] are stratified in depth, i.e., with respect to the minimum order of Lie brackets required for generating a tangent space from basis vector fields. The complexity of sub-Riemannian (SR) structures grows substantially with increasing step. At present, SR structures of step at most three have been examined in detail [2–8]. Accordingly, a task of great interest is to systematically investigate SR structures of step 4. The study of the simplest of these structures, namely, a nilpotent SR structure with growth vector (2, 3, 5, 8) (see Example 2 below) was initiated in [9–11]. Below, we obtain a complete classification of nilpotent SR structures and distributions with growth vector (2, 3, 5, 6). It is shown that all such structures are quotient structures of a nilpotent SR structure with growth vector (2, 3, 5, 8).

1. SUB-RIEMANNIAN QUOTIENT STRUCTURES

In this paper, all Lie algebras are considered over the field $\mathbb{R}$.

Definition 1. A nilpotent Lie algebra $\mathfrak{g}$ is called a *Carnot algebra* if

(i) $\mathfrak{g}$ is graded:

$$\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \dots \oplus \mathfrak{g}^{(s)},$$

$$[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)}, \quad \mathfrak{g}^{(k)} = \{0\} \quad \text{for } k > s, \quad (1)$$

(ii) $\mathfrak{g}$ is generated by the first component:

$$\text{Lie}(\mathfrak{g}^{(1)}) = \mathfrak{g}. \quad (2)$$

The corresponding connected simply connected Lie group is called a *Carnot group*.

Conditions (i) and (ii) are equivalent to the condition

$$[\mathfrak{g}^{(1)}, \mathfrak{g}^{(i)}] = \mathfrak{g}^{(i+1)}, \quad \mathfrak{g}^{(k)} = \{0\} \quad \text{for } k > s.$$

Definition 2. The *growth vector* of a Carnot algebra $\mathfrak{g}$ is defined as

$$(n_1, \dots, n_s), \quad n_j = \sum_{i=1}^j \dim \mathfrak{g}^{(i)}.$$

Here, $(n_1, \dots, n_s)$ is the growth vector of a left-invariant distribution on the Lie group of the Lie algebra $\mathfrak{g}$ generated by the subspace $\mathfrak{g}^{(1)} \subset \mathfrak{g}$.

Let M be a smooth manifold. A *sub-Riemannian structure* on M [1] is a pair (Δ, g) consisting of a vector distribution $\Delta \subset TM$ and a scalar product g in Δ .

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. A *left-invariant SR structure* on G consists of a left-invariant distribution on G and a left-invariant scalar product in the distribution. This structure is specified by a subspace $\Delta \subset \mathfrak{g}$ and a scalar product g in Δ . In this case, we say that (Δ, g) is an *SR structure in the algebra* $\mathfrak{g}$.

Left-invariant SR structures on Carnot groups arise as nilpotent approximations of general SR structures on smooth manifolds [1].

Definition 3. Let (Δ, g) be an SR structure in a Lie algebra $\mathfrak{g}$, and let $\mathfrak{i} \subset \mathfrak{g}$ be an ideal such that $\Delta \cap \mathfrak{i} = \{0\}$. Let $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{i}$ be a quotient algebra and $\pi: \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ be a

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canonical projection. Define $\tilde{\Delta} = \pi(\Delta)$ and $\tilde{g}(\pi(X), \pi(Y)) = g(X, Y)$ for $X, Y \in \Delta$. Then $(\tilde{\Delta}, \tilde{g})$ is an SR structure in $\tilde{g}$, which will be called a *quotient structure* of the SR structure (Δ, g) .

Example 1. Let $\mathfrak{g}^5$ be a free nilpotent Lie algebra of step 3 with two generators (Cartan algebra); this is the Carnot algebra with growth vector $(2, 3, 5)$. There exists a basis $\mathfrak{g}^5 = \text{span}(X_1, \dots, X_5)$ in which the nonzero Lie brackets are

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5.$$

Consider an SR structure (Δ, g) in $\mathfrak{g}^5$ with an orthonormal frame (X_1, X_2) [8]. Sequentially choosing the subspaces $\mathbb{R}X_5$, $\text{span}(X_4, X_5)$, and $\text{span}(X_3, X_4, X_5)$ as an ideal $\mathfrak{i} \subset \mathfrak{g}^5$, we obtain SR quotient structures in the Engel algebra (growth vector $(2, 3, 4)$) [7], Heisenberg algebra (growth vector $(2, 3)$) [2], and the two-dimensional commutative algebra $\mathbb{R}^2$ (growth vector (2)).

Example 2. Let $\mathfrak{g}^8$ be a free nilpotent Lie algebra of step 4 with two generators; this is the Carnot algebra with growth vector $(2, 3, 5, 8)$, which will be called a nilpotent $(2, 3, 5, 8)$ -algebra. There exists a basis $\mathfrak{g}^8 = \text{span}(X_1, \dots, X_8)$ in which the nonzero Lie brackets are given by

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \quad (3)$$

$$[X_1, X_4] = X_6, \quad [X_1, X_5] = [X_2, X_4] = X_7, \quad [X_2, X_5] = X_8. \quad (4)$$

Consider an SR structure (Δ, g) in $\mathfrak{g}^8$ with an orthonormal frame (X_1, X_2) [9–11]. It is easy to see that this SR structure is unique, up to a Lie algebra automorphism, SR structure in $\mathfrak{g}^8$ of rank 2 satisfying the condition $\text{Lie}(\Delta) = \mathfrak{g}^8$; we call it a nilpotent SR $(2, 3, 5, 8)$ -structure. In what follows, we will use a dual basis in the Lie coalgebra $(\mathfrak{g}^8)^*$:

$$\omega_1, \dots, \omega_8 \in (\mathfrak{g}^8)^*, \quad \omega_i(X_j) = \delta_{ij}, \quad i, j = 1, \dots, 8.$$

The goal of this paper is, given a structure (Δ, g) , to describe all SR quotient structures for two-dimensional ideals $\mathfrak{i} \subset \mathfrak{g}^8$. These are exactly nilpotent SR structures with growth vector $(2, 3, 5, 6)$.

It is easy to see that a two-dimensional subspace $\mathfrak{i} \subset \mathfrak{g}^8$ is an ideal if and only if $\mathfrak{i} \subset Z(\mathfrak{g}) = \text{span}(X_6, X_7, X_8)$. Therefore, any quotient algebra $\mathfrak{g}^8/\mathfrak{i}$ by the two-dimensional ideal $\mathfrak{i}$ is a Carnot algebra with growth vector $(2, 3, 5, 6)$. Let us describe such algebras.

2. CARNOT ALGEBRAS WITH GROWTH VECTOR $(2, 3, 5, 6)$

Theorem 1. (i) *In any Carnot algebra $\mathfrak{g}$ with growth vector $(2, 3, 5, 6)$, there is a basis $\mathfrak{g} = \text{span}(X_1, \dots, X_6)$ in which all nonzero Lie brackets have the form*

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \quad (5)$$

$$[X_1, X_4] = \alpha X_6, \quad [X_1, X_5] = [X_2, X_4] = \beta X_6, \quad [X_2, X_5] = \gamma X_6, \quad (6)$$

$$(\alpha, \beta, \gamma) \in \mathbb{R}^3 \setminus \{0\}.$$

(ii) *Any two Carnot algebras with growth vector $(2, 3, 5, 6)$ that have proportional triples (α, β, γ) in (6) are isomorphic to each other. Accordingly, such algebras are denoted by $\mathfrak{g}_{\alpha:\beta:\gamma}^6$, $(\alpha : \beta : \gamma) \in \mathbb{R}P^2$.*

(iii) *The following isomorphism of Lie algebras holds:*

$$\mathfrak{g}_{\alpha:\beta:\gamma}^6 \cong \mathfrak{g}^8 / (\ker \omega \cap Z(\mathfrak{g}^8)), \\ \omega = \alpha\omega_6 + \beta\omega_7 + \gamma\omega_8 \neq 0.$$

Each basis in the algebra $\mathfrak{g}_{\alpha:\beta:\gamma}^6$ with multiplication table (5), (6) is associated with a quadratic form $Q(x, y) = \alpha x^2 + 2\beta xy + \gamma y^2$. Depending on the sign of its discriminant $s = \text{sgn}(\alpha\gamma - \beta^2) \in \{0, \pm 1\}$, the algebra $\mathfrak{g}_{\alpha:\beta:\gamma}^6$ is called

- parabolic if $s = 0$,
- elliptic if $s = 1$,
- hyperbolic if $s = -1$.

Remark 1. *Depending on the number $s = \text{sgn}(\alpha\gamma - \beta^2)$, the topology of the sets of triples $(\alpha : \beta : \gamma) \in \mathbb{R}P^2$ is as follows:*

$$A_P = \{(\alpha : \beta : \gamma) \in \mathbb{R}P^2 \mid s = 0\},$$

$$A_E = \{(\alpha : \beta : \gamma) \in \mathbb{R}P^2 \mid s = 1\},$$

$$A_H = \{(\alpha : \beta : \gamma) \in \mathbb{R}P^2 \mid s = -1\}.$$

Topologically, A_P is a circle, A_E is an open disk, and $A_H = \mathbb{R}P^2 \setminus (A_P \cup A_E)$ is a projective plane with a cutout hole, i.e., a Möbius strip.

Below, we give several examples of Carnot algebras with growth vector $(2, 3, 5, 6)$, together with nonzero Lie brackets in the corresponding basis. Recall that the notation $N_{6,2,*}$ for these algebras was used in [12, 13]. A classification of nilpotent Lie algebras of dimension ≤ 7 was obtained in [12], and all Carnot algebras of dimension ≤ 7 were described in [13].

Example 3. The parabolic algebra $\mathfrak{g}_{1:0:0}^6 = N_{6,2,7} = \text{span}(X_1, \dots, X_6)$,

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4,$$

$$[X_2, X_3] = X_5, \quad [X_1, X_4] = X_6.$$

Example 4. The hyperbolic algebra $\mathfrak{g}_{1:0:(-1)}^6 = N_{6,2,5} = \text{span}(X_1, \dots, X_6)$,

$$\begin{aligned} [X_1, X_2] &= X_3, & [X_1, X_3] &= X_4, \\ [X_2, X_3] &= X_5, & [X_1, X_4] &= -[X_2, X_5] = X_6. \end{aligned}$$

Example 5. The elliptic algebra $\mathfrak{g}_{1:0:1}^6 = N_{6,2,5a} = \text{span}(X_1, \dots, X_6)$,

$$\begin{aligned} [X_1, X_2] &= X_3, & [X_1, X_3] &= X_4, \\ [X_2, X_3] &= X_5, & [X_1, X_4] &= [X_2, X_5] = X_6. \end{aligned}$$

Theorem 2. (i) The algebras $\mathfrak{g}_{1:0:0}^6$, $\mathfrak{g}_{1:0:(-1)}^6$, and $\mathfrak{g}_{1:0:1}^6$ are pairwise nonisomorphic. Any algebra $\mathfrak{g}_{\alpha:\beta:\gamma}^6$, $(\alpha : \beta : \gamma) \in \mathbb{R}P^2$, is isomorphic to one of these algebras.

(ii) The number $\mathbf{s} = \text{sgn}(\alpha\gamma - \beta^2) \in \{0, \pm 1\}$ is an invariant of the Carnot algebra $\mathfrak{g}$; $\mathbf{s}$ is called the signature of $\mathfrak{g}$.

(ii) The algebras $\mathfrak{g}_{\alpha:\beta:\gamma}^6$ are separated by the signature $\mathbf{s}$:

$$\mathfrak{g}_{\alpha:\beta:\gamma}^6 \cong \mathfrak{g}_{1:0:0}^6 \Leftrightarrow \mathbf{s} = 0 \quad (\text{parabolic algebra}), \quad (7)$$

$$\mathfrak{g}_{\alpha:\beta:\gamma}^6 \cong \mathfrak{g}_{1:0:(-1)}^6 \Leftrightarrow \mathbf{s} = -1 \quad (\text{hyperbolic algebra}), \quad (8)$$

$$\mathfrak{g}_{\alpha:\beta:\gamma}^6 \cong \mathfrak{g}_{1:0:1}^6 \Leftrightarrow \mathbf{s} = 1 \quad (\text{elliptic algebra}). \quad (9)$$

Thus, the signature $\mathbf{s} \in \{0, \pm 1\}$ is an invariant separating three classes of isomorphism of the algebras $\mathfrak{g}_{\alpha:\beta:\gamma}^6$.

Remark 2. Item (i) in Theorem 2 was first proved in [13]. We proved it independently, together with an algorithm for reducing the multiplication table in the (2, 3, 5, 6)-algebra to the normal form in Examples 3–5.

To find a basis change in the algebra $\mathfrak{g} = \mathfrak{g}_{\alpha:\beta:\gamma}^6$ that transforms the basis into one of the normal forms indicated in Examples 3–5, it suffices to reduce the quadratic form $Q(x, y) = \alpha x^2 + 2\beta xy + \gamma y^2$ to a sum of squares, to apply this change to the basis (X_1, X_2) of the space $\mathfrak{g}^{(1)}$, and to normalize the vector X_6 generating the space $\mathfrak{g}^{(4)}$.

3. NILPOTENT (2, 3, 5, 6)-DISTRIBUTIONS AND SUB-RIEMANNIAN STRUCTURES

Definition 4. Let $\mathfrak{g}$ be a Carnot algebra with growth vector (2, 3, 5, 6). A nilpotent sub-Riemannian (2, 3, 5, 6)-structure is a sub-Riemannian structure (Δ, g) in $\mathfrak{g}$ satisfying the conditions

$$\dim \Delta = 2, \quad \text{Lie}(\Delta) = \mathfrak{g}, \quad (10)$$

or, equivalently, the equality $\Delta \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)} \oplus \mathfrak{g}^{(4)} = \mathfrak{g}$. The corresponding plane $\Delta \subset \mathfrak{g}$ is called a nilpotent (2, 3, 5, 6)-distribution in $\mathfrak{g}$.

The vector (2, 3, 5, 6) is the growth vector of the left-invariant distribution on the Lie group G of the Lie algebra $\mathfrak{g}$ defined by the plane $\Delta \subset \mathfrak{g}$.

Theorem 3. Let $\mathfrak{g}$ be a Carnot algebra with growth vector (2, 3, 5, 6) of signature $\mathbf{s} \in \{0, \pm 1\}$. For any nilpotent (2, 3, 5, 6)-distribution $\Delta \subset \mathfrak{g}$, there exists a frame $\Delta = \text{span}(X_1, X_2)$ such that

$$\begin{aligned} [X_1, X_2] &= X_3, & [X_1, X_3] &= X_4, & [X_2, X_3] &= X_5, \\ [X_1, X_4] &= X_6, & [X_1, X_5] &= [X_2, X_4] = 0, \\ [X_2, X_5] &= \mathbf{s}X_6. \end{aligned}$$

Theorem 4. Let $\mathfrak{g}$ be a Carnot algebra with growth vector (2, 3, 5, 6). For any nilpotent SR (2, 3, 5, 6)-structures in $\mathfrak{g}$, there exists an orthonormal frame (X_1, X_2) and a number $v \in [-1, 1]$ for which

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \quad (11)$$

$$\begin{aligned} [X_1, X_4] &= X_6, & [X_1, X_5] &= [X_2, X_4] = 0, \\ [X_2, X_5] &= vX_6. \end{aligned} \quad (12)$$

The number $v \in [-1, 1]$ in (12) is called the canonical parameter of the SR structure (Δ, g) . It is easy to see that v is equal to the ratio of the smaller (in absolute value) eigenvalue of Q to the larger eigenvalue. Moreover, $\mathbf{s} = \text{sgn} v$.

Theorem 5. The canonical parameter $v \in [-1, 1]$ is an invariant of the nilpotent SR (2, 3, 5, 6)-structure.

Let (Δ, g) be an SR structure on a manifold M . The corresponding SR distance d [1] transforms M into a metric space. Recall that an isometry between metric spaces (M, d) and $(\tilde{M}, \tilde{d})$ is a mapping $F: M \rightarrow \tilde{M}$ such that

$$d(x, y) = \tilde{d}(F(x), F(y)), \quad x, y \in M.$$

The metric group [15] is a Lie group with a left-invariant distance inducing a manifold topology on this group. In particular, any Carnot group is a metric group. It was proved in [15] that

any isometry between metric groups is an analytic mapping;

any isometry between connected nilpotent metric groups is an affine mapping (i.e., the composition of a left shift and an isomorphism).

The following theorem provides a classification of left-invariant SR (2, 3, 5, 6)-structures up to isometries of corresponding Carnot groups.

Theorem 6. Let $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ be nilpotent Carnot (2, 3, 5, 6)-algebras, and let G and $\tilde{G}$ be corresponding Carnot groups. Let (Δ, g) and $(\tilde{\Delta}, \tilde{g})$ be nilpotent SR (2, 3, 5, 6)-structures in $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, v and $\tilde{v}$ be corresponding canoni-

cal parameters, and d and $\tilde{d}$ be corresponding left-invariant SR metrics on G and $\tilde{G}$.

The metric spaces (G, d) and $(\tilde{G}, \tilde{d})$ are isometric if and only if $v = \tilde{v}$.

Theorem 7. Let $\mathfrak{g} = \mathfrak{g}^8$ be the $(2, 3, 5, 8)$ -algebra from Example 2 with a basis $(X_1, \dots, X_8)$ according to (3), (4), and let (Δ, g) be an SR structure in $\mathfrak{g}$ with orthonormal frame (X_1, X_2) . Let $\mathfrak{i} = \ker \omega \cap Z(\mathfrak{g})$ with $\omega = \alpha\omega_6 + \beta\omega_7 + \gamma\omega_8 \neq 0$ be any 2-dimensional subspace of $Z(\mathfrak{g})$. Then the SR quotient structure $(\tilde{\Delta}, \tilde{g})$ is one of the SR $(2, 3, 5, 6)$ -structures in $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{i}$:

- (i) in the parabolic case $\mathbf{s} = 0$, we have $\tilde{\mathfrak{g}} \cong \mathfrak{g}_{1:0:0}^6$;
- (ii) in the hyperbolic case $\mathbf{s} = -1$, we have $\tilde{\mathfrak{g}} \cong \mathfrak{g}_{1:0:(-1)}^6$;
- (iii) in the elliptic case $\mathbf{s} = 1$, we have $\tilde{\mathfrak{g}} \cong \mathfrak{g}_{1:0:1}^6$.

Conversely, any nilpotent $(2, 3, 5, 6)$ -structure in each of these algebras is realized as a quotient structure of a nilpotent $(2, 3, 5, 8)$ -structure (Δ, g) .

Remark 3. An SR structure in the elliptic algebra $\mathfrak{g}_{1:0:1}^6$ with canonical parameter $v = 1$ was considered in [14]. For this structure, it was proved that the vertical subsystem of the Hamiltonian system in the Pontryagin maximum principle [1] is Liouville integrable. This Hamiltonian system was integrated in [11].

4. CASIMIR FUNCTIONS AND SYMPLECTIC FOLIATIONS

Let us recall some basic concepts of symplectic geometry. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ be the Lie coalgebra (the dual of the space $\mathfrak{g}$). Any function $h \in C^\infty(\mathfrak{g}^*)$ is called a Hamiltonian. For any Hamiltonians f and g , the Poisson bracket is defined as

$$\{f, g\}(\lambda) = \langle \lambda, [df_\lambda, dg_\lambda] \rangle, \quad \lambda \in \mathfrak{g}^*.$$

The Casimir function (on a submanifold $M \subset \mathfrak{g}^*$) is any Hamiltonian f commuting with all Hamiltonians in the sense of the Poisson bracket:

$$\begin{aligned} \{f, h\}(\lambda) &= 0, \quad h \in C^\infty(\mathfrak{g}^*), \\ \lambda &\in \mathfrak{g}^* \quad (\text{accordingly } \lambda \in M). \end{aligned}$$

A symplectic foliation on $\mathfrak{g}^*$ is a partition of $\mathfrak{g}^*$ into symplectic leaves $L(\lambda)$, which are orbits of the coadjoint action of the Lie group G of the Lie algebra $\mathfrak{g}$ in $\mathfrak{g}^*$:

$$L(\lambda) = \{(Ad_x^*)(\lambda) \mid x \in G\}, \quad \lambda \in \mathfrak{g}^*.$$

The rank of the Poisson bracket $\{\cdot, \cdot\}$ at a point $\lambda \in \mathfrak{g}^*$ is defined as the dimension of the symplectic leaf passing through λ and is denoted by $\text{rank}(\lambda)$. The symplectic leaves and Casimir functions are invariants of the Hamiltonian vector field

$$\dot{\lambda} = \mathbf{h}(\lambda), \quad \lambda \in \mathfrak{g}^*,$$

for any Hamiltonian h , so they are important for analyzing left-invariant Hamiltonians on the Lie group of the Lie algebra $\mathfrak{g}$, in particular, for studying left-invariant optimal control problems on this Lie group.

For coalgebras $\mathfrak{g}^*$ of all Carnot algebras $\mathfrak{g}$ with growth vector $(2, 3, 5, 6)$, we describe their Casimir functions and symplectic foliations. In each of the algebras $\mathfrak{g} = \mathfrak{g}_{1:0:0}^6, \mathfrak{g}_{1:0:(-1)}^6$, and $\mathfrak{g}_{1:0:1}^6$, we use a canonical basis $\mathfrak{g} = \text{span}(X_1, \dots, X_6)$ according to Examples 3–5 and corresponding linear Hamiltonians $h_1, \dots, h_6 \in \mathfrak{g}^{**}$, $h_i(\lambda) = \langle \lambda, X_i \rangle$. In each of the cases presented below, the symplectic leaf $L(\lambda)$ is a connected component of common level surfaces of the Casimir functions on corresponding submanifolds in $\mathfrak{g}^*$.

4.1. Parabolic Algebra $\mathfrak{g} = \mathfrak{g}_{1:0:0}^6$

- (1) If $h_5 h_6 \neq 0$, then $\text{rank}(\lambda) = 4$.

Casimir functions: h_5, h_6 .

- (2) If $h_6 = 0$ and $h_3^2 + h_4^2 + h_5^2 \neq 0$, then $\text{rank}(\lambda) = 2$.

Casimir functions: $h_4, h_5, \frac{h_3^2}{2} + h_1 h_5 - h_2 h_4$.

- (3) If $h_5 = 0$ and $h_3^2 + h_4^2 + h_6^2 \neq 0$, then $\text{rank}(\lambda) = 2$.

Casimir functions: $h_6, C_1 = h_3 h_6 - \frac{h_4^2}{2}, C_2 = h_2 h_6^2 - C_1 h_4 - \frac{h_4^3}{6}$.

- (4) If $h_3 = \dots = h_6 = 0$, then $\text{rank}(\lambda) = 0$.

Casimir functions: h_1, h_2 .

4.2. Hyperbolic Algebra $\mathfrak{g} = \mathfrak{g}_{1:0:(-1)}^6$

- (1) If $h_6 \neq 0$, then $\text{rank}(\lambda) = 4$.

Casimir functions: $h_6, h_3 h_6 + \frac{h_4^2 - h_5^2}{2}$.

- (2) If $h_6 = 0$ and $h_3^2 + h_4^2 + h_5^2 \neq 0$, then $\text{rank}(\lambda) = 2$.

Casimir functions: $h_4, h_5, \frac{h_3^2}{2} + h_1 h_5 - h_2 h_4$.

- (3) If $h_3 = \dots = h_6 = 0$, then $\text{rank}(\lambda) = 0$.

Casimir functions: h_1, h_2 .

4.3. Elliptic Algebra $\mathfrak{g} = \mathfrak{g}_{1:0:1}^6$

- (1) If $h_6 \neq 0$, then $\text{rank}(\lambda) = 4$.

Casimir functions: $h_6, h_3 h_6 - \frac{h_4^2 + h_5^2}{2}$.

- (2) If $h_6 = 0$ and $h_3^2 + h_4^2 + h_5^2 \neq 0$, then $\text{rank}(\lambda) = 2$.

Casimir functions: $h_4, h_5, \frac{h_3^2}{2} + h_1 h_5 - h_2 h_4$.

(3) If $h_3 = \dots = h_6 = 0$, then $\text{rank}(\lambda) = 0$.

Casimir functions: h_1, h_2 .

CONCLUSIONS

The following results were obtained in this paper.

We described the Carnot algebras with growth vector (2, 3, 5, 6), i.e., quotient algebras of a free nilpotent Carnot algebra $\mathfrak{g}^8$ of step 4 with two generators. Previously, it was known that there are three normal forms of such algebras: parabolic, hyperbolic, and elliptic. An invariant separating these algebras was found, namely, the signature $s \in \{0, \pm 1\}$. Additionally, a change of basis transforming the multiplication table into one of three normal forms was described.

We studied the left-invariant sub-Riemannian structures with growth vector (2, 3, 5, 6), i.e., quotient structures of a (unique) sub-Riemannian structure with growth vector (2, 3, 5, 8) by the two-dimensional subspace of the center of the algebra $\mathfrak{g}^8$. A classification of (2, 3, 5, 6)-structures was obtained up to isometries: all of them are uniquely parametrized by a canonical parameter $v \in \{0, \pm 1\}$ such that $s = \text{sgn } v$. A classification of left-invariant (2, 3, 5, 6)-distributions was obtained: in each of the (2, 3, 5, 6)-algebras (parabolic, hyperbolic, elliptic) there exists a unique, up to isomorphism, distribution of this type.

For each Carnot algebra $\mathfrak{g}$ with growth vector (2, 3, 5, 6), the rank of the Poisson bracket, Casimir functions, and a symplectic foliation in the Lie coalgebra $\mathfrak{g}^*$ were calculated.

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