ISSN 1064-5624, Doklady Mathematics, 2021, Vol. 103, No. 1, pp. 44–46. © The Author(s), 2021. This article is an open access publication.
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Problem of Determining the Anisotropic Conductivity in Electrodynamic Equations

Corresponding Member of the RAS **V. G. Romanov**[a,b,*]

Received November 3, 2020; revised December 2, 2020; accepted December 7, 2020

Abstract—For a system of electrodynamic equations, the inverse problem of determining an anisotropic conductivity is considered. It is supposed that the conductivity is described by a diagonal matrix $\sigma(x) = \mathrm{diag}(\sigma_1(x), \sigma_2(x), \sigma_3(x))$ with $\sigma(x) = 0$ outside of the domain $\Omega = \{x \in \mathbb{R}^3 \mid |x| < R\}$, $R > 0$, and the permittivity ε and the permeability μ of the medium are positive constants everywhere in $\mathbb{R}^3$. Plane waves coming from infinity and impinging on an inhomogeneity localized in Ω are considered. For the determination of the unknown functions $\sigma_1(x)$, $\sigma_2(x)$, and $\sigma_3(x)$, information related to the vector of electric intensity is given on the boundary S of the domain Ω. It is shown that this information reduces the inverse problem to three identical problems of X-ray tomography.

Keywords: Maxwell equations, anisotropy, conductivity, plane waves, inverse problem, tomography
DOI: 10.1134/S1064562421010099

Consider the nonstationary Maxwell equations

$$\begin{aligned} \mathrm{curl}\mathbf{H} &= \varepsilon\mathbf{E}_t + \sigma(x)\mathbf{E}, \\ \mathrm{curl}\mathbf{E} &= -\mu\mathbf{H}_t, \quad \mathrm{div}\mathbf{H} = 0. \end{aligned} \tag{1}$$

Here, $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{H} = (H_1, H_2, H_3)$ are the electric and magnetic field strengths, $\sigma(x) = \mathrm{diag}(\sigma_1(x), \sigma_2(x), \sigma_3(x))$ is a positive semidefinite diagonal matrix, and ε and μ are positive constants. Assume that $\sigma(x) = 0$ outside the domain $\Omega = \{x \in \mathbb{R}^3 \mid |x| < R\}$, where $R > 0$.

Let $c = \frac{1}{\sqrt{\varepsilon\mu}}$ denote the velocity of propagation of electromagnetic waves. Let $\nu = (\nu_1, \nu_2, \nu_3)$, $|\nu| = 1$, and $\mathbf{j}$ be the unit vector orthogonal to ν, i.e., $\mathbf{j} \cdot \nu = 0$.

For Maxwell's equations (1) in a homogeneous medium ($\sigma(x) = 0$), there exist solutions of the form

$$\begin{aligned} \mathbf{E}^0(x, t, \nu, \mathbf{j}) &= \mathbf{j} f\left(t + t_0 - \frac{x \cdot \nu}{c}\right), \\ \mathbf{H}^0(x, t, \nu, \mathbf{j}) &= \frac{\nu \times \mathbf{j}}{\mu c} f\left(t + t_0 - \frac{x \cdot \nu}{c}\right), \end{aligned} \tag{2}$$

where $t_0 = \min\limits_{x \in \Omega} \frac{x \cdot \nu}{c} = -\frac{R}{c}$ and $f(t)$ is an arbitrary generalized function. Each such solution represents a plane wave propagating in the direction of the vector ν and is a weak solution of Maxwell's equations for a homogeneous medium.

Consider the Cauchy problem for an anisotropic medium:

$$\begin{aligned} &\mathrm{curl}\mathbf{H} = \varepsilon\mathbf{E}_t + \sigma(x)\mathbf{E}, \quad \mathrm{curl}\mathbf{E} = -\mu\mathbf{H}_t, \\ &\mathbf{E}\big|_{t<0} = \mathbf{E}^0(x, t, \nu, \mathbf{j}), \quad \mathbf{H}\big|_{t<0} = \mathbf{H}^0(x, t, \nu, \mathbf{j}), \end{aligned} \tag{3}$$

where $\mathbf{E}^0(x, t, \nu, \mathbf{j})$ and $\mathbf{H}^0(x, t, \nu, \mathbf{j})$ are defined by formulas (2) and $f(t)$ is a smooth function such that $f(t) \equiv 0$ for $t \le 0$ and $f(+0) \ne 0$. Thus, $\mathbf{E}(x, t, \nu, \mathbf{j}) = 0$ and $\mathbf{H}(x, t, \nu, \mathbf{j}) = 0$ for all $x \cdot \nu \ge ct_0$ and $t < 0$. Let $\Sigma(\nu) =: \{x \in \mathbb{R}^3 \mid x \cdot \nu = ct_0\}$ be the plane corresponding to the front of a plane wave at the time $t = 0$, when this front touches the domain Ω.

Let $S = \{x \in \mathbb{R}^3 \mid |x| = R\}$ be the boundary of Ω and $S^+(\nu) = \{x \in S \mid x \cdot \nu > 0\}$ be its shadow part with respect to light propagating in the direction ν.

Below, problem (3) is considered for three different vectors $\mathbf{j}^k$, $k = 1, 2, 3$, and for corresponding orthogonal vectors ν^k depending on the angular parameter φ, namely,

[a] *Sobolev Institute of Mathematics, Siberian Branch, Russian Academy of Sciences, Novosibirsk, 630090 Russia*
[b] *Mathematical Center in Akademgorodok, Novosibirsk State University, Novosibirsk, 630090 Russia*
**e-mail: romanov@math.nsc.ru*

$$\begin{aligned}
&\mathbf{j}^1 = (1,0,0), && \nu^1(\varphi) = (0,\cos\varphi,\sin\varphi), && \varphi\in[0,\pi],\\
&\mathbf{j}^2 = (0,1,0), && \nu^2(\varphi) = (\cos\varphi,0,\sin\varphi), && \varphi\in[0,\pi],\\
&\mathbf{j}^3 = (0,0,1), && \nu^3(\varphi) = (\cos\varphi,\sin\varphi,0), && \varphi\in[0,\pi].
\end{aligned} \tag{4}$$

Inverse problem. Find $\sigma(x)$ from functions $E_k(x,t,\nu^k(\varphi),\mathbf{j}^k)$, $k=1,2,3$, given for all $x\in S^+(\nu^k(\varphi))$, $\varphi\in[0,\pi]$, and $t\in[0,T_k(x,\varphi)]$, where $T_k(x,\varphi)=\dfrac{x\cdot\nu^k(\varphi)}{c}-t_0+\delta_0$ and $\delta_0>0$ is an arbitrary number (possibly small). In other words, the task is to find $\sigma(x)$ from the given functions

$$\begin{gathered}
F_k(x,t,\varphi) = E_k(x,t,\nu^k(\varphi),\mathbf{j}^k), \qquad k=1,2,3,\\
x\in S^+(\nu^k(\varphi)), \quad \varphi\in[0,\pi], \quad t\in[0,T_k(x,\varphi)].
\end{gathered} \tag{5}$$

For stationary electrodynamic equations, inverse problems of determining conductivity representing a one-variable function were studied by Tikhonov [1–4] and Cagniard [5]. For nonstationary equations, the theory of inverse electrodynamic problems based on the full system of Maxwell equations was developed in [6–8]. The problem of determining the permittivity of an anisotropic medium was considered in [9]. Additionally, phaseless inverse problems of determining permittivity from the magnitude of the electric or magnetic component of a stationary electromagnetic field were studied (see [10] and the review article in [11]).

The following result holds for the inverse problem stated above.

Theorem 1. *Suppose that the matrix* $\sigma(x)$ *belongs to* $C^2(\mathbb{R}^3)$ *and vanishes outside* Ω, *while the function* $f(t)$ *has the form* $f(t)=\hat f(t)\theta_0(t)$, *where* $\hat f(t)$ *is a smooth function of class* $C^2[0,\infty)$, $\hat f(0)\neq 0$, *and* $\theta_0(t)$ *is the Heaviside step function, i.e.,* $\theta_0(t)=1$ *for* $t\geq 0$ *and* $\theta_0(t)=0$ *for* $t<0$. *Then all elements of* $\sigma(x)$ *in* Ω *are uniquely determined by data* (5).

The study of the inverse problem is based on analyzing the structure of the solution to problem (3). In this case, it is convenient to use an integro-differential equation for the vector $\mathbf{E}(x,t,\nu,\mathbf{j})$. To derive it, we apply the curl operator to the second equation in (3) and use the first equation to eliminate the emerging term $\operatorname{curl}\mathbf{H}_t$. Then we obtain the equation

$$(-\Delta+\nabla\operatorname{div})\mathbf{E} = -\mu\varepsilon\mathbf{E}_{tt}-\mu\sigma(x)\mathbf{E}_t. \tag{6}$$

Computing $\operatorname{div}\mathbf{E}$ with the help of the first equation in (3) yields

$$\operatorname{div}\mathbf{E}(x,t,\nu,\mathbf{j}) = -\frac{1}{\varepsilon}\operatorname{div}\int_{-\infty}^{t}\sigma(x)\mathbf{E}(x,\tau,\nu,\mathbf{j})d\tau. \tag{7}$$

It follows from (6), (7), and (3) that the function $\mathbf{E}$ is a solution of the Cauchy problem

$$\begin{gathered}
c^{-2}\mathbf{E}_{tt}-\Delta\mathbf{E}+\mu\sigma(x)\mathbf{E}_t\\
-\frac{1}{\varepsilon}\nabla\operatorname{div}\int_{-\infty}^{t}\sigma(x)\mathbf{E}(x,\tau,\nu,\mathbf{j})d\tau = 0,\\
\mathbf{E}\big|_{t<0} = \mathbf{E}^0(x,t,\nu,\mathbf{j}).
\end{gathered} \tag{8}$$

The following result holds for problem (8).

Theorem 2. *Suppose that the matrix* $\sigma(x)$ *and the function* $f(t)$ *satisfy the conditions of Theorem* 1. *Then the function* $\mathbf{E}(x,t,\nu,\mathbf{j})$ *for* $t\geq 0$ *can be represented in the form*

$$\begin{gathered}
\mathbf{E}(x,t,\nu,\mathbf{j}) = \alpha(x,\nu,\mathbf{j})\theta_0\left(t+t_0-\frac{x\cdot\nu}{c}\right)\\
+\hat{\mathbf{E}}(x,t,\nu,\mathbf{j})\theta_1\left(t+t_0-\frac{x\cdot\nu}{c}\right),
\end{gathered} \tag{9}$$

where $\theta_1(t)=t\theta_0(t)$, *the function* $\alpha(x,\nu,\mathbf{j})$ *is a solution of the Cauchy problem*

$$\begin{gathered}
\frac{2}{c}(\nu\cdot\nabla)\alpha+\mu\sigma(x)\alpha-\frac{1}{\varepsilon c^2}\nu((\sigma(x)\alpha)\cdot\nu) = 0,\\
\alpha\big|_{x\cdot\nu=ct_0} = \mathbf{j}\hat f(0),
\end{gathered} \tag{10}$$

and $\hat{\mathbf{E}}(x,t,\nu,\mathbf{j})$ *is a bounded function of* x *and* t *for* $t\in\left[\dfrac{x\cdot\nu}{c}-t_0,T\right]$ *for any* $T>0$.

Equation (10) is derived by substituting representation (9) into Eq. (8) and equating the coefficients of $\delta\left(t+t_0-\dfrac{x\cdot\nu}{c}\right)$ to zero, while the initial data for the function $\alpha(x,\nu,\mathbf{j})$ follow from the initial data for $\mathbf{E}(x,t,\nu,\mathbf{j})$. The function $\alpha(x,\nu,\mathbf{j})$ is the amplitude of $\mathbf{E}(x,t,\nu,\mathbf{j})$ at the electromagnetic wave front, i.e., at $t=\dfrac{x\cdot\nu}{c-t_0}$. Equation (10) is a vector ordinary differential equation along any ray $x=x^0+s\nu$, $s\in\mathbb{R}^1$, starting at the arbitrary point $x^0\in\mathbb{R}^3$. For case (4), the components of the vectors $\alpha(x,\nu^k(\varphi),\mathbf{j}^k)=(\alpha_1(x,u^k(\varphi),\mathbf{j}^k),\alpha_2(x,\nu^k(\varphi),\mathbf{j}^k),\alpha_3(x,\nu^k(\varphi),\mathbf{j}^k))$ can be computed in explicit form, namely,

$$\begin{aligned}
&\alpha_k(x,\nu^k(\varphi),\mathbf{j}^k)\\
&= \hat f(0)\exp\left(-\frac{\mu c}{2}\int_0^\infty\sigma_k(x-s\nu^k(\varphi))ds\right), \qquad k=1,2,3.
\end{aligned}$$

The same components can be computed using data (5) of the inverse problem:

$$\begin{gathered}
\alpha_k(x,\nu^k(\varphi),\mathbf{j}^k) = g_k(x,\varphi) =: \lim_{t\to(x\cdot\nu^k(\varphi))/c-t_0}F_k(x,t,\varphi),\\
x\in S^+(\nu^k(\varphi)), \quad \varphi\in[0,\pi], \quad k=1,2,3.
\end{gathered}$$

These formulas imply that the integrals

$$\int_0^\infty \sigma_k(x - s\nu^k(\varphi))ds = -\frac{2}{\mu c}\ln\frac{g_k(x,\varphi)}{\hat{f}(0)}, \quad k = 1,2,3, \tag{11}$$

are known for all $x \in S^+(\nu^k(\varphi))$ and $\varphi \in [0,\pi]$.

Thus, for each $k = 1,2,3$, the right-hand side of (11) is known along any straight line intersecting Ω in the direction $\nu^k(\varphi)$. By varying φ, we conclude that, in each section of Ω by the plane $x_k = \text{const}$, the integrals along all possible straight lines lying in this plane are known. As a result, we obtain an X-ray tomography problem for determining $\sigma_k(x)$, $k = 1,2,3$. It is well known that this problem is uniquely solvable (see [12–14]). This implies Theorem 1 on the uniqueness of a solution to the inverse problem and an algorithm for its solution.

FUNDING

This work was supported by the Mathematical Center in Akademgorodok at Novosibirsk State University (contract no. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation).

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REFERENCES

1. A. N. Tikhonov, Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz. **10** (3), 213–231 (1946).
2. A. N. Tikhonov, Dokl. Akad. Nauk SSSR **60** (5), 797–800 (1949).
3. A. N. Tikhonov, Dokl. Akad. Nauk SSSR **73** (2), 295–297 (1950).
4. A. N. Tikhonov, USSR Comput. Math. Math. Phys. **5** (3), 207–211 (1965).
5. L. Cagniard, Geophysics **18** (3), 605–635 (1953).
6. V. G. Romanov and S. I. Kabanikhin, *Inverse Problems in Geoelectrics* (Nauka, Moscow, 1991) [in Russian].
7. A. L. Karchevsky, J. Inv. Ill-Posed Probl. **17** (4), 385–402 (2009).
8. V. G. Romanov, Sib. Math. J. **52** (4), 682–695 (2011).
9. V. G. Romanov, Comput. Math. Math. Phys. **60** (6), 1037–1044 (2020).
10. A. L. Karchevsky and V. A. Dedok, J. Appl. Ind. Math. **12** (3), 470–478 (2018).
11. V. G. Romanov, Comput. Math. Math. Phys. **60** (6), 1045–1062 (2020).
12. S. Helgason, *The Radon Transform* (Birkhäuser, Boston, 1980).
13. F. Natterer, *The Mathematics of Computerized Tomography* (SIAM, Philadelphia, PA, 2001).
14. D. Finch, Inverse Probl. **2** (2), 197–203 (1986).

Translated by I. Ruzanova