On Global Solvability of Nonlinear Equations with Parameters

A. V. Arutyunov^{a,b} and S. E. Zhukovskiy^{a,*}

Presented by Academician of the RAS A.B. Kurzhanski August 14, 2020

Received August 18, 2020; revised December 11, 2020; accepted December 12, 2020

Abstract—We consider smooth mappings acting from one Banach space to another and depending on a parameter belonging to a topological space. Under various regularity assumptions, sufficient conditions for the existence of global and semilocal continuous inverse and implicit functions are obtained. We consider applications of these results to the problem of continuous extension of implicit functions and to the problem of coincidence points of smooth and continuous compact mappings.

Keywords: global implicit function, implicit function extension, coincidence point

DOI: 10.1134/S1064562421010014

Given Banach spaces X and Y , a topological space Σ , and a continuous mapping $f : X \times \Sigma \rightarrow Y$, we consider the equation

$$f(x, \sigma) = 0, \quad (1)$$

where $x \in X$ is the unknown and $\sigma \in \Sigma$ is a parameter. A continuous function $g(\cdot)$ defined on Σ or on a subset of Σ and satisfying the identity $f(g(\sigma), \sigma) \equiv 0$ is called an implicit function.

This paper presents global implicit function theorems, i.e., conditions under which, for all values of $\sigma \in \Sigma$, Eq. (1) is solvable and its solution $x = g(\sigma)$ depends continuously on σ . As a special case, we obtain a global (and semilocal) inverse function theorem providing sufficient conditions for the existence of a continuous right inverse of a given smooth mapping defined on a given ball in Y . As applications of the implicit function theorem, we derive results concerning a continuous extension of an implicit function, an ε -implicit function, and the existence of coincidence points.

A global inverse function theorem goes back to Hadamard, who proved in [1] (see also [2]) that, if $X = Y = \mathbb{R}^n$, and a continuously differentiable mapping $F : X \rightarrow Y$ is uniformly nonsingular, i.e., the lin-

ear operator $\frac{\partial F}{\partial x}(x)$ is invertible for each $x \in X$ and the

function $\left\| \frac{\partial F}{\partial x}(x)^{-1} \right\|$ is bounded on X , then F is a diffeo-

morphism. Here and below, $\|\cdot\|$ denotes the norm of a linear bounded operator and the norm in the spaces X and Y .

Equation (1) is of particular interest when X and Y are not linearly topologically isomorphic spaces, for example, when $X = \mathbb{R}^n$, $Y = \mathbb{R}^k$, and $n > k$. In this case, global implicit and inverse function theorems were obtained in [3] assuming that the mapping f is smooth in x . A global inverse function theorem for locally Lipschitz mappings was derived in [4].

Let us extend the result of [3] to Hilbert spaces. For this purpose, we first introduce the necessary concepts.

As usual, $\mathcal{L}(X, Y)$ is the space of linear bounded operators A acting from a Banach space X to a Banach space Y , and $\mathcal{SL}(X, Y)$ is the set of all surjective operators $A \in \mathcal{L}(X, Y)$. By $B(x, r)$, we denote a closed ball in X of radius $r \geq 0$ centered at the point $x \in X$, and the same notation is used for balls in Y .

Given a linear operator $A \in \mathcal{L}(X, Y)$, we define

$$\text{cov} A := \sup\{\alpha \geq 0 : B(0, \alpha) \subset AB(0, 1)\}.$$

The Banach open mapping theorem means that $\text{cov} A > 0$ if and only if $A \in \mathcal{SL}(X, Y)$. The Michael selection theorem (see [5]) guarantees that, if $A \in \mathcal{SL}(X, Y)$,

^a Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, 117997 Russia

^b Institute for Information Transmission Problems of the Russian Academy of Sciences (Kharkevich Institute), Moscow, Russia

*e-mail: s-e-zhuk@yandex.ru

then the equation $Ax = y$ has a continuous solution $x = x(y)$ on Y .

Assume that the mapping $f(\cdot, \sigma)$ is differentiable for each $\sigma \in \Sigma$. For an arbitrary continuous function $\varphi: \Sigma \rightarrow X$ and $t \geq 0$, we define

$$\alpha_\varphi(t) := \inf \left\{ \text{cov} \frac{\partial f}{\partial x}(x, \sigma): x \in B(\varphi(\sigma), t), \sigma \in \Sigma \right\}.$$

Theorem 1. Suppose that X and Y are Hilbert spaces; the mapping $f: X \times \Sigma \rightarrow Y$ is continuous; and, for each fixed $\sigma \in \Sigma$, the mapping $f(\cdot, \sigma)$ is twice continuously differentiable with respect to x and the mappings $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ are continuous on $X \times \Sigma$.

Then, for any continuous function $\varphi: \Sigma \rightarrow X$ satisfying at least one of the conditions

$$\int_0^{+\infty} \alpha_\varphi(t) dt = +\infty \quad (2)$$

or

$$\sup_{\sigma \in \Sigma} \|f(\varphi(\sigma), \sigma)\| < \int_0^{+\infty} \alpha_\varphi(t) dt, \quad (3)$$

there exists a continuous function $g = g_\varphi: \Sigma \rightarrow X$ such that

$$f(g(\sigma), \sigma) = 0 \quad \forall \sigma \in \Sigma, \quad (4)$$

$$\int_0^{\|g(\sigma) - \varphi(\sigma)\|} \alpha_\varphi(t) dt \leq \|f(\varphi(\sigma), \sigma)\| \quad \forall \sigma \in \Sigma. \quad (5)$$

Theorem 1 is proved by applying methods of the theory of ordinary differential equations; more specifically, the proof consists in an analysis of solutions to some Cauchy problem based on the mapping f (see [3]).

Of particular interest and complexity is the situation when X and Y are Banach spaces. In this case, the above-described method of reducing Eq. (1) to a Cauchy problem cannot be applied, since there exist Banach spaces X and Y and a linear operator $A \in \mathcal{L}(X, Y)$ such that A does not have a continuous linear (or even nonlinear, but locally Lipschitz) right inverse mapping (for more details, see [6]). Accordingly, the problem of a global implicit function in Banach spaces requires a different approach.

As such an approach, it is possible to use a method based on sufficient conditions for the existence of a minimum of lower semicontinuous functionals (see [7, 8]). By applying the results of [7, 8], the following assertion can be proved.

Let $\pi: \Sigma \rightarrow \mathbb{R}_+$ be a given continuous function (where $\mathbb{R}_+$ is the set of nonnegative real numbers). For $d > 0$, let

$$\Sigma(d) := \{\sigma \in \Sigma: \pi(\sigma) < d\}.$$

The following assumptions are made about the mapping $f: X \times \Sigma \rightarrow Y$.

Assumption A1. The mapping f is continuous, the mapping $f(\cdot, \sigma)$ is continuously differentiable with respect to x for any fixed $\sigma \in \Sigma$, and the mapping $\frac{\partial f}{\partial x}: X \times \Sigma \rightarrow \mathcal{L}(X, Y)$ is continuous.

Since f is differentiable with respect to x , we conclude that, for any $(x, \sigma) \in X \times \Sigma$ and $\xi \in X$, it holds that

$$f(x + \xi, \sigma) = f(x, \sigma) + \frac{\partial f}{\partial x}(x, \sigma)\xi + o(x, \sigma; \xi),$$

where $o: X \times \Sigma \times X \rightarrow Y$ is a mapping for which

$$\forall (x, \sigma) \in X \times \Sigma, \quad \forall \varepsilon > 0 \quad \exists \delta > 0:$$

$$\|o(x, \sigma; \xi)\| \leq \varepsilon \|\xi\| \quad \forall \xi \in B(0, \delta).$$

Assumption A2. For every $d > 0$, the mapping f is uniformly differentiable with respect to x in the sense that

$$\forall \varepsilon > 0 \quad \exists \delta > 0: \quad \|o(x, \sigma; \xi)\| \leq \varepsilon \|\xi\|$$

$$\forall (x, \sigma) \in B(0, d) \times \Sigma(d), \quad \forall \xi \in B(0, \delta).$$

Assumption A3. For every $d > 0$, the derivative of f with respect to x is uniformly bounded in the sense that

$$\exists c = c(d) \geq 0: \quad \left\| \frac{\partial f}{\partial x}(x, \sigma) \right\| \leq c$$

$$\forall (x, \sigma) \in B(0, d) \times \Sigma(d).$$

Note that, if the spaces X and Y are finite-dimensional and Σ is compact, then A2 and A3 follow from A1. Similarly, if X and Y are finite-dimensional, $\Sigma = Y$, and f can be represented in the form $f(x, \sigma) \equiv F(x) - \sigma$, where F is continuously differentiable, then Assumptions A1–A3 hold true.

Theorem 2. Let X and Y be Banach spaces and f be a mapping satisfying Assumptions A1–A3.

Then, for any continuous function $\varphi: \Sigma \rightarrow X$ such that either (2) or (3) holds and for any $\varepsilon > 0$, there exists a continuous function $g = g_{\varphi, \varepsilon}: \Sigma \rightarrow X$ satisfying relations (4) and

$$\int_0^{\|g(\sigma) - \varphi(\sigma)\|} \alpha_\varphi(t) dt \leq (1 + \varepsilon) \|f(\varphi(\sigma), \sigma)\| \quad \forall \sigma \in \Sigma. \quad (6)$$

Theorems 1 and 2 are global, i.e., they guarantee the existence of an implicit function g defined on the entire Σ . If the space Σ in the definition of the function α_φ and in relations (2), (3) is replaced by a given subset $\hat{\Sigma} \subset \Sigma$, then Theorems 1 and 2 guarantee the existence of an implicit function on $\hat{\Sigma}$, so they become semilocal.

If assumptions (2) and (3) are both violated, they can be satisfied by decreasing the set Σ (and, accordingly, reducing the left-hand side of (3) and increasing the right-hand side of (3)). In this context, the following natural question arises: under what conditions does the set $\hat{\Sigma}$ exist? Smoothness and continuity considerations imply that, if there is a point $(x_0, \sigma_0) \in X \times \Sigma$ such that $f(x_0, \sigma_0) = 0$ and $\frac{\partial f}{\partial x}(x_0, \sigma_0) \in \mathcal{GL}(X, Y)$, then there exists a neighborhood $\hat{\Sigma}$ of the point σ_0 in which (3) holds for $\Sigma = \hat{\Sigma}$. By using this argument, it is easy to derive local implicit function theorems from Theorems 1 and 2.

A simple sufficient condition under which relation (2) holds is

$$\gamma := \inf \left\{ \text{cov} \frac{\partial f}{\partial x}(x, \sigma) : (x, \sigma) \in X \times \Sigma \right\} > 0. \quad (7)$$

This condition corresponds to the assumptions of the Hadamard theorem, but it is more burdensome than condition (2), which was, in fact, used in [9, 10].

Below, we give an example in which condition (2) holds for any bounded function φ , while the condition (7) is violated. Let $u: \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary continuous even function such that $u(0) > 0$, u decreases on $[0, +\infty)$, and

$$u(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty, \quad \int_0^{+\infty} u(t) dt = +\infty.$$

Note that u can be specified, for example, as $u(t) = (1 + |t|^p)^{-1}$, $t \in \mathbb{R}$, where $p \in (0, 1]$.

Define

$$f(x, \sigma) := \int_0^x u(t) dt - \sigma, \quad (x, \sigma) \in \mathbb{R} \times \mathbb{R}.$$

Obviously, the mapping f is continuously differentiable.

Additionally, $\frac{\partial f}{\partial x}(x, \sigma) = u(x) \rightarrow 0$ as $x \rightarrow \infty$ and, hence, condition (7) is violated. At the same time, $\alpha_\varphi(t) \geq u(t + c_\varphi)$ for any $t \geq 0$, where $c_\varphi := \sup\{|\varphi(\sigma)| : \sigma \in \mathbb{R}\}$; hence, condition (2) is satisfied.

From Theorem 2, we can derive its semilocal version.

Corollary 1. *Suppose that X and Y are Banach spaces and the mapping f satisfies Assumptions A1–A3.*

Then, given any continuous mapping $\varphi: \Sigma \rightarrow X$ and any $r > 0$ for which

$$\begin{aligned} \gamma &= \gamma(r, \varphi) \\ &:= \inf \left\{ \text{cov} \frac{\partial f}{\partial x}(x, \sigma) : (x, \sigma) \in B(\varphi(\sigma), r) \times \Sigma \right\} > 0, \\ \sup_{\sigma \in \Sigma} \|f(\varphi(\sigma), \sigma)\| &< \gamma r, \end{aligned}$$

for any $\varepsilon > 0$, there exists a continuous function $g: \Sigma \rightarrow X$ such that

$$\begin{aligned} f(g(\sigma), \sigma) &= 0 \quad \forall \sigma \in \Sigma, \\ \|g(\sigma) - \varphi(\sigma)\| &\leq \frac{1 + \varepsilon}{\gamma} \|f(\varphi(\sigma), \sigma)\| \quad \forall \sigma \in \Sigma. \end{aligned}$$

For an inverse function theorem, conditions (2) and (3) take a simpler form. Below is a corresponding inverse function theorem following from Theorem 2.

Let X and Y be Banach spaces and $F: X \rightarrow Y$ be a given continuously differentiable mapping.

Theorem 3. *Assume that the derivative $\frac{\partial F}{\partial x}$ is bounded on any bounded set and the mapping F is uniformly differentiable, i.e.,*

$$\forall r > 0, \quad \forall \varepsilon > 0 \quad \exists \delta > 0:$$

$$\begin{aligned} \left\| F(x + \xi) - F(x) - \frac{\partial F}{\partial x}(x) \xi \right\| &\leq \varepsilon \|\xi\| \\ \forall x \in B(0, r), \quad \forall \xi \in B(0, \delta). \end{aligned}$$

Then, given any point $x_0 \in X$ and any $R > 0$ satisfying

$$R < \int_0^{+\infty} \left(\inf_{x \in B(x_0, t)} \text{cov} \frac{\partial F}{\partial x}(x) \right) dt, \quad (8)$$

for any $\varepsilon > 0$ there exists a continuous function $G = G_{x_0, \varepsilon, R}: B(F(x_0), R) \rightarrow X$ such that

$$F(G(y)) = y, \quad (9)$$

$$\int_0^{\|G(y) - x_0\|} \left(\inf_{x \in B(x_0, t)} \text{cov} \frac{\partial F}{\partial x}(x) \right) dt \leq (1 + \varepsilon) \|F(x_0) - y\| \quad (10)$$

for all $y \in B(F(x_0), R)$.

In particular, if

$$\int_0^{+\infty} \left(\inf_{x \in B(x_0, t)} \text{cov} \frac{\partial F}{\partial x}(x) \right) dt = +\infty, \quad (11)$$

then there exists a continuous function $G: Y \rightarrow X$ satisfying conditions (9) and (10) for all $y \in Y$.

Theorem 3 is semilocal, since it guarantees the existence of a continuous inverse function G defined on a given ball $B(F(x_0), R) \subset Y$. Similar to Theorem 3, an assertion for Hilbert spaces can be derived from Theorem 1.

Corollary 1 implies the following version of Theorem 3, which is also a semilocal inverse function result.

Corollary 2. *Assume that the mapping F is uniformly differentiable and the derivative $\frac{\partial F}{\partial x}$ is bounded on any bounded set.*

Then, given any point $x_0 \in X$ and any $r > 0$ for which

$$\gamma = \gamma(r, x_0) := \inf_{x \in B(x_0, r)} \text{cov} \frac{\partial F}{\partial x}(x) > 0,$$

for any $\varepsilon > 0$ there exists a continuous function $G =$

$$G_{x_0, \varepsilon, r}: B\left(F(x_0), \frac{\gamma r}{1 + \varepsilon}\right) \rightarrow X \text{ such that}$$

$$F(G(y)) = y,$$

$$\|G(y) - x_0\| \leq \frac{1 + \varepsilon}{\gamma} \|y - F(x_0)\|$$

$$\text{for all } y \in B\left(F(x_0), \frac{\gamma r}{1 + \varepsilon}\right).$$

Under the assumptions of Theorem 3, an inverse G of F that is smooth or at least satisfies the Lipschitz condition in a neighborhood of the point $F(x_0)$ may not exist even if F is infinitely differentiable. This conclusion is explained by the fact that, as was noted above, there exist linear operators from one Banach space to another that do not have a Lipschitz right inverse. For some classes of Banach spaces X and Y , including the class of all Hilbert spaces, global theorems on smooth and locally Lipschitz inverse and implicit functions were obtained in [11].

An important feature of the above-presented results is a priori estimates (5), (6), and (9) obtained for implicit and inverse functions. These estimates have various applications.

We begin with the following theorem on continuous extension of an implicit function.

Theorem 4. *Suppose that the topological space Σ is Hausdorff and paracompact and the mapping f satisfies Assumptions A1–A3 and (7).*

Then, for any closed subset $C \subset \Sigma$ and any continuous function $\varphi: C \rightarrow X$ for which

$$f(\varphi(\sigma), \sigma) = 0 \quad \forall \sigma \in C,$$

there exists a continuous function $g: \Sigma \rightarrow X$ such that

$$g(\sigma) = \varphi(\sigma) \quad \forall \sigma \in C, \quad f(g(\sigma), \sigma) = 0 \quad \forall \sigma \in \Sigma.$$

Another application of Theorem 2 is the following theorem on an approximate implicit function.

Given $\varepsilon > 0$, a continuous mapping $\varphi: \Sigma \rightarrow X$ is called an ε -implicit function if $\|f(\varphi(\sigma), \sigma)\| \leq \varepsilon \quad \forall \sigma \in \Sigma$. In other words, φ is an “implicit function up to ε .”

Proposition 1. *Suppose that a mapping f satisfies Assumptions A1–A3 and (7). Then, for any ε -implicit function φ , there exists a continuous function $g = g_\varphi: \Sigma \rightarrow X$ such that*

$$f(g(\sigma), \sigma) = 0, \quad \|g(\sigma) - \varphi(\sigma)\| \leq \frac{\varepsilon}{\gamma} \quad \forall \sigma \in \Sigma.$$

The last property is known as Ulam–Hyers stability (see [12]).

The following result is a corollary to Theorem 3 and represents a generalization of the Brouwer and Schauder fixed point theorems to the coincidence point problem. Recall that a point $\xi \in X$ is called a coincidence point of two mappings $F, \Phi: X \rightarrow Y$ if $F(\xi) = \Phi(\xi)$.

Proposition 2. *Suppose that a mapping $F: X \rightarrow Y$ satisfies the assumptions of Theorem 3.*

Then, for any point $x_0 \in X$, any $r > 0$, and any completely continuous mapping $\Phi: B(x_0, r) \rightarrow Y$ for which

$$\sup_{x \in B(x_0, r)} \|\Phi(x) - F(x_0)\| < \int_0^r \left(\inf_{x \in B(x_0, t)} \operatorname{cov} \frac{\partial F}{\partial x}(x) \right) dt, \quad (12)$$

there exists a point $\xi \in B(x_0, r)$ such that $F(\xi) = \Phi(\xi)$.

Obviously, (12) holds for some $r > 0$ if the mapping $\Phi: X \rightarrow Y$ is bounded and (11) is satisfied. Relation (11) holds if there exists $\gamma > 0$ for which $\operatorname{cov} \frac{\partial F}{\partial x}(x) \geq \gamma \quad \forall x \in X$.

FUNDING

This work was supported by the Russian Science Foundation, project no. 20-11-20131.

OPEN ACCESS

This article is distributed under the terms of the Creative Commons Attribution 4.0 International Public License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

REFERENCES

1. J. Hadamard, *Bull. Soc. Math. France* **34**, 71–84 (1906).
2. J. Ortega and W. Rheinboldt, *Iterative Solution of Nonlinear Equations of Several Variables* (McGraw-Hill, New York, 1970).
3. A. V. Arutyunov and S. E. Zhukovskiy, *Differ. Equations* **55** (4), 437–448 (2019).
4. A. V. Arutyunov, A. F. Izmailov, and S. E. Zhukovskiy, *J. Optim. Theory Appl.* **185**, 679–699 (2020).
5. E. Michael, *Ann. Math.* **63** (2), 361–382 (1956).
6. I. G. Tsar’kov, *Russ. Math. Surv.* **50** (2), 453–454 (1995).
7. M. Fabian and D. Preiss, *Comment. Math. Univ. Carol.* **28**, 311–324 (1987).
8. A. V. Arutyunov, *Proc. Steklov Inst. Math.* **291**, 24–37 (2015).
9. R. Plastock, *Trans. Am. Math. Soc.* **200**, 169–183 (1974).
10. A. A. Abramov and L. F. Yukhno, *Comput. Math. Math. Phys.* **55** (11), 1794–1801 (2015).
11. I. G. Tsar’kov, *Russ. Acad. Sci. Sb. Math.* **79** (2), 287–313 (1994).
12. S. M. Ulam, *A Collection of Mathematical Problems* (Interscience, New York, 1960).

Translated by I. Ruzanova