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MATHEMATICS

A Family of Monotone Methods for the Numerical Solution of Three-Dimensional Diffusion Problems on Unstructured Tetrahedral Meshes

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A one-parameter family of finite-volume methods is proposed for three-dimensional diffusion problems with a heterogeneous anisotropic diffusion tensor. The family is based on a nonlinear approximation of the diffusive flux. The monotonicity of these schemes is proved without imposing any constraints on the mesh and the coefficients of the problem.

The simulation of substance transport in porous media [1] necessitates the discretization of the diffusion operator. In such problems, the diffusion tensor is strongly heterogeneous and anisotropic and the geometry of the computational domain requires the use of unstructured condensing meshes. Under these conditions, the solutions produced by some modern numerical schemes [2] exhibit unphysical oscillations and negative values. Negative solution values may lead to incorrectly computed chemical interactions between the substances. As a result, the scheme becomes nonconservative.

This paper presents a method for constructing computational schemes that is based on the nonlinear approximation of the diffusive flux proposed for two-dimensional problems by Le Potier [3] and modified by the author. The resulting schemes are conservative and monotone; i.e., they ensure that the solution is nonnegative for nonnegative sources and nonnegative Dirichlet boundary conditions [4, Section 2.4]. The method proposed can be used to construct explicit and implicit conservative monotone schemes for the nonstationary diffusion equation.

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NONLINEAR FINITE-VOLUME METHOD

Let Ω be a convex polyhedral domain in $\mathbf{R}^3$ with boundary $\partial\Omega$. Consider the stationary diffusion equation with homogeneous Dirichlet boundary conditions in the mixed formulation:

$$\nabla \cdot \vec{r} = f, \qquad (1a)$$

$$\vec{r} = -\mathcal{D}\nabla C \quad \text{in} \quad \Omega, \qquad (1b)$$

$$C|_{\partial\Omega} = 0. \qquad (1c)$$

Here, C is the concentration of the substance, $\vec{r}$ is the diffusive flux, f is the source function, and $\mathcal{D}$ is a symmetric positive definite diffusion tensor of dimension 3×3 that is piecewise constant in Ω. In the computational domain Ω, we construct a conformal tetrahedral mesh ε_h such that the diffusion tensor is constant on each of its elements T. Let N_T be the number of tetrahedra $T \in \varepsilon_h$, N_P be the number of vertices, N_e be the total number of faces, and N_B be the number of external faces in ε_h.

The mass conservation law (1a) can be integrated over an element $T \in \varepsilon_h$ by using Green's identity:

$$\int_{\partial T} \vec{r} \cdot \vec{n}\, ds = \int_T f\, dx \quad \forall T \in \varepsilon_h, \qquad (2)$$

where $\vec{n}$ denotes the unit outward normal to ∂T. Let $\vec{n}_e$ be an outward normal to the face e of T whose length is numerically equal to the surface area of the corresponding face; i.e., $|n_e| = |e|$. Relation (2) can be rewritten as

$$\sum_{e \in \partial T} \vec{r}_e \cdot \vec{n}_e\, ds = \int_T f\, dx \quad \forall T \in \varepsilon_h, \qquad (3)$$

where $\vec{r}_e$ is the mean diffusive flux density through the face e:

$$\vec{r}_e = \frac{1}{|e|}\int_e \vec{r}\,ds.$$

The diffusive flux $\vec{r}_e \cdot \vec{n}_e$ through e can be approximated as follows. For each $T \in \varepsilon_h$ and each external face e, we introduce their degrees of freedom. The set of support points of these degrees of freedom is defined as $\mathbb{B} = \{X_j\}_{j=1}^{N_T+N_B}$. For each tetrahedron T, $\mathbb{B}$ includes some point X_T inside T (its coordinates will be specified later). For each e, $\mathbb{B}$ includes its center of mass X_e. Since Ω is convex, for any internal vertex O_i of ε_h, there exist four points $X_{i,j}$ ($j = 1, 2, 3, 4$) from $\mathbb{B}$ such that O_i lies inside the tetrahedron formed by them (the nearest points are picked). Therefore, there are nonnegative coefficients $\lambda_{i,j}$ satisfying the conditions $\sum_{j=1}^{4}\lambda_{i,j}\cdot\overrightarrow{O_iX_{i,j}} = 0$ and $\sum_{j=1}^{4}\lambda_{i,j} = 1$. The coefficients $\lambda_{i,j} \ge 0$ are used for linear interpolation of the concentration at interior nodes of the initial mesh from its values at points of $\mathbb{B}$:

$$C_{O_i} = \sum_{j=1}^{4}\lambda_{i,j}C_{X_{i,j}}. \qquad (4)$$

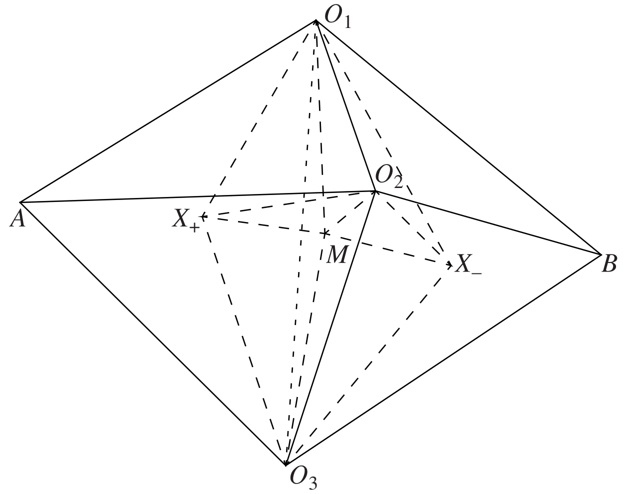
Consider two neighboring tetrahedra $T_+ = AO_1O_2O_3$ and $T_- = BO_1O_2O_3$ of the initial mesh ε_h (see figure); X_+ and X_- are the corresponding elements in $\mathbb{B}$, $\mathcal{D}_+$, and $\mathcal{D}_-$ are diffusion tensors, and V^+ and V^- are their volumes. Let M be the center of mass of the common face e, $e = O_1O_2O_3$. We introduce the following notation (here and below, i, j, and k are assumed to be different; i.e., $\{i, j, k\} = \{1, 2, 3\}, \{2, 1, 3\}, \{3, 1, 2\}$):

T_i^+ and T_i^- are the tetrahedra $X_+MO_jO_k$ and $X_-MO_jO_k$, respectively, and V_i^+ and V_i^- are their respective volumes.

$\vec{n}_e$ is the normal to the common face $O_1O_2O_3$ that is external with respect to T_+.

$\vec{n}_{ei}^+$ and $\vec{n}_{ei}^-$ are the normals to the face MO_jO_k that is external with respect to T_i^+ and T_i^-, respectively.

$\vec{n}_{ij}^+$ and $\vec{n}_{ij}^-$ are the normals to the respective faces MX_+O_k and MX_-O_k that are external with respect to T_i^+ and T_i^-, respectively.



Geometric constructions for the nonlinear finite-volume method.

$\vec{n}_i^+$ and $\vec{n}_i^-$ are the normals to the respective faces $X_+O_jO_k$ and $X_-O_jO_k$ that are external with respect to T_i^+ and T_i^-, respectively.

The lengths of all the above normals are numerically equal to the surface areas of the corresponding faces.

Each pair of tetrahedra T_i^+ and T_i^- is associated with an auxiliary variable $C_{M,i}$ that is the substance concentration at the point M. The diffusive flux $\vec{r}_i^*$ (here and below, the star denotes either a plus or a minus) on each tetrahedron T_i^* is defined by using Green's identity $\int_{T_i^*}\mathcal{D}_*^{-1}\vec{r}\,dx = -\int_{\partial T_i^*}C\vec{n}\,ds$, integrating it to second-order accuracy, and taking into account $\vec{n}_i^* + \vec{n}_{ei}^* + \vec{n}_{ij}^* + \vec{n}_{ik}^* = 0$:

$$V_i^*\mathcal{D}_*^{-1}\vec{r}_i^* = \frac{1}{3}(\vec{n}_i^*C_{M,i} + \vec{n}_{ei}^*C_{X_*} + \vec{n}_{ij}^*C_{O_j} + \vec{n}_{ik}^*C_{O_k}). \qquad (5)$$

The introduced degrees of freedom $C_{M,i}$ are eliminated using the assumption of flux continuity through e: $\vec{r}_i^+ \cdot \vec{n}_e = \vec{r}_i^- \cdot \vec{n}_e$. As a result, the flux in (5) is defined in terms of the concentrations C_{X_+} and C_{X_-} at the points X_+ and X_- and in terms of C_{O_j} and C_{O_k}, for which we use linear interpolation (4). The total diffusive flux $\vec{r}_e \cdot \vec{n}_e$ through e is represented as a linear combination of three fluxes $\vec{r}_i^+ \cdot \vec{n}_e$:

$$\vec{r}_e \cdot \vec{n}_e = \mu_1^e\vec{r}_1^+ \cdot \vec{n}_e + \mu_2^e\vec{r}_2^+ \cdot \vec{n}_e + \mu_3^e\vec{r}_3^+ \cdot \vec{n}_e. \qquad (6)$$

To determine the coefficients μ_i^e $(i = 1, 2, 3)$, we set the following conditions on diffusive flux (6) through e.

(i) If the values $\dfrac{\vec{r}_i^+ \cdot \vec{n}_e}{|\vec{n}_e|}$ approximate the diffusive flux density, then $\dfrac{\vec{r}_e \cdot \vec{n}_e}{|\vec{n}_e|}$ is also its approximation:

$$\sum_{j=1}^{3} \mu_j^e = 1. \qquad (7)$$

(ii) The approximation stencil for the flux is two-point and nonlinear:

$$\vec{r}_e \cdot \vec{n}_e = K_+(C_{O_1}, C_{O_2}, C_{O_3})C_{X_+} - K_-(C_{O_1}, C_{O_2}, C_{O_3})C_{X_-}. \qquad (8)$$

This condition is ensured by the equation

$$(a_{12}C_{O_2} + a_{13}C_{O_3})\mu_1^e + (a_{21}C_{O_1} + a_{23}C_{O_3})\mu_2^e + (a_{31}C_{O_1} + a_{32}C_{O_2})\mu_3^e = 0, \qquad (9)$$

where

$$a_{ij} = \frac{(\mathcal{D}_+\vec{n}_j^+, \vec{n}_a)(\mathcal{D}_-\vec{n}_i^-, \vec{n}_a) - (\mathcal{D}_-\vec{n}_j^-, \vec{n}_a)(\mathcal{D}_+\vec{n}_i^+, \vec{n}_a)}{(\mathcal{D}_+\vec{n}_i^+, \vec{n}_a)V_i^- - (\mathcal{D}_-\vec{n}_i^-, \vec{n}_a)V_i^+}.$$

Equations (7) and (9) define a family of solutions with parameter p^e:

$$\mu_1^e(p^e) = \mu_1^e(0) + p^e[C_{O_1}(a_{31} - a_{21}) + C_{O_2}a_{32} - C_{O_3}a_{23}], \qquad (10a)$$

$$\mu_2^e(p^e) = \mu_2^e(0) + p^e[C_{O_2}(a_{12} - a_{32}) + C_{O_3}a_{13} - C_{O_1}a_{31}], \qquad (10b)$$

$$\mu_3^e(p^e) = \mu_3^e(0) + p^e[C_{O_3}(a_{23} - a_{13}) + C_{O_1}a_{21} - C_{O_2}a_{12}]. \qquad (10c)$$

Here, $\mu_1^e(0)$, $\mu_2^e(0)$, and $\mu_3^e(0)$ comprise a particular solution to system (7), (9):

$$\mu_i^e(0) = \frac{[(\mathcal{D}_-\vec{n}_i^-, \vec{n}_e)V_i^+ - (\mathcal{D}_+\vec{n}_i^+, \vec{n}_e)V_i^-]C_{O_i}}{\sum_{j=1}^{3}[(\mathcal{D}_-\vec{n}_j^-, \vec{n}_e)V_j^+ - (\mathcal{D}_+\vec{n}_j^+, \vec{n}_e)V_j^-]C_{O_j}}. \qquad (11)$$

Remark 1. Coefficients (11) are identical to those in the two-dimensional nonlinear finite-volume method with the volumes replaced by areas. In the two-dimensional case, μ_1^e and μ_2^e are unique and precisely determined by conditions (7) and (8) of the two-point diffusive flux approximation.

If the face $O_1O_2O_3$ belongs to the boundary of the domain, Green's identity on the tetrahedron $X_+O_1O_2O_3$ with volume V^+ yields the equation

$$V^+\mathcal{D}^{-1}\vec{r} = \frac{1}{3}(C_{X_+}\vec{n}_e + C_{O_1}\vec{n}_1^+ + C_{O_2}\vec{n}_2^+ + C_{O_3}\vec{n}_3^+), \qquad (12)$$

where C_{O_i} $(i \in \{1, 2, 3\})$ are known from the boundary conditions. Since (1c) is a homogeneous Dirichlet boundary condition, for the external face e, we can write

$$\vec{r}_e \cdot \vec{n}_e = K_{B_+}C_{X_+}, \quad K_{B_+} = \frac{(\mathcal{D}\vec{n}_e, \vec{n}_e)}{3V^+}. \qquad (13)$$

Thus, the diffusive flux $\vec{r}_e \cdot \vec{n}_e$ is defined by formulas (6), (10), and (5) for internal mesh faces and by formula (13) for external mesh faces. The formulation of the method is completed by substituting the flux expressions into mass conservation law (3).

PROPERTIES OF THE METHOD

Discretization (3) produces a nonlinear system of equations

$$A(C_X)C_X = F, \qquad (14)$$

where F is the vector with elements $F_i = \int_{T_i} f\,dx$ and C_X is the N_T-vector of unknown concentrations at the points X_T of $\mathbb{B}$. System (14) is solved using the Picard iteration

$$A(C_X^k)C_X^{k+1} = F \qquad (15)$$

with some initial approximation C_X^0. To construct monotone schemes, we define the location of a point $X_T \in \mathbb{B}$ corresponding to an arbitrary tetrahedron $T = ABCD$ in the initial mesh ε_h with faces a, b, c, and d opposite to A, B, C, and D, respectively. Let $\vec{R}_A$, $\vec{R}_B$, $\vec{R}_C$, and $\vec{R}_D$ be the position vectors of the corresponding vertices of T. The vectors $\vec{n}_a$, $\vec{n}_b$, $\vec{n}_c$, and $\vec{n}_d$ are outward normals to the faces. Their lengths are numerically equal to the surface areas of the corresponding faces. Define

$$X_T = \frac{\vec{R}_A\|\vec{n}_a\|_{\mathcal{D}} + \vec{R}_B\|\vec{n}_b\|_{\mathcal{D}} + \vec{R}_C\|\vec{n}_c\|_{\mathcal{D}} + \vec{R}_D\|\vec{n}_d\|_{\mathcal{D}}}{\|\vec{n}_a\|_{\mathcal{D}} + \|\vec{n}_b\|_{\mathcal{D}} + \|\vec{n}_c\|_{\mathcal{D}} + \|\vec{n}_D\|_{\mathcal{D}}}, \qquad (16)$$

where $\|\vec{n}_\beta\|_{\mathcal{D}} = \sqrt{(\mathcal{D}\vec{n}_\beta, \vec{n}_\beta)}$ and $\beta \in \{a, b, c, d\}$. Note that, for an isotropic tensor, expression (16) gives the coordinates of the center of the sphere inscribed in T.

Theorem 1. *Let the right-hand side in system* (14) *of the nonlinear finite-volume method be nonnegative* (*i.e.*, $F_i \geq 0$); *the support points of the degrees of freedom on the tetrahedra be given by formula* (16); *the initial approximation be* $(C_X^0)_i \geq 0$; *and, for any internal face e, the nonnegative values* μ_i^e, $i \in \{1, 2, 3\}$) *be chosen from solutions* (10a)–(10c) *on every Picard iteration* (15).

Then all the iterative approximations to C_X *are nonnegative*:

$$(C_X^k)_i \geq 0, \quad i = 1, 2, \ldots, N_T, \quad \forall k \geq 0.$$

Proof. Assume that the matrix $A(C_X)$ is monotone for any nonnegative vector C_X. Then the solution C_X^{k+1} to system (15) is also a nonnegative vector. Taking into account $(C_X^0)_i \geq 0$, we find by induction that $(C_X^k)_i \geq 0$, $\forall k \geq 0$, $\forall i = 1, 2, \ldots, N_T$.

Let us prove that the matrix $A(C_X)$ is monotone for any nonnegative vector C_X. Consider the coefficients $K_+(C_{O_1}, C_{O_2}, C_{O_3})$, $K_-(C_{O_1}, C_{O_2}, C_{O_3})$, and K_{B_+} in expressions (8) and (13) for the diffusive flux through a face. The coefficient K_{B_+} is positive because $\mathcal{D}$ is positive definite. Plugging (5) (after eliminating $C_{M,i}$) into (6) gives formulas for K_+ and K_-:

$$K_+ = \sum_{i=1}^{3} \mu_i^e \cdot \frac{(\mathcal{D}_+\vec{n}_e, \vec{n}_e)}{3V^+} \times \frac{(\mathcal{D}_-\vec{n}_i^-, \vec{n}_e)V_i^+}{(\mathcal{D}_-\vec{n}_i^-, \vec{n}_e)V_i^+ - (\mathcal{D}_+\vec{n}_i^+, \vec{n}_e)V_i^-},$$

$$K_- = -\sum_{i=1}^{3} \mu_i^e \cdot \frac{(D_-\vec{n}_e, \vec{n}_e)}{3V^-} \times \frac{(D_+\vec{n}_i^+, \vec{n}_e)V_i^-}{(\mathcal{D}_-\vec{n}_i^-, \vec{n}_e)V_i^+ - (\mathcal{D}_+\vec{n}_i^+, \vec{n}_e)V_i^-}.$$

For K_+ and K_- to be positive, it is sufficient to show that

$$(\mathcal{D}_-\vec{n}_i^-, \vec{n}_e) > 0, \quad (\mathcal{D}_+\vec{n}_i^+, \vec{n}_e) < 0. \qquad (17)$$

Consider the tetrahedron $ABCD \in \varepsilon_h$ with faces a, b, c, and d opposite to the vertices A, B, C, and D, respectively, and with normals $\vec{n}_a$, $\vec{n}_b$, $\vec{n}_c$, and $\vec{n}_d$ to these faces (the lengths of the normals are numerically equal to the surface areas of the corresponding faces). The point X_T inside the tetrahedron is defined by formula (16). Let $\vec{n}_{ab}$ be defined as the normal (external with respect to X_TBCD) to the plane X_TCD, $\vec{n}_{bc}$ be defined as the normal (external with respect to X_TACD) to the plane X_TAD, and so on for $\vec{n}_{\beta\gamma}$, where $\beta, \gamma \in \{a, b, c, d\}$, $\beta \neq \gamma$. Since the length of a normal is not important for the proof of (17), $\vec{n}_{ab}$ can be calculated as

$$\vec{n}_{ab} = \frac{1}{2}(\|\vec{n}_a\|_{\mathcal{D}} + \|\vec{n}_b\|_{\mathcal{D}} + \|\vec{n}_c\|_{\mathcal{D}} + \|\vec{n}_d\|_{\mathcal{D}})(\overrightarrow{CX_T} \times \overrightarrow{DX_T}). \qquad (18)$$

For the vectors $\overrightarrow{CX_T}$ and $\overrightarrow{DX_T}$, we have the expressions

$$\overrightarrow{CX_T} = \frac{\overrightarrow{CA}\|\vec{n}_a\|_{\mathcal{D}} + \overrightarrow{CB}\|\vec{n}_b\|_{\mathcal{D}} + \overrightarrow{CD}\|\vec{n}_d\|_{\mathcal{D}}}{\|\vec{n}_a\|_{\mathcal{D}} + \|\vec{n}_b\|_{\mathcal{D}} + \|\vec{n}_c\|_{\mathcal{D}} + \|\vec{n}_d\|_{\mathcal{D}}},$$

$$\overrightarrow{DX_T} = \frac{\overrightarrow{DA}\|\vec{n}_a\|_{\mathcal{D}} + \overrightarrow{DB}\|\vec{n}_b\|_{\mathcal{D}} + \overrightarrow{DC}\|\vec{n}_c\|_{\mathcal{D}}}{\|\vec{n}_a\|_{\mathcal{D}} + \|\vec{n}_b\|_{\mathcal{D}} + \|\vec{n}_c\|_{\mathcal{D}} + \|\vec{n}_d\|_{\mathcal{D}}}.$$

Substituting them into vector product (18) gives

$$\vec{n}_{ab} = \vec{n}_b\|\vec{n}_a\|_{\mathcal{D}} - \vec{n}_a\|\vec{n}_b\|_{\mathcal{D}}.$$

Let us show that $(\mathcal{D}\vec{n}_a, \vec{n}_{ab}) < 0$ and $(\mathcal{D}\vec{n}_b, \vec{n}_{ab}) > 0$ by using the Cauchy–Schwarz inequality

$$(\mathcal{D}\vec{n}_a, \vec{n}_{ab}) = (\mathcal{D}\vec{n}_a, \vec{n}_b)\|\vec{n}_a\|_{\mathcal{D}} - (\mathcal{D}\vec{n}_a, \vec{n}_a)\|\vec{n}_b\|_{\mathcal{D}} = \|\vec{n}_a\|_{\mathcal{D}}[(\vec{n}_a, \vec{n}_b)_{\mathcal{D}} - \|\vec{n}_a\|_{\mathcal{D}}\|\vec{n}_b\|_{\mathcal{D}}] < 0. \qquad (19)$$

Here, $(\cdot, \cdot)_{\mathcal{D}}$ is the scalar product in the metric defined by the tensor $\mathcal{D}$. Similarly, we can prove $(\mathcal{D}\vec{n}_b, \vec{n}_{ab}) > 0$ and inequalities of the form $(\mathcal{D}\vec{n}_\beta, \vec{n}_{\beta\gamma}) < 0$ and $(\mathcal{D}\vec{n}_\gamma, \vec{n}_{\beta\gamma}) > 0$, where $\beta \neq \gamma$ and $\beta, \gamma \in \{a, b, c, d\}$. In (17), $\vec{n}_i^-$ and $\vec{n}_i^+$ are replaced by the corresponding vectors $\vec{n}_{\beta\gamma}$ and $\vec{n}_e$ is replaced by $\vec{n}_\beta$ or $\vec{n}_\gamma$. Then, using (19), we prove (17). Therefore, K_+ and K_- are positive. Thus, the matrix $A(C_X)$ has the following properties.

(i) All the diagonal elements of $A(C_X)$ are positive.

(ii) All the off-diagonal elements of $A(C_X)$ are nonpositive.

(iii) The matrix is column diagonally dominant; this diagonal dominance is strict for columns corresponding to elements that have faces on the boundary of the computational domain.

Therefore, $A^T(C_X)$ is an M-matrix and all the elements of $(A^T(C_X))^{-1}$ are nonnegative. Since the transposition and inversion of matrices are commuting operations, we have $(A^T(C_X))^{-1} = (A^{-1}(C_X))^T$. Therefore, all the elements of $A^{-1}(C_X)$ are nonnegative and $A(C_X)$ is monotone.

Remark 2. The validity of (17) implies that $\mu_i^e \geq 0$, $i \in \{1, 2, 3\}$) required in the assumption of the theorem can always be chosen by setting $p^e = 0$ $\forall e$ in (10a)–(10c).

The range of p^e for which μ_i^e are positive is an interval; it may degenerate into the point $p^e = 0$ when two of the three C_{O_i} are zero. If $C_{O_i} = 0\ \forall i \in \{1, 2, 3\}$, then solution (10a)–(10c) is always positive and does not depend on p^e.

Remark 3. The point X_T given by (16) is a solution to the system of six equations determining the equality of the angles in the $\mathfrak{D}$-metric between the vectors $\vec{n}_\beta$, $\vec{n}_{\beta\gamma}$ and $\vec{n}_\gamma, -\vec{n}_{\beta,\gamma}$, where $\beta, \gamma \in \{a, b, c, d\}$ and $\beta \neq \gamma$.

Corollary 1. *Consider the nonstationary diffusion equation*

$$\frac{\partial C}{\partial t} - \nabla \cdot \mathfrak{D} \nabla C = f$$

with a nonnegative right-hand side, a nonnegative initial condition, and a nonnegative Dirichlet boundary condition. The nonlinear finite-volume method is used to construct the implicit scheme

$$\left(\frac{V}{\Delta t} + A(C_X^{n+1})\right) C_X^{n+1} = V C_X^n + F^{n+1},$$

where V is a diagonal matrix of elements' volumes and F involves the right-hand side and the boundary conditions. At every time step, the system is solved by the Picard method

$$\left(\frac{V}{\Delta t} + A(C_X^{n+1,k})\right) C_X^{n+1,k+1} = \frac{V}{\Delta t} C_X^n + F^{n+1},$$

$$k = 1, 2, \ldots, \quad C_X^{n+1,0} = C_X^n.$$

If $\mu_i^e\ \forall e, i \in \{1, 2, 3\}$ *are positive, then* $(C_X^{n+1,k})_j \geq 0$ *for* $j = 1, 2, \ldots, N_T$ *and* $k = 1, 2, \ldots$.

Corollary 2. *For the explicit scheme*

$$\frac{V}{\Delta t} C_X^{n+1} = \left(\frac{V}{\Delta t} - A(C_X^n)\right) C_X^n + F^{n+1}.$$

The solution C_X^{n+1} *can be made nonnegative by choosing a sufficiently small* Δt *ensuring that the diagonal elements of* $\frac{V}{\Delta t} - A(C^n)$ *are nonnegative* (*its off-diagonal elements are obviously nonnegative*). *Moreover*, $\Delta t \sim h^2$ (*where h is the size of a quasi-uniform mesh*), *which is similar to the stability condition for explicit schemes.*

Although the convergence of the discrete solution to the solution of differential problem (1a)–(1c) is not proved, test computations have revealed that the nonlinear finite-volume method with coefficients (11) has quadratic convergence with respect to the concentration and linear convergence with respect to diffusion fluxes.

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