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Generalizing Stratonovich–Weyl Axioms for Composite Systems

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Abstract—The statistical model of quantum mechanics is based on the mapping between operators on the Hilbert space and functions on the phase space. This map can be implemented by an operator that satisfies physically motivated Stratonovich–Weyl axioms. Arguments are given in favour of a certain extension of the axioms, provided that there is a priori knowledge about the composite nature of the quantum system.

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1. INTRODUCTION

Everyday experience tells us that a "composite" is something admitting partition into an "elementary" which is considered to be primary and, therefore, not dividable any further. Moreover, it is assumed that, when a "composite" object is divided into its "elementary" parts, all properties of the "composite" object are determined by the characteristics of its constituents via certain rules of composition. Several fundamental notions of classical physics originated from these intuitive assertions and were afterwards wrapped into rigorous mathematical concepts of measure space. In particular, in classical statistical mechanics a vague idea of the "whole & part" is embodied in the measure-theoretic framework of the theory of probability where the "composition" rule is encoded in the *principle of additivity of probability distributions*, one of Kolmogorov's axioms of the probability space[1].

(A.I) Non-negativity of the probability measure, $\mathbb{P}(A) \geq 0$;

(A.II) Finite norm, $\mathbb{P}(\Omega) = 1$;

(A.III) σ-additivity for pairwise disjoint sets, $\mathbb{P}(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$.

The last axiom has a profound impact on the description of composite objects in classical theory. Along with the assumption on the Boolean character of events, (A.III) results in an intuitively expected representation of the classical phase space of a composite system in the form of the Cartesian product of the phase spaces of subsystems A and B,

$$\Omega_{A\times B} = \Omega_A \times \Omega_B. \tag{1}$$

However, even at the beginning of the quantum theory era, scientists realized that the aforementioned concepts would fail to describe a quantum world (cf. [1–3]). In particular, in the phase-space formulation of the quantum theory one cannot maintain the non-negativity axiom A.I and the composition rule (1) as universal laws without encountering a contradiction [4–6]. To avoid such a contradiction, a generalization of (A.I)–(A.III) was suggested through the replacement of the σ-algebra by the non-Boolean lattice which meant a fundamental transformation of conventional Boolean logic into non-Boolean quantum logic (see e.g., [1, 7]). As a result of the change of the classical paradigm, the notion of probability distribution functions has been converted into the concept of quasiprobability distributions.

In the present note, remaining in the framework of the phase-space formulation of quantum mechanics, the analog of (1) for composite finite-dimensional quantum systems will be discussed. The Stratonovich-Weyl (SW) axioms [8] will be complemented by a new axiom of additivity for systems a priori known to be compound ones.

2. COMPOSITE QUANTUM SYSTEMS

In quantum theory the analog of the Cartesian product (1) of the classical phase spaces of systems A and B is a tensor product of the corresponding Hilbert

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[1] Under the probability space I assume the measure space $(\Omega, \mathscr{F}, \mathbb{P})$, such that the sample space (the space of all states of the system) Ω is a $2n$-dimensional symplectic manifold, the space of events $\mathscr{F} \subset \Omega$ is represented by elements of σ-algebra, and the probability measure $\mathbb{P} : \mathscr{F} \to [0,1]$ of event $A \in \mathscr{F}$ is given by the Lebesgue integral, $\mathbb{P}(A) = \int_A \mu(dz)\rho(z)$, with the probability distribution function $\rho(z)$.

spaces $\mathcal{H}_{AB} \subset \mathcal{H}_A \otimes \mathcal{H}_B$. The partial trace operation, $\mathrm{Tr}_B : \mathcal{H}_A \otimes \mathcal{H}_B \mapsto \mathcal{H}_A$, $\mathrm{Tr}_A : \mathcal{H}_A \otimes \mathcal{H}_B \mapsto \mathcal{H}_B$, allows us to extract information on each subsystem from the density matrix ϱ_{AB} of the whole system in the form of the density matrices of subsystems ϱ_A and ϱ_B :

$$\begin{aligned} \mathrm{Tr}_B : \varrho_{AB} &\mapsto \varrho_A := \mathrm{Tr}_B \varrho_{AB}, \\ \mathrm{Tr}_A : \varrho_{AB} &\mapsto \varrho_B := Tr_A \varrho_{AB}. \end{aligned} \tag{2}$$

Aiming to construct the quasiprobability distributions of the subsystems for a given composite system, we will propose the idea to use the partial trace operation (2) not only for the states, but for their duals, the Stratonovich–Weyl kernels.

3. THE STRATONOVICH–WEYL PRINCIPLES

The Wigner quasiprobability distribution $W_\varrho(\Omega_N)$ of an N-dimensional quantum system in a mixed state is defined by the pairing of density matrix ϱ and the Stratonovich–Weyl kernel $\Delta(\Omega_N)$, which is defined on the symplectic space Ω_N:

$$W_\varrho(\Omega_N) = \mathrm{tr}(\varrho \Delta(\Omega_N)). \tag{3}$$

The SW kernel $\Delta(\Omega_N)$ determines a proper quasiprobability distribution provided the following axioms are satisfied (see e.g., [8, 9] and references therein):

(1) **Reconstruction** of state ϱ by integrating the Wigner function over a phase space:

$$\varrho = \int_{\Omega_N} \mathrm{d}\Omega_N \, \Delta(\Omega_N) W_\varrho(\Omega_N); \tag{4}$$

(2) **Hermicity** of the SW kernel, $\Delta(\Omega_N) = \Delta(\Omega_N)^\dagger$;

(3) **Finite norm** of a state given by the integral of the Wigner distribution:

$$\mathrm{tr}[\varrho] = \int_{\Omega_N} d\Omega_N W_\varrho(\Omega_N), \quad \int_{\Omega_N} d\Omega_N \, \Delta(\Omega_N) = 1; \tag{5}$$

(4) **Covariance**: The unitary transformation $\Delta'(\Omega_N) = U(\alpha)^\dagger \Delta(\Omega_N) U(\alpha)$ induces the symplectic change of coordinates $\mathbf{z}' = T_\alpha \mathbf{z}$, $T_\alpha \in Sp(d_N)$, $\mathbf{z} = \{z_1, z_2, \ldots, z_{d_N}\} \in \Omega_N$.

According to [9], the axioms (1)–(4) are fulfilled if

(1) The SW kernel $\Delta(\Omega_N)$ in (3) is an element of the dual space:

$$\begin{aligned} \mathfrak{P}_N^* = \{X \in M_N(\mathbb{C}) | \, X = X^\dagger, \\ \mathrm{tr}(\Delta(\Omega_N)) = 1, \quad \mathrm{tr}(\Delta(\Omega_N)^2) = N\}. \end{aligned} \tag{6}$$

The space of solutions to (6), i.e., the moduli space, is set by an isotropy group, $\mathcal{P}_N := \mathfrak{P}_N^* / \mathrm{Iso}_{SU(N)}(\Delta)$. For a regular $SU(N)$-stratum $\mathcal{P}_N$ is a polyhedron on $\mathbb{S}_{N-2}(1)$.

(2) The phase space Ω_N is identified with the orbit $\Omega_N = \mathrm{SU(N)} / \mathrm{Iso}_{\mathrm{SU(N)}}(\Delta)$.

Now, leaving (1)–(4) in force, the 5th axiom on composite quantum systems is proposed:

(5) **Composite systems axiom:** *Let* ϱ_{AB} *and* $\Delta(\Omega_N)$ *be a density matrix and a SW kernel of composite system* $\mathcal{H}_{A\times B}$*, respectively, then the Wigner functions of states* ϱ_A *and* ϱ_B *are constructed by pairing* (3) *with partially reduced matrices* $\Delta_A := \mathrm{Tr}_B \Delta$ and $\Delta_B := \mathrm{Tr}_A \Delta$*, i.e.,* $W_{\varrho_A} = \mathrm{Tr}\left(\varrho_A \Delta_{N_A}\right)$ and $W_{\varrho_B} = \mathrm{Tr}\left(\varrho_B \Delta_{N_B}\right)$.

The extended system of axioms (1)–(5) allows us to render the following assertions:

- The dual state space $\mathfrak{P}_{A\times B}^*$ of a binary composite system with N_A- and N_B-dimensional subsystems is $(N_A^2 N_B^2 - 4)$-dimensional subspace of $\mathfrak{P}_N^*$, defined as

$$\begin{aligned} \mathfrak{P}_{A\times B}^* = \{X \in \mathfrak{P}_N^*, N = N_A N_B | \, \mathrm{Tr}_A(\mathrm{Tr}_B X)^2 = N_A, \\ \mathrm{Tr}_B(\mathrm{Tr}_A X)^2 = N_B\}. \end{aligned} \tag{7}$$

- The phase space $\Omega_{A\times B}$ of a composite system is determined by the Local Unitary (LU) group, $\mathrm{LU} := SU(N_A) \times SU(N_B) \subset SU(N)$, corresponding to the factorization $\mathcal{H}_A \times \mathcal{H}_B$, and by the isotropy group of the SW kernel: $\Omega_{A\times B} := \mathrm{LU} / \mathrm{Iso}_{\mathrm{LU}}(\Delta)$;
- The moduli space is the factor space $\mathcal{P}_{A\times B} := \mathfrak{P}_{A\times B}^* / \mathrm{LU}$.

Below, the statements (i)–(iii) are illustrated for a simplest binary system of a pair of qubits.

4. EXEMPLIFYING A 2-QUBIT SYSTEM

Let us consider the space of full rank 4×4 density matrices $\varrho \in \mathfrak{P}_4$. Bearing in mind that the system is composed from 2-qubits, an adapted representation of a state is given in terms of the Bloch vectors of qubits $\boldsymbol{\xi}_\mathbf{A}$ and $\boldsymbol{\xi}_\mathbf{B}$ along with a 3×3 correlation matrix $\mathcal{C}$:

$$\varrho = \frac{1}{4}\mathbb{I}_4 + \frac{\sqrt{6}}{4}[\boldsymbol{\xi}_\mathbf{A} \cdot \boldsymbol{\sigma}_\mathbf{A} + \boldsymbol{\xi}_\mathbf{B} \cdot \boldsymbol{\sigma}_\mathbf{B} + \mathcal{C}_{ij} \sigma_i \otimes \sigma_j]. \tag{8}$$

In (8), $\boldsymbol{\sigma}_\mathbf{A} = (\sigma_{10}, \sigma_{20}, \sigma_{30})$ and $\boldsymbol{\sigma}_\mathbf{B} = (\sigma_{01}, \sigma_{02}, \sigma_{03})$ denote the elements of the Fano basis $\sigma_{\mu\nu} := \sigma_\mu \otimes \sigma_\nu$ of the $\mathfrak{su}(4)$ algebra constructed out of Pauli matrices $\sigma_\mu = (\mathbb{I}_2, \sigma_1, \sigma_2, \sigma_3)$. Similarly, vectors $\boldsymbol{\eta}_\mathbf{A}$ and $\boldsymbol{\eta}_\mathbf{B}$ and a real 3×3 matrix $\mathcal{E}$ define the SW kernel $\Delta \in \mathfrak{P}_4$:

$$\Delta = \frac{1}{4}\mathbb{I}_4 + \frac{\sqrt{30}}{4}[\boldsymbol{\eta}_\mathbf{A} \cdot \boldsymbol{\sigma}_\mathbf{A} + \boldsymbol{\eta}_\mathbf{B} \cdot \boldsymbol{\sigma}_\mathbf{B} + \mathcal{E}_{ij} \sigma_i \otimes \sigma_j]. \tag{9}$$

If a 4-level system is elementary, the master Eqs. (6) impose the following condition:

$$\boldsymbol{\eta}_\mathbf{A}^2 + \boldsymbol{\eta}_\mathbf{B}^2 + \mathrm{tr}(\mathscr{E}\mathscr{E}^T) = 1/2. \quad (10)$$

However, if it is known that the system consists of two qubits, according to (7), instead of (10) the individual norms of vectors $\boldsymbol{\eta}_\mathbf{A}$, $\boldsymbol{\eta}_\mathbf{B}$ and the matrix $\mathscr{E}$ are fixed:

$$\boldsymbol{\eta}_\mathbf{A}^2 = \frac{1}{10},\ \boldsymbol{\eta}_\mathbf{B}^2 = \frac{1}{10},\ \mathrm{tr}(\mathscr{E}\mathscr{E}^T) = \frac{3}{10}. \quad (11)$$

Therefore, the 2-qubit's dual space is $\mathfrak{P}^*_{2\times2} \subset \mathfrak{P}^*_4$ such that $\dim(\mathfrak{P}^*_{2\times2}) = 15 - 3 = 12$. The phase space Ω_4, corresponding to an elementary 4-level system, is one of the $SU(4)$-orbits $\Omega_4 = SU(4)/\mathrm{Iso}_{\mathrm{SU(4)}}(\Delta)$ on $\mathfrak{P}_4$. When a 4-level system is considered to be a joint of 2 qubits, then according to assertion (ii) the LU transformations and isotropy group $\mathrm{Iso}_{LU}(\Delta)$ of the SW kernel (9) define the phase space of 2-qubit $\Omega_{2\times2}$, as well as its moduli space $\mathcal{P}_{2\times2}$. In accordance with the 2-qubit Hilbert space factorization, the LU transformations form the subgroup $K = SU(2)\times SU(2) \subset SU(4)$ and the orbits of K on $\mathfrak{P}_4$ define $\Omega_{2\times2}$. Hence, the issue of describing the 2 qubit phase space is reduced to the mathematical problem of classifying admissible types of K-orbits on $\mathfrak{P}_4$. Being restricted by the required volume of this note I will illustrate the construction of a 2-qubit SW kernel for only one class of 6-dimensional phase space $\Omega_{2\times2}$ and the moduli space $\mathcal{P}_{2\times2}$ In order to explicitly describe these spaces, it is convenient to decompose $g \in SU(4)$ into three factors:

$$g := K\mathscr{A}T^3, \quad \text{with} \quad \mathscr{A} = \exp\mathfrak{a}\exp\mathfrak{a}', \quad T^3\text{—maximal torus}. \quad (12)$$

In (12) $\mathfrak{a}$ and $\mathfrak{a}'$ are the Abelian subalgebras in the direct sum decomposition of the algebra:

$$\mathfrak{su}(4) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{k}', \quad \mathfrak{k} := \mathfrak{su}(2) \oplus \mathfrak{su}(2),\ \ \mathfrak{k}' := \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus u(1), \quad (13)$$

such that the following commutator relations hold[2]:

$$[\mathfrak{a}',\mathfrak{a}] \subset \mathfrak{l},\ \ [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k},\ \ [\mathfrak{k}',\mathfrak{k}'] \subset \mathfrak{k}',\ \ [\mathfrak{k},\mathfrak{k}'] \subset \mathfrak{a} \oplus \mathfrak{a}'. \quad (14)$$

Using (12) for unitary factor U in SVD of $4\Delta_4 = U\left(\mathbb{I}_4 + \sqrt{60}\sum_{\lambda_\alpha\in\mathfrak{k}'}\lambda_\alpha\mu_\alpha\right)U^\dagger$, one can be convinced that the master Eqs. (11) describe a bundle of a unit 2-sphere and two ellipsoids E_A and E_B in the space $\mathcal{P}_4$ with the Cartesian coordinates $\boldsymbol{\mu} = \{\mu_3, \mu_6, \mu_{15}\}$:

$$\boldsymbol{\mu}\boldsymbol{\mu}^T = 1,\ \ E_A : \boldsymbol{\mu}\mathbb{A}\boldsymbol{\mu}^T = 1,\ \ E_B : \boldsymbol{\mu}\mathbb{B}\boldsymbol{\mu}^T = 1. \quad (15)$$

The 3×3 matrices $\mathbb{A}$ and $\mathbb{B}$ in (15) are constructed out of the adjoint matrix $\mathscr{A}\lambda_\nu\mathscr{A}^\dagger = O_{\nu\mu}\lambda_\mu$:

$$\mathbb{A}_{\alpha\beta} := 5\sum_{i=1,2,3} O_{\alpha i}O^T_{i\beta},\ \ \mathbb{B}_{\alpha\beta} := 5\sum_{i=4,5,6} O_{\alpha i}O^T_{i\beta}. \quad (16)$$

The moduli space $\mathcal{P}_{2\times2}$ of a 2-qubit system is determined from analyzing the following pairwise characteristic polynomials of ellipsoids E_A, E_B and a unit 2-sphere:

$$f_{E_A\cap\mathbb{S}_2}(t) = \det(t\mathbb{I}_3 + \mathbb{A}),\ \ f_{E_B\cap\mathbb{S}_2}(t) = \det(t\mathbb{I}_3 + \mathbb{B}), \quad f_{E_A\cap E_B}(t) = \det(t\mathbb{A} + \mathbb{B}). \quad (17)$$

According to [10], each characteristic polynomial in (17) always has at least one negative root. Moreover, the ellipsoids and the 2-sphere overlap iff characteristic polynomials have no positive roots. More on the geometric properties of $\mathcal{P}_{2\times2}$ will be given elsewhere.

5. CONCLUSIONS

Discussing the phase-space approach to elementary vs composite quantum systems, it is worth emphasizing the common principle used: the underlying symmetry of a system dictates the construction of the basic quantities in both cases. In the first case, the global $SU(N)$ symmetry sets the phase space $\Omega_N := SU(N)/\mathrm{Iso}_{SU(N)}(\Delta)$, as well as the moduli space $\mathcal{P}_N$, via the master equations (6) on the orbit space $\mathfrak{P}^*_N/SU(N)$. In the second case, dealing with another pair of spaces $(\mathfrak{P}_{A\times B}, \mathfrak{P}^*_{A\times B})$, the local sub-symmetry, $LU := SU(N_A)\times SU(N_B) \subset SU(N)$, comes into play once more. The latter defines the phase space $\Omega_{N_A\times N_B}$ and the moduli space $\mathcal{P}_{N_A\times N_B}$ via the extended master Eqs. (7) on the orbit space $\mathfrak{P}^*_{A\times B}/LU$.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

[2] Note that if $\{\lambda_1, \ldots, \lambda_{15}\} = \frac{i}{2}\{\sigma_{10}, \sigma_{20}, \sigma_{30}, \sigma_{01}, \sigma_{02}, \sigma_{03}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{22}, \sigma_{23}, \sigma_{31}, \sigma_{32}, \sigma_{33}\}$, then $\mathfrak{a} = \mathrm{span}\{\lambda_{11}, \lambda_9\lambda_{13}\}$, $\mathfrak{a}' = \mathrm{span}\{\lambda_4, \lambda_1\lambda_7\}$ while the torus algebra is $\mathfrak{k}' := \mathrm{span}\{\lambda_3, \lambda_6\lambda_{15}\}$, and the algebra of subgroup K reads $\mathfrak{k} := \mathrm{span}\{-\lambda_{14}, \lambda_2, -\lambda_8; -\lambda_5, \lambda_{12}, -\lambda_{10}\}$.

REFERENCES

1. G. Birkhoff and J. von Neumann, Ann. Math. **37**, 823 (1936).
2. R. Healey and H. Gomes, "Holism and Nonseparability in Physics," Ed. by E. N. Zalta, in *The Stanford Encyclopedia of Philosophy* (2022).
3. M. P. Seevinck, Stud. Hist. Phil. Mod. Phys. **35**, 693 (2004).
4. E. Wigner, Phys. Rev. **40**, 749 (1932).
5. J. G. Kirkwood, Phys. Rev. **44**, 31 (1933).
6. P. A. M. Dirac, Rev. Mod. Phys. **171**, 195 (1945).
7. D. Ellerman, Entropy **20**, 679 (2018).
8. R. L. Stratonovich, Sov. Phys. JETP **4**, 891 (1957).
9. V. Abgaryan and A. Khvedelidze, Symmetry **13**, 1013 (2021).
10. W. Wang et al., Comp. Aided Geom. Design **18**, 531 (2001).