Some Discrete Tomography Problems in Hypergraph Model Interpretation

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Abstract—In this paper we consider discrete tomography problems with an additional requirement of nonrepeatability of rows of the binary matrix to be reconstructed; as well as discrete tomography problems with given pairwise projections. Representing the problems in the hypergraph model and pointing out their equivalence to the basic definitions, we state the following results: (i) nonconvexity of the set of hypergraphic sequences of simple hypergraphs with n vertices and m hyperedges in the n-dimensional $m + 1$-valued lattice, (ii) characterization of monotone Boolean functions associated with degree sequences of 3-/(n–3)-uniform hypergraphs, (iii) formulation of discrete tomography problems with paired projections, their connection to hypergraph degree sequence problem with generalized degrees, a solution for a particular case.

Keywords: discrete tomography, paired projections, hypergraphic sequences, generalized degrees, lattice, nonconvexity

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INTRODUCTION

Discrete models, where only some partial characteristics of objects are given, are associated with problems of existence, reconstruction and description of objects according to these given characteristics. These problems arise in various scientific fields and applications such as image recognition, network modeling, design of experiments, and many others.

In this paper, in frame of the above mentioned class we consider discrete tomography models. A discrete object (e.g., a set of cells of the n-dimensional integer lattice Z^n) is accessible through a finite set of its projections, and the main problems are formulated as [19]:

(i) Consistency: does there exist a set $T \in Z^d$ with given projections.

(ii) Reconstruction: construct a $T \in Z^d$ from its projections when T exists.

It is known that the consistency/reconstruction problems of discrete tomography are NP-complete even for the case of Z^2 for 3 nonparallel projections [17]. Subsets of Z^2 can be represented as binary images or binary matrices. In the case of two orthogonal (horizontal and vertical) projections, the problem has polynomial complexity [16, 25] but the number of solutions can be large [24]. Any prior knowledge/constraints regarding the reconstructed images may reduce the search space of possible solutions. The existence problem under various additional constraints (convexity, connectivity, etc.) is studied in [3, 4, 11, 37]; NP-completeness is proved for particular cases; but some other cases can be solved by polynomial algorithms; and there are also open problems in the sense of complexity. It is therefore important to develop efficient algorithms at least for approximate solution of the problems, and to investigate the constraints whose presence/absence makes the problem easy or hard.

In this paper, we consider the case of orthogonal projections with an additional requirement of nonrepeatability of objects—rows of the binary matrix. Discrete tomography problems with pairwise projections are also considered.

We will also present the considered problems in the hypergraph model, pointing out their equivalence to the basic definitions. In the hypergraph model, the vertical projection of a binary matrix X corresponds to the degree sequence of a hypergraph having X as its incidence matrix, and horizontal projection corresponds to the sizes of its hyperedges. Thus, in terms of hypergraphs, the case of horizontal and vertical projections can be formulated as existence/construction of a hypergraph with a given degree sequence and with given sizes of hyperedges. The nonrepeatability of rows requires the hypergraph to be simple.

The existence of simple uniform hypergraphs with a given degree sequence was a long-standing open problem [1, 2, 5–7, 10, 22, 27–30]; in 2018, the NP-completeness of the problem was proved [12]. However, the problem has not lost its relevance, and research in this area is ongoing. New sufficient conditions on the degree sequences of uniform hypergraphs

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were obtained in [15]. An asymptotic enumeration formula is found in [9] for simple r-uniform hypergraphs with a given degree sequence. In [13] a rejection sampling algorithm is described and analyzed for sampling simple uniform hypergraphs with a given degree sequence. An asymptotic enumeration formula for the number of simple r-uniform hypergraphs with a given degree sequence is found in [18], when the number of edges is sufficiently large. In [26], it is proved that the existence of simple hypergraphs with a given degree sequence (without given sizes of hyperedges) is not easier than the case of uniform hypergraphs.

Characterization of $D_m(n)$, the set of all degree sequences of simple hypergraphs with n vertices and m hyperedges, is investigated in [27–29]. In terms of discrete tomography, $D_m(n)$ is a set of nonnegative integer vectors of length n that can serve as vertical projections of binary matrices with n columns and m distinct rows. Structures, properties, and a number of related results have also been derived for $D_m(n)$ [30, 33].

In this paper, we present the following results:

(i) results concerning the description of $D_m(n)$; in particular, it is proved that $D_m(n)$, being a subset of the n-dimensional $m+1$-valued grid Ξ_{m+1}^n, is not a convex set in Ξ_{m+1}^n; also, the characterization of the smallest convex set that contains $D_m(n)$ is given;

(ii) a characterization of monotone Boolean functions associated to degree sequences of 3-/(n–3)-uniform hypergraphs is obtained;

(iii) generalized degree sequences of hypergraphs are introduced, and a connection to discrete tomography problems with paired projections is outlined.

The rest of the paper is organized as follows. Section 2 presents necessary definitions, preliminaries, and basic concepts. A characterization of $D_m(n)$ in terms of its boundary elements and monotone Boolean functions is given in Section 3. The proof of nonconvexity of $D_m(n)$ in n-dimensional $m+1$-valued grid Ξ_{m+1}^n, as well as characterization of the smallest convex set containing $D_m(n)$ is presented in Section 4. Section 5 contains a partial result related to 3-/(n–3)-uniform hypergraphs. Generalized degree sequences of hypergraps are introduced in Section 6, the relation with discrete tomography problems is established, and some problems with paired projections are investigated. Concluding remarks and potential future research directions are given in Conclusion section.

2. PRELIMINARIES

2.1. Hypergraph Degree Sequences

A *hypergraph* H is a pair (V, E), where V is the vertex set of H, and E, the set of hyperedges, is a collection of nonempty subsets of V [2]. The *degree* of a vertex v of H, denoted by $d(v)$, is the number of hyperedges in H containing v. A hypergraph H is *simple* if it has no repeated hyperedges. A hypergraph H is r-*uniform* if all hyperedges contain r-vertices.

Let $V = \{v_1, v_2, \ldots, v_n\}$. $D(H) = (d(v_1), d(v_2), \ldots, d(v_n))$ is the *degree sequence* of hypergraph H. A sequence $d = (d_1, d_2, \ldots, d_n)$ is hypergraphic if there is a simple hypergraph H with the degree sequence d. Undefined terms can be found in [2].

For a given m, $0 < m \le 2^n$, let $H_m(n)$ denote the set of all simple hypergraphs $([n], E)$, where $[n] = \{1, 2, \ldots, n\}$, and $|E| = m$; and $D_m(n)$ denote the set of all hypergraphic sequences of hypergraphs in $H_m(n)$.

2.2. Generalized Degree Sequences

Let H be a hypergraph with the vertex set $[n]$.

Generalized degree of a pair of vertices (i, j) of H is the number of hyperedges of H, that contain both vertices i and j [34].

In terms of incidence matrices, the generalized degree of a pair of vertices (i, j) is the number of intersecting 1s in the ith and jth the columns of the incidence matrix of H.

2.3. Associated Vectors of Partitions

Let $B^n = \{(x_1, \ldots, x_n) | x_i \in \{0,1\}, i = 1, \ldots, n\}$ denote the set of vertices of the n-dimensional binary (unit) cube. Vertices of B^n are obtained by assigning values to the binary variables $x_1, x_2, \ldots, x_n$.

We define *partition/splitting* of B^n into two $(n-1)$-dimensional subcubes according to the values of the binary variables; for arbitrary x_i:

$$B_{x_i=0}^{n-1} = \{(x_1, \ldots, x_n) \in B^n \,|\, x_i = 0\}$$

and $$B_{x_i=1}^{n-1} = \{(x_1, \ldots, x_n) \in B^n \,|\, x_i = 1\}.$$

Any subset $\mathcal{M} \subseteq B^n$ will be partitioned into

$$\mathcal{M}_{x_i=1} \subseteq B_{x_i=1}^{n-1} \quad \text{and} \quad \mathcal{M}_{x_i=0} \subseteq B_{x_i=0}^{n-1}.$$

An integer vector $S = (s_1, \ldots, s_n)$ is called *associated vector of partitions* of the set $\mathcal{M} \subseteq B^n$, if $s_i = |\mathcal{M}_{x_i=1}|$, $i = 1, \ldots, n$ [28].

In general, different sets may have the same associated vector of partitions. The number of sets of size $|\mathcal{M}|$ is $C_{2^n}^{|\mathcal{M}|}$; the number of associated vectors has not yet been estimated.

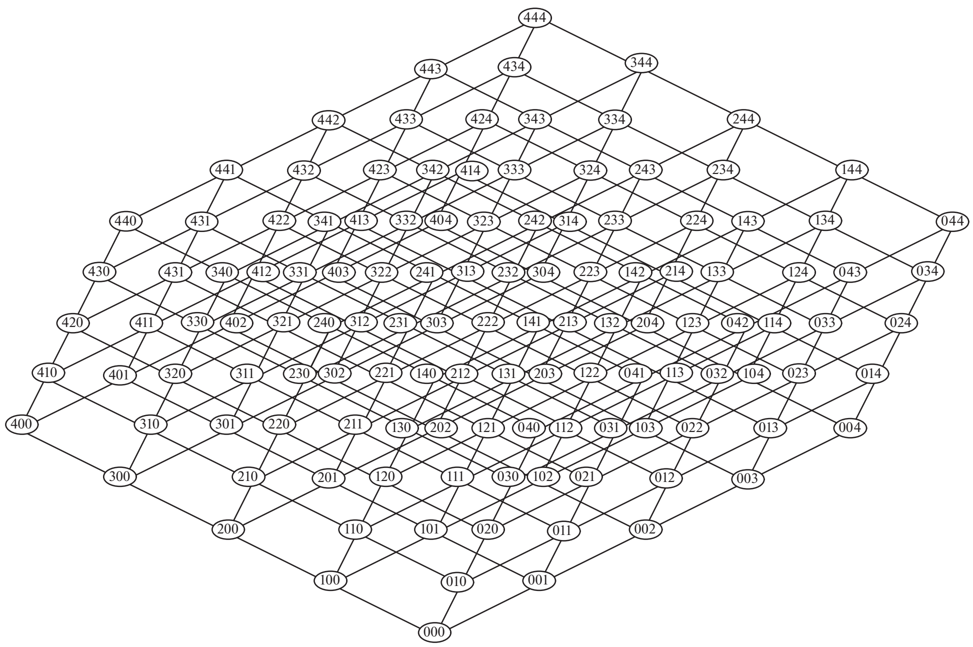

Fig. 1. Hasse diagram of Ξ_5^3.

2.4. Relation with the Hypergraph Degree Sequences

Consider the power set $\mathcal{P}([n])$ of $[n]$ and its partial order by inclusion: for arbitrary $a, b \subseteq [n]$, a preceded b if and only if $a \subseteq b$. Identify subsets of $[n]$ with binary sequences/vectors of length n such that the ith entry equals "1" if and only if the ith element of $[n]$ is included in the subset. In this manner, 1-1 correspondence between $\mathcal{P}([n])$ and B^n is established; and each $\mathcal{M} \subseteq B^n$ can be identified with an element of $\mathcal{P}([n])$. In other words, each $\mathcal{M} \subseteq B^n$ can be identified with a simple hypergraph $\mathcal{H}$ on the vertex set $[n]$, whose edges are nonrepetitive and are determined by the elements of $\mathcal{M}$. The degree of ith vertex is equal to $|\mathcal{M}_{x_i=1}|$; and the associated vector of partitions of $\mathcal{M}$ corresponds to the degree sequence of $\mathcal{H}$. In [35, 36] hypergraph enumeration problems are considered, but in observed hypergraphs edge repetitions are allowed.

2.5. Monotone Boolean Functions

Boolean function $f : B^n \to \{0,1\}$ is called *monotone* if for every two vertices $\alpha, \beta \in B^n$, if $\alpha \prec \beta$ then $f(\alpha) \le f(\beta)$. Vertices of B^n, where f takes the value "1" are called *units* or *true points* of the function; vertices, where f takes the value "0" are called *zeros* or *false points* of the function [1].

3. CHARACTERIZATION OF $D_m(n)$

In this section we bring several structures, properties, results related to the characterization of hypergraphic sequences (partially, from [27, 28]), which will be used to prove theorems in the next section.

Clearly, every sequence of length n with all integer components between 0 and m, can serve potentially as a degree sequence of such hypergraphs. If repeated hyperedges are allowed, then the corresponding set of degree sequences coincides with $\Xi_{m+1}^n = \{(a_1, \ldots, a_n) | 0 \le a_i \le m\}$.

Thus, $D_m(n) \subseteq \Xi_{m+1}^n$. $(\Xi_{m+1}^n, \preccurlyeq)$ is a partial ordered set with the component-wise partial order on Ξ_{m+1}^n: $(a_1, \ldots, a_n) \preccurlyeq (b_1, \ldots, b_n)$ if and only if $a_i \le b_i$ for all i. Undefined terms can be found in [8]. Figure 1 demonstrates the Hasse diagram of Ξ_5^3.

3.1. Boundary Cases Depending on the Value of m

If $m = 0$ then $D_m(n)$ consists of a single vector, all components of which are equal to 0.

If $m = 1$ then Ξ_{m+1}^n coinsides with B^n; any of 2^n elements of B^n is an element of $D_m(n)$; $|D_m(n)| = 2^n$.

If $m = 2^n$ then $D_m(n)$ consists of a single vector, all components of which are equal to 2^{n-1}.

3.2. Classes of Elements in Ξ^n_{m+1}

We distinguish several classes of elements.

3.2.1. Middle elements. $m_{mid+} = \left(\left\lceil \frac{m}{2} \right\rceil, \dots, \left\lceil \frac{m}{2} \right\rceil\right)$ and $m_{mid-} = \left(\left\lfloor \frac{m}{2} \right\rfloor, \dots, \left\lfloor \frac{m}{2} \right\rfloor\right)$ are called *middle elements* of Ξ^n_{m+1}.

3.2.2. Opposite elements in Ξ^n_{m+1}**.** Two elements $d, \overline{d}$ of Ξ^n_{m+1} are called *opposite* if one can be obtained from the other by inversions of component values, i.e., if $d = (d_1, \dots, d_n)$, then $\overline{d} = (m - d_1, \dots, m - d_n)$.

3.2.3. Boundary elements of $D_m(n)$**.** $(d_1, \dots, d_n) \in D_m(n)$ is an *upper boundary/lower boundary/*element of $D_m(n)$ if no $(a_1, \dots, a_n) \in \Xi^n_{m+1}$ with $(a_1, \dots, a_n) > (d_1, \dots, d_n)$/with $(a_1, \dots, a_n) < (d_1, \dots, d_n)$/belongs to $D_m(n)$.

Let $\hat{D}_{\max}$ and $\check{D}_{\min}$ denote the sets of upper and lower boundary elements of $D_m(n)$, respectively.

3.2.4. Interval/subgrid in Ξ^n_{m+1}**.** For a pair of elements d', d'' of Ξ^n_{m+1} with $d' \le d''$, $E(d', d'')$ denotes the minimal subgrid/interval in Ξ^n_{m+1} spanned by these elements, i.e., $E(d', d'') = \{a \in \Xi^n_{m+1} | d' \le a \le d''\}$.

The following preliminary results are from [27, 28].

Lemma 1. *$d = (d_1, \dots, d_i, \dots, d_n)$ belongs to $D_m(n)$ if and only if $\overline{d}_i = (d_1, \dots, m - d_i, \dots, d_n)$ belongs to $D_m(n)$, for arbitrary $i, 1 \le i \le n$.*

Lemma 2. *For each element $\hat{d} \in \hat{D}_{\max}$ there exists its opposite element $\overline{\hat{d}} \in \check{D}_{\min}$, and vice versa. Thus,* $|\hat{D}_{\max}| = |\check{D}_{\min}|$.

Lemma 3. *For every element $\hat{d} = (\hat{d}_1, \dots, \hat{d}_n)$ of $\hat{D}_{\max} \hat{d}_i \ge m - \hat{d}_i$; and for every element $\check{d} = (\check{d}_1, \dots, \check{d}_n)$ of $\check{D}_{\min}$ $\check{d}_i \le m - \check{d}_i$, $i = 1, \dots, n$.*

Let $d_{\min}$ denote the element of $\hat{D}_{\max}$, which has the minimum rank among all elements of $\hat{D}_{\max}$, $r(d_{\min}) = \min_{d \in \hat{D}_{\max}} r(d)$.

Lemma 4. *$d_{\min}$ has components equal to m, if* $m \le 2^{n-1}$.

Theorem 1. $D_m(n) = \bigcup_{\hat{D} \in \hat{D}_{\max}, \check{D} \in \check{D}_{\min}} E(\check{D}, \hat{D})$, *where* $(\hat{D}, \check{D})$ *are pairs of opposite elements.*

It is worth noting the relation of $\hat{D}_{\max}$ to the monotone Boolean functions defined on B^n. Each subset of vertices of B^n can be identified with the set of units of some Boolean function; and thus, monotone Boolean functions represent a specific class of sets in B^n. Let M_m denote the class of m-sets in B^n represented by monotone Boolean functions (sets of the units of the function) with m units, and let $D_{M_m}(n)$ denote the class of corresponding associated vectors of partitions.

Theorem 2.

$$\hat{D}_{\max} \subseteq D_{M_m}(n).$$

4. NONCONVEXITY OF $D_m(n)$ IN Ξ^n_{m+1}

For the case of k-uniform hypergraphs, the convex hull of degree sequences was investigated in [20, 21, 23].

Ξ^n_{m+1} is an n-dimensional integral polytope, — a convex polytope, vertices of which have all integer coordinates between 0 to m. Undefined terms can be found in [14].

The intervals $E(\check{D}, \hat{D})$, where $(\hat{D}, \check{D})$ are pairs of opposite elements, are convex subsets in Ξ^n_{m+1}, by definition.

In this section, we prove that $D_m(n)$, being a union of convex sets $E(\check{D}, \hat{D})$, is not convex in Ξ^n_{m+1}. The smallest convex subset of Ξ^n_{m+1}, containing $D_m(n)$ is also characterized.

Theorem 3. *$D_m(n)$ is convex for $m = 1$, $2^n - 1$, 2^n, and not convex for* $1 < m < 2^n - 1$.

Proof. (a) $m = 1$.

According to Theorems 1 and 2, $D_m(n) = \bigcup_{\hat{D} \in \hat{D}_{\max}, \check{D} \in \check{D}_{\min}} E(\check{D}, \hat{D})$; and $\hat{D}_{\max} \subseteq D_{M_m}(n)$. Thus, elements of $\hat{D}_{\max}$ are among the associated vectors of partitions coming from monotone Boolean functions. For $m = 1$ there exists a unique monotone Boolean function with the single unit vertex $(1, 1, \dots, 1)$ of B^n. Therefore, $\hat{D}_{\max}$ consists of the single element $(m, m, \dots, m)$, and this is the only possible case that $\hat{D}_{\max}$ contains $(m, m, \dots, m)$. According to Lemma 2, $\check{D}_{\min}$ contains the single element $(0, 0, \dots, 0)$. Then, $D_m(n) = E((0, 0, \dots, 0), (m, m, \dots, m))$, which coincides with Ξ^n_{m+1}.

(b) $m = 2^n$.

There exists a unique monotone Boolean function, with the set of unit vertices coinciding with the whole B^n. $\hat{D}_{\max}$ consists of a single element with all components equal to 2^{n-1}.

(c) $m = 2^n - 1$.

There exists a unique monotone Boolean function, the set of unit vertices of which coincides with $B^n \backslash \{(0,0,\dots,0)\}$. $\hat{D}_{\max}$ consists of a single element with components equal to 2^{n-1}.

Thus, in (b) and (c), $\hat{D}_{\max}$ consists of a single element with components equal to 2^{n-1}, and this is the only possible case that $\hat{D}_{\max}$ contains such an element. Hence, $D_m(n) = E\left((2^{n-1},\dots,2^{n-1}), (2^{n-1},\dots,2^{n-1})\right)$.

Thus, we proved that in (a)–(c), $D_m(n)$ is convex.

(d) $1 < m < 2^n - 1$.

Let $\hat{D}_{\max}$ consists of r elements (it follows that $\check{D}_{\min}$ consists of r elements, as well): $\hat{D}_{\max} = \{\hat{D}_1,\dots,\hat{D}_r\}$, $\check{D}_{\min} = \{\check{D}_1,\dots,\check{D}_r\}$, where $\hat{D}_i$, $\check{D}_i$ are opposite elements.

We prove that there exist $\check{D}_i \in \check{D}_{\min}$ and $\hat{D}_j \in \hat{D}_{\max}$, $i \neq j$ such that $E(\check{D}_i, \hat{D}_j)$ is not contained in $D_m(n)$.

Firstly, we notice that $\check{D}_i \le \hat{D}_j$ for arbitrary i, j, since the components' values of $\hat{D}_j$ are greater or equal to the middle value $m/2$, and the components' values of $\check{D}_i$- are less than or equal to the middle value $m/2$ (according to Lemma 3). Consider the following cases:

(1) $m \le 2^{n-1}$.

Let $\hat{D}_j$ be a minimal element of $\hat{D}_{\max}$ (assume that components are in decreasing order): $\hat{D}_j = \left(m, \hat{d}_2^j, \dots, \hat{d}_n^j\right)$ (according to Lemma 4, it has m valued component). Consider another element $\hat{D}_i = \left(\hat{d}_1^i, \hat{d}_2^i, \dots, \hat{d}_n^i\right)$ of $\hat{D}_{\max}$, where $\hat{d}_1^i < m$. Such an element exists—it can simply be the vector obtained from $\hat{D}_j$ by components permutation, taking into account also that all the components of $\hat{D}_j$ cannot be equal to m.

Consider the opposite to $\hat{D}_i$ element: $\check{D}_i = \left(m - \hat{d}_1^i, m - \hat{d}_2^i, \dots, m - \hat{d}_n^i\right)$, and replace the first component with m; we obtain $\left(m, m - \hat{d}_2^i, \dots, m - \hat{d}_n^i\right)$, which belongs to $E(\check{D}_i, \hat{D}_j)$, but does not belong to $D_m(n)$, since according to Lemma 1, $\left(m, \hat{d}_2^i, \dots, \hat{d}_n^i\right)$ should belong to $D_m(n)$, which contradicts the fact that $\hat{D}_i$ is an element of $\hat{D}_{\max}$.

(2) $m > 2^{n-1}$.

The proof is similar to the previous case, taking into account that all components of $\hat{D}_{\max}$ cannot be equal to 2^{n-1}, besides the case of $m = 2^n - 1$.

□

As an example, consider $D_4(3)$ in Ξ_5^3 given in Fig. 2. $(0,2,2)$ and $(3,3,3)$ belong to $D_4(3)$, and $(0,2,2) < (3,3,3)$. However, the elements $(0,3,2)$, (0, 2, 3), (0, 3, 3) of $\Xi 3_5^n$, which are greater than $(0,2,2)$, and less than $(3,3,3)$, do not belong to $D_4(3)$.

We denote by $C_{D_m(n)}$. We denote this set by $C_{D_m(n)}$ he smallest convex subset of Ξ_{m+1}^n, containing $D_m(n)$.

Theorem 4. $C_{D_m(n)} = \bigcup_{i=1}^r \bigcup_{j=1}^r E(\check{D}_i, \hat{D}_j)$.

Proof. It is clear that $D_m(n) \subseteq \bigcup_{i=1}^r \bigcup_{j=1}^r E(\check{D}_i, \hat{D}_j)$. Now, we prove that $\bigcup_{i=1}^r \bigcup_{j=1}^r E(\check{D}_i, \hat{D}_j)$ is a convex set in Ξ_{m+1}^n, and there is no smaller set in Ξ_{m+1}^n, that contains $D_m(n)$.

Firstly, we prove that $\bigcup_{i=1}^r \bigcup_{j=1}^r E(\check{D}_i, \hat{D}_j)$ is convex in Ξ_{m+1}^n.

Let $a, b \in \bigcup_{i=1}^r \bigcup_{j=1}^r E(\check{D}_i, \hat{D}_j)$, and $a < b$; we prove that the interval $[a,b] = \{c \in \Xi_{m+1}^n | a \le c \le b\}$ belongs to $\bigcup_{i=1}^r \bigcup_{j=1}^r E(\check{D}_i, \hat{D}_j)$, as well. If a, b are boundary elements (upper or lower), or belong to some $E(\check{D}_i, \hat{D}_i)$, then the proof is evident. Suppose that a, b are not boundary elements, and $a \in E(\check{D}_i, \hat{D}_i)$, $b \in E(\check{D}_j, \hat{D}_j)$, $i \neq j$. In this case, every element c from $[a,b]$ belongs to $E(\check{D}_i, \hat{D}_j)$ /taking into account that $\check{D}_i \le \hat{D}_j$, for arbitrary $1 \le i, j \le r$/.

On the other hand, $\bigcup_{i=1}^r \bigcup_{j=1}^r E(\check{D}_i, \hat{D}_j) \subseteq C_{D_m(n)}$, which implies that there is no smaller set in Ξ_{m+1}^n, that contains $D_m(n)$.

□

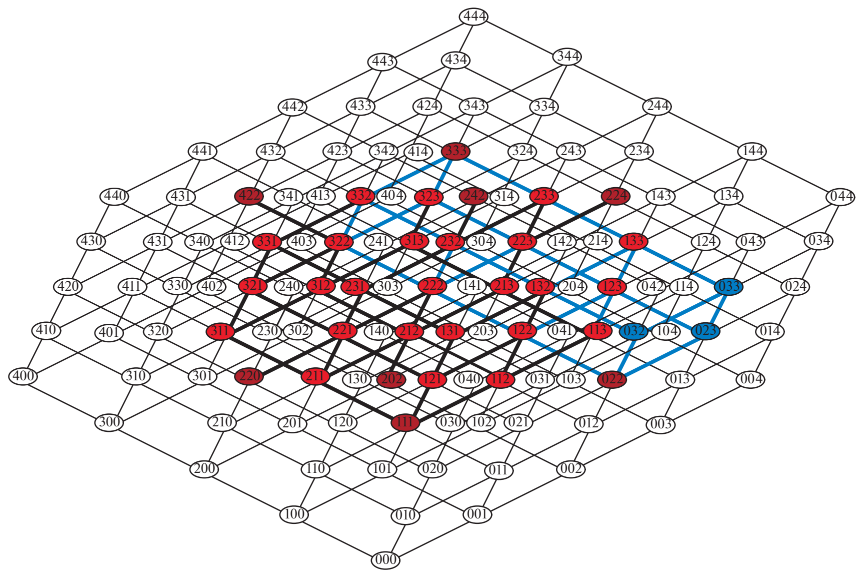

Fig. 2. Nonconvexity example.

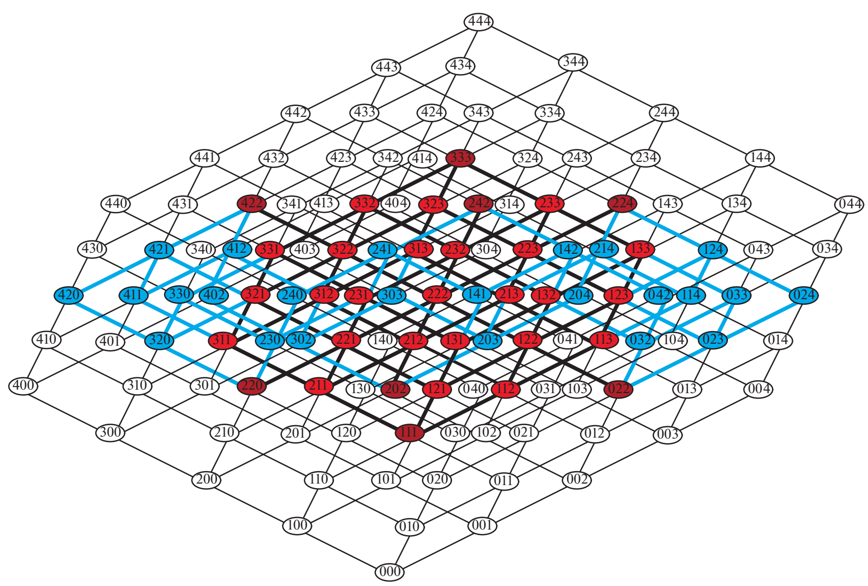

Fig. 3. Elements of $C_{D_4(3)}$ are colored (red and blue); elements of $D_4(3)$ are in red color.

Figure 3 demonstrates $C_{D_4(3)}$ in Ξ_5^3.

5. BOOLEAN FUNCTIONS RELATED TO 3-/(*n*–3)-UNIFORM HYPERGRAPHS

In this section we introduce a partial result related to 3-uniform hypergraphs; we characterize monotone Boolean functions, whose associated vectors of partitions (more precisely, associated vectors of partitions of the sets of their units) correspond to the degree sequences of 3-/(*n*–3)-uniform hypergraphs.

Let $D_m^k(n)$ denote the set of degree sequences of simple k-uniform hypergraphs with n vertices and m hyperedges.

Consider the set $D_m^{n-3}(n)$ (or, the set $D_m^3(n)$). Note that in this case $m \le C_n^3$.

All elements of $D_m^{n-3}(n)$ are located on the $m \cdot (n-3)$th layer of Ξ_{m+1}^n, namely in those parts, which are included in $\bigcup_{\hat{D} \in \hat{D}_{\max}} E(\check{D}, \hat{D})$.

1	1	0	1	1
1	1	0	1	1
0	1	0	1	1
1	0	0	1	1
1	1	0	1	1
1	0	0	1	1
1	1	1	1	1
0	1	1	1	0

Fig. 4. $A\_VP = (4,1,6,6,2,6,5,2,1,7)$, $N\_VP = (4,2,2,7)$.

Theorem 5. *For every sequence* $d \in D_m^{n-3}(n)$, *there exists some monotone Boolean function from* M_m *whose lower units are on the* $(n-3)$*th and/or higher than* $(n-3)$*th layers of* B^n, *and which has associated vector of partitions greater than* d.

Proof.

Let $d \in D_m^{n-3}(n)$, and $\mathcal{M}$ is an m-set on the $(n-3)$ — the layer of B^n, whose associated vector of partitions is d. Transform $\mathcal{M}$ into the set of units of monotone Boolean function according to the following algorithm:

Input: $\mathcal{M}$, output: the required set $\mathcal{M}'$.

Algorithm ***TRANSFORM_MONOTONE (TM)***:

$\mathcal{M}' := \varnothing$;

While $(\mathcal{M} \neq \varnothing)$

{Consider current vertex $v \in \mathcal{M}$;

$\mathcal{M} := \mathcal{M} \backslash \{v\}$;

If v does not have upper neighbors in B^n, which does not belong to $\mathcal{M}'$ then

$\{\mathcal{M}' := \mathcal{M}' \cup \{v\}\}$;

Otherwise:

{While v has upper neighbors in B^n, which does not belong to $\mathcal{M}'$

{Consider current neighbor v'; $v := v'$};

}

$\mathcal{M}' := \mathcal{M}' \cup \{v\}$;

}

Then, $\mathcal{M}'$ will correspond to monotone function, and $\mathcal{M}'$ has associated vector greater than d.

□

In this manner, if we consider the characterization of $D_m^{n-3}(n)$ in terms of monotone Boolean functions, then only those functions are necessary, whose lower units are located on $(n-3)$—the and/or upper layers of B^n.

6. DISCRETE TOMOGRAPHY PROBLEMS WITH PAIRED PROJECTIONS RELATED TO GENERALIZED DEGREES OF HYPERGRAPHS

In this section we introduce pair-based projections, establish a connection with the generalized degree sequences of hypergraphs, and investigate related problems.

(1) $S = \left(s_1, s_2, \ldots, s_{C_n^2}\right)$ is called *A_VP projection* of a binary matrix $X = \{x_{i,j}\}$ of size $m \times n$, if s_k is the number of intersecting 1s in the kth pair of columns, for $k = 1,\ldots,C_n^2$ (supposed that the pairs are enumerated beforehand in the lexicographic order).

(2) $S = (s_1, s_2, \ldots, s_{n-1})$ is called *N_VP projection* of a binary matrix $X = \{x_{i,j}\}$ of size $m \times n$, if s_k is the number of intersecting 1s of the kth and $(k+1)$th columns of X, for $k = 1,\ldots,n-1$.

Note that (1) corresponds to the generalized degree sequence of a hypergraph consisting of degrees of all pairs of vertices; and (2) corresponds to the generalized degree sequence of a hypergraph consisting of pairs of consecutive vertices $(k, k+1)$.

An example is given in Fig. 4.

Based on these projections, we formulate the following problems.

HV_AVP. Given nonnegative integer vectors $R = (r_1, r_2, \ldots, r_m)$, $C = (c_1, c_2, \ldots, c_n)$ and $S = \left(s_1, s_2, \ldots, s_{C_n^2}\right)$. Is there a binary matrix $X = \{x_{i,j}\}$ of size $m \times n$, such that R, C and S are its horizontal, vertical and *A_VP* projections?

HV_NVP. Given nonnegative integer vectors $R = (r_1, r_2, \ldots, r_m)$, $C = (c_1, c_2, \ldots, c_n)$ and $S = (s_1, \ldots, s_{n-1})$. Is there a binary matrix $X = \{x_{i,j}\}$ of size $m \times n$, such that R, C, and S are its horizontal, vertical, and *N_VP* projections.

HV_ΣNVP. Given nonnegative integer vectors $R = (r_1, r_2, \ldots, r_m)$, $C = (c_1, c_2, \ldots, c_n)$ and natural number k. Is there a binary matrix $X = \{x_{i,j}\}$ of size $m \times n$, such that R, C are its horizontal and vertical projections, and the sum of components of *N_VP* projection equals k.

Complexities of $\boldsymbol{HV\_AVP}$ and $\boldsymbol{HV\_\Sigma NVP}$ are investigated in [34], and NP-completeness are proved using the relations with the “Subsuming degree sequence problem” [10] for $\boldsymbol{HV\_AVP}$, and “Existence of horizontal convex binary matrix with given horizontal and vertical projections” [17] for $\boldsymbol{HV\_\Sigma NVP}$; both are known NP-complete problems.

To investigate the complexity of $\boldsymbol{HV\_NVP}$, we present it as a system of integer constraints [34]:

$$\begin{aligned} &(1)\quad \sum_{i=1}^{m} x_{i,j} = c_j, \quad j = 1,\dots,n \\ &(2)\quad \sum_{j=1}^{n} x_{i,j} = r_i, \quad i = 1,\dots,m \qquad (HV\_NVP) \\ &\sum_{i=1}^{m} \min(x_{i,j}, x_{i,j+1}) = s_j, \quad j = 1,\dots,n-1 \\ &x_{i,j} \in \{0, 1\}, \end{aligned}$$

where (1) provides vertical projection, (2) provides horizontal projection, and (3) provides N_VP projection of $X = \{x_{i,j}\}$, respectively. We may add also $s_j \le \min(c_j, c_{j+1})$

On the other hand, let us denote by $\boldsymbol{H\_CONVEX}$ the problem of existence of a horizontal convex binary matrix with given horizontal and vertical projections. It can be presented as:

$$\begin{aligned} &(1)\ \sum_{i=1}^{m} x_{i,j} = c_j, \ \ j = 1,\dots,n; \\ &(2)\ \sum_{j=1}^{n} x_{i,j} = r_i,\ i = 1,\dots,m; \qquad (H\_CONVEX) \\ &(3)\ \sum_{i=1}^{m}\sum_{j=1}^{n-1} \min(x_{i,j}, x_{i,j+1}) = \sum_{i=1}^{m}(r_i - 1); \\ &x_{i,j} \in \{0,1\}. \end{aligned}$$

If in some row i of X the number of intersecting 1s of the kth and $(k+1)$th columns of X, $k = 1,\dots,n-1$, equals r_i, then ith row is horizontally convex. Therefore, (3) in (H_CONVEX) provide the horizontal convexity of the matrix.

If we could reduce polynomially the NP-complete problem (H_CONVEX) to (HV_NVP) then we would prove the NP-completeness of HV_NVP.

Let $I(H\_CONVEX)$ be an arbitrary instance of H_CONVEX.

$$I(H\_CONVEX){:}\ R = (r_1, r_2, \dots, r_m),\quad C = (c_1, c_2, \dots, c_n).$$

And compose the following series of HV_NVP instances:

$$\{I(HV\_NVP)\}{:}\ R = (r_1, r_2, \dots, r_m),\ C = (c_1, c_2, \dots, c_n),$$

$$A = \{S = (s_1, \dots, s_{n-1})\} \,|\, \sum_{j=1}^{n-1} s_j = \sum_{i=1}^{m}(r_i - 1)\}.$$

We need to show that if $I(H\_CONVEX)$ is a positive instance of $\_CONVEX$ if and only if there is a positive instance in $\{I(HV\_NVP)\}$, i.e., there exists $S = (s'_1, \dots, s'_{n-1})$ in A such that $R = (r_1, r_2, \dots, r_m)$, $C = (c_1, c_2, \dots, c_n)$, $S = (s'_1, \dots, s'_{n-1})$ is a positive instance of HV_NVP.

Suppose $I(H\_CONVEX)$ is positive, i.e. there exists $X = \{x_{i,j}\}$ with $R = (r_1, r_2, \dots, r_m)$, $C = (c_1, c_2, \dots, c_n)$ and with $r_i - 1$ consecutive 1s in the ith row.

Thus, $\sum_{i=1}^{m}(r_i - 1) = \sum_{i=1}^{m}\sum_{j=1}^{n-1}\min(x_{i,j}, x_{i,j+1}) = \sum_{j=1}^{n-1}\sum_{i=1}^{m}\min(x_{i,j}, x_{i,j+1})$.

Denote $s'_j = \sum_{i=1}^{m}\min(x_{i,j}, x_{i,j+1})$ for $j = 1,\dots,n-1$. Then, $S' = (s'_1, \dots, s'_{n-1})$ belongs to A, and thus, C, R, S' is a positive instance of HV_NVP.

The converse is obvious because if $R = (r_1, r_2, \dots, r_m)$, $C = (c_1, c_2, \dots, c_n)$, $S' = (s'_1, \dots, s'_{n-1})$, where $\sum_{j=1}^{n-1} s'_j = \sum_{i=1}^{m}(r_i - 1)$, is a positive instance of HV_NVP, then the matrix must be h-convex.

For the polynomiality of the reduction we need to count the cardinality of

$$A = \{S = (s_1, \dots, s_{n-1})\} \,|\, \sum_{j=1}^{n-1} s_j = \sum_{i=1}^{m}(r_i - 1)\}.$$

Several simplifications are done in [34], however, the explicit formula is not obtained.

In this regard, the question of investigating the complexity of the following special case of $\boldsymbol{H\_CONVEX}$ arose.

$\boldsymbol{H\_CONVEX\_1}$. Given nonnegative integer vectors $R = (r_1, r_2, \dots, r_m)$, $C = (c_1, c_2, \dots, c_n)$, where either $m - c_1$ or $m - c_n$ is constant. Is there a h-convex binary matrix $X = \{x_{i,j}\}$ of size $m \times n$, such that R and C are its horizontal and vertical projections.

Theorem 6. *H_CONVEX_1 can be solved in polynomial time.*

Proof. Suppose $m - c_1$ is constant (the case when $m - c_n$ is constant is similar). We construct h-convex binary matrix X in the following way. The first row must contain $m - c_1$ 0s. There are $C_m^{m-c_1}$ possibilities for putting $m - c_1$ zeros (consequently, c_1 1s on the first column). For each of them the corresponding $i_1, i_2, \dots, i_{c_1}$ rows are constructed uniquely in the following way – $r_{i_1}, r_{i_2}, \dots, r_{i_{c_1}}$ 1s followed by $n - r_{i_1}, n - r_{i_2}, \dots, n - r_{i_{c_1}}$

0s. Let X_1 denote the matrix consisting of these constructed rows, and X_2 denote the matrix consisting of the remaining $m - c_1$ rows. Then, the vertical projection $C' = (c_1' = c_1, c_2', \ldots, c_n')$ can be easily found. The vertical projection of X_2 must be $C'' = (c_1'' = 0$, $c_2'' = c_2 - c_2'$, ..., $c_n'' = c_n - c_n')$. Each row of X_2 must be convex, thus there are $n - r_j + 1$ possibilities for the first 1 in j –th row, $j \neq i_1, i_2, \ldots, i_{c_1}$. For each of them we can count the vertical projection. If it equals $C'' = (c_1'' = 0, c_2'' = c_2 - c_2', \ldots, c_n'' = c_n - c_n')$, then the required matrix exists. Since the number of rows of X_2 is constant, the problem can be solved in polynomial time.

CONCLUSIONS

In this paper, we considered discrete tomography problems with an additional requirement of nonrepeatability of objects: rows of the binary matrix to be reconstructed; as well as discrete tomography problems with given pairwise projections. We represented the problems in the hypergraph model, and also, as systems of integer linear inequalities/constraints, indicating their equivalence to the corresponding problems. We obtained the following results: (i) proved that the set of degree sequences of simple hypergraphs with n vertices and m hyperedges is nonconvex in the n-dimensional $m+1$-valued lattice, (ii) obtained characterizations of monotone Boolean functions associated with degree sequences of 3-/(n–3)-uniform hypergraphs, (iii) we formulated and investigated some discrete tomography problems with paired projections in terms of hypergraphs and generalized degrees, as well as in terms of systems of integer linear constraints.

In future studies we will consider consistency and reconstruction problems of discrete tomography with paired projections involving diagonal/antidiagonal projections.

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CONFLICT OF INTEREST

The author of this work declares that she has no conflicts of interest.

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