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Mathematical Modeling of Household Economy in Russia

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Abstract—An optimal control problem modeling the economic behavior of a representative household is studied. The existence of its solution is proved, necessary optimality conditions in the form of the Pontryagin–Clarke maximum principle are obtained, and an optimal control synthesis is constructed. The model is identified against Russian statistical data. The model is used to analyze consumer credit in Russia and its influence on the household economy under COVID-19 pandemic conditions.

Keywords: Ramsey model, consumer credit, optimal control synthesis, maximum principle, mathematical modeling, regression analysis

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1. INTRODUCTION

Consumer credits have become an important element of economic relations in Russia in the 21st century. Figure 1 shows the dynamics of consumer debt as a percentage of GDP.

Under the conditions when most of the population has low actual income, consumer credit stimulates economic activities of the population, maintains the solvent demand of households, and has a positive impact on the GDP growth rate. In consolidated balance sheets of commercial banks, the fraction of consumer credit, which is one of the most profitable assets, reached 18% in 2019. Almost half of the credit debt of individuals consists of unsecured consumer credit. Over 2017–2019, the number of debtors grew from 34 to 40 million people and the ratio of the interest rates of the banking system to the cumulative income of the population increased from 3 to 6%. The growth of the total debt burden is nonuniformly distributed among various segments of the population. Available studies show that, on average, various debt burden indicators have the largest values in the first two, fifth, and sixth deciles of the population income distribution (see [1, 2]). Credit overdue occurs mainly among borrowers in the first two deciles, while a high ratio of the credit payments to the borrowers’ income is observed in the fifth and sixth deciles. Additionally, the debt structure is varied in different regions of the country. A spatial analysis of Russian regions in terms of their credit-saving behavior was performed in [3]. It was noted that the tendency to take credit for consumption financing is typical for high-level poverty regions or regions characterized by high standards of consumption during the 2000s. Among Russian regions of the second type, Moscow and St. Petersburg stand out as regions in which a significant portion of the population is able to save and maintain consumption standards without falling into over-indebtedness.

In the fall of 2019, the Russian Federation Government (see [4]) discussed the problem of consumer credit as an asset of commercial banks, as well as consumer debt restructuring measures as a condition for borrower default. The decrease in the actual income of the population due to the COVID-19 pandemic has aggravated this problem. Below, various aspects of this problem and the influence of credit policies on them are analyzed using mathematical models.

Fig. 1. Population's debts.

2. MODELING OF HOUSEHOLD BEHAVIOR WITH THE HELP OF A RAMSEY-TYPE OPTIMAL CONTROL PROBLEM

The modeling of economic household behavior is based on Ramsey's work (see [5]). Ramsey-type models in the form of optimal control problems were investigated, for example, in [6, 7].

Assume that economic decisions made by a typical household are boundedly rational, i.e., the household makes decisions concerning consumption expenditures, savings, and credit in order to maximize discounted consumption taking into account the budgetary constraints. Additionally, we suppose that the household cannot predict variations in the economic environment and makes decisions assuming that the inflation rate and the deposit and credit interest rates remain unchanged. Assume that the household income minus the deposit interest rate (S) is described by the equation

$$\frac{dS}{dt} = \gamma S,$$

$$S(t_0) = S_0,$$

where t is time, γ is the income growth rate (not necessarily positive), and the consumer price index $p(t)$ grows at the rate j , i.e., $p(t) = pe^{jt}$. Let $D(t)$ denote savings in the form of deposits with interest rate r_D , $L(t)$ is the consumer debt with interest rate r_L , $C(t)$ is the household consumption, $p(t)$ is the consumer price index, and $M(t)$ consists of cash and demand deposit accounts. The supply of money $M_0(t)$ necessary for household consumption expenditures $p(t)C(t)$ is modeled by Fisher's equation of exchange $M_0(t) = \theta p(t)C(t)$, where $1/\theta > 0$ is the velocity of money. The dynamics of the money supply is described by the balance equation

$$\frac{dM_0(t)}{dt} = S(t) - p(t)C(t) + H_L(t) - H_D(t).$$

Here, $H_L(t)$ is consumer credit and $H_D(t)$ is investment in a deposit account. Negative values of $H_L(t)$ mean debt repayments, while negative values of $H_D(t)$ mean withdrawal of money from the deposit account. The consumer debt is governed by the equation

$$\frac{dL(t)}{dt} = H_L(t) + r_L L(t).$$

Savings in the form of deposits are described by the equation

$$\frac{dD(t)}{dt} = H_D(t) + r_D D(t).$$

The absence of arbitrage in the market of savings and consumer credit assumes that $r_L > r_D > 0$, $L(t) \geq 0$, and $D(t) \geq 0$. The financial state of the household is specified by the quantity $X(t) = M_0(t) + D(t) - L(t)$. Rational use of financial opportunities means that

$$D(t) = \max\{0, X(t) - M_0(t)\} = (X(t) - M_0(t))_+, \quad L(t) = (M_0(t) - X(t))_+.$$

It follows that

$$\frac{dX}{dt} = Se^{\gamma(t-t_0)} - \frac{1}{\theta} M_0 + r_D(X - M_0)_+ - r_L(M_0 - X)_+.$$

Thus, the dynamics of the household's financial state and income is described by the controlled dynamical system

$$\frac{dX}{dt} = Se^{\gamma(t-t_0)} - \frac{1}{\theta} M_0 + r_D(X - M_0)_+ - r_L(M_0 - X)_+, \quad X(t_0) = x_0.$$

Here, the money supply $M_0 \geq 0$ is a control determining the consumption expenditures according to Fisher's equation of exchange. Assume that the household seeks to maximize the discounted utility function with a constant risk aversion:

$$\int_{t_0}^T (C(t))^\alpha e^{-\delta_0(t-t_0)} dt \rightarrow \max.$$

Here, $\delta_0 > 0$ is the discount factor, $0 < \alpha < 1$ is a parameter determining the risk aversion coefficient, and T is the time horizon. Since the money supply and consumption expenditures can be increased using consumer credit, we need to impose constraints on the credit debt, determining liquidity conditions for the financial state x_0 . Assume that $r_L > r_D > \gamma$ and $\delta_0 > \max(\alpha\gamma, \alpha r_D)$. Suppose that the financial state x_0 is liquid by the time T if there exists a control ensuring that $x(T) \geq 0$, i.e., the liquidity conditions are given by

$$S + (r_L - \gamma)x_0 > 0, \quad T - t_0 > \left\{ \frac{1}{r_L} \ln \left[\frac{S}{S + (r_L - \gamma)x_0} \right] \right\}_+.$$

Thus, the economic behavior of the household is modeled by the optimal control problem

$$\begin{aligned} \int_{t_0}^T \left(\frac{M_0(t)}{\theta p(t)} \right)^\alpha e^{-\delta_0(t-t_0)} dt &\rightarrow \max_{M(t) \geq 0}, \\ \frac{dX}{dt} &= Se^{\gamma(t-t_0)} - \frac{1}{\theta} M_0 + r_D(X - M_0)_+ - r_L(M_0 - X)_+, \\ X(t_0) &= x_0, \quad X(T) \geq 0, \quad M_0(t) \geq 0. \end{aligned}$$

Define $x(t) = X(t)e^{-\gamma(t-t_0)}$, $M(t) = M_0(t)e^{-\gamma(t-t_0)}$, and $\delta = \delta_0 - \alpha\gamma$. Without loss of generality, we set $t_0 = 0$.

Theorem 1. Suppose that

$$S + (r_L - \gamma)x_0 > 0, \quad T > \left\{ \frac{1}{r_L} \ln \left[\frac{S}{S + (r_L - \gamma)x_0} \right] \right\}_+.$$

Then the optimal control problem

$$\int_0^T M^\alpha e^{-(\delta - \alpha\gamma)t} dt \rightarrow \max, \quad (1)$$

$$\frac{dx}{dt} = S - \gamma x - \frac{1}{\theta} M + r_D(x - M)_+ - r_L(M - x)_+, \quad (2)$$

$$x(t_0) = x_0, \quad x(T) \geq 0, \quad (3)$$

$$M(t) \geq 0, \quad (4)$$

has a solution.

Proof. Under the conditions

$$S + (r_L - \gamma)x_0 > 0, \quad T > \left[\frac{1}{r_L} \ln \left(\frac{S}{S + (r_L - \gamma)x_0} \right) \right]_+,$$

by virtue of the solution to the Cauchy problem

$$\frac{dx}{dt} = S - \gamma x + r_D(x)_+, \quad x(0) = x_0,$$

the control $M(t) \equiv 0$ generates a trajectory $x(t)$ satisfying all constraints of the optimal control problem. If $x_0 \geq 0$, then constraints (2) and (4) of the optimal control problems imply that

$$\frac{dx}{dt} \leq S + (r_D - \gamma), \quad x(0) = x_0.$$

According to Gronwall's inequality, we have the estimate

$$x(t) \leq -\frac{S}{r_D - \gamma} + \left(x_0 + \frac{S}{r_D - \gamma} \right) e^{(r_D - \gamma)t} \leq \left((x_0)_+ + \frac{S}{r_D - \gamma} \right) e^{(r_D - \gamma)T}, \quad 0 \leq t \leq T.$$

The case $x_0 < 0$ is considered in a similar manner and leads to the same estimate:

$$x(t) \leq \left((x_0)_+ + \frac{S}{r_D - \gamma} \right) e^{(r_D - \gamma)T}, \quad 0 \leq t \leq T,$$

whence

$$x(T) - x_0 \leq ST + [(r_D - \gamma)T(x_0)_+ + ST] e^{(r_D - \gamma)T} - \frac{1}{\theta} \int_0^T M(t) dt.$$

Since $x(T) \geq 0$, we have

$$\frac{1}{\theta} \int_0^T M(t) dt \leq (x_0)_+ + ST + [(r_D - \gamma)T(x_0)_+ + ST] e^{(r_D - \gamma)T}.$$

Thus, the admissible controls satisfying the constraints of the control problem are uniformly bounded in the L_1 norm; moreover, since $0 \leq M^\alpha \leq \max\{1, M\}$ for $0 < \alpha < 1$, the functional of the optimal control problem is bounded on the set of trajectories satisfying its constraints, and its supremum is denoted by A . Therefore, there exists a maximizing sequence $\{M_i(t) | i = 1, 2, \dots\}$ of admissible controls:

$$\lim_{i \rightarrow +\infty} \int_0^T (M_i(t))^\alpha e^{-(\delta - \alpha\gamma)t} dt = A.$$

By the Komlós theorem (see [8]), the sequence of functions $\{M_i(t) | i = 1, 2, \dots\}$ bounded in the L_1 norm has a subsequence $\{M_{i_n}(t) | n = 1, 2, \dots\}$ such that its Cesàro means converge almost everywhere on the interval $[0, T]$ to a function $M_{\text{opt}}(t)$, i.e.,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n M_{i_k}(t) = M_{\text{opt}}(t) \quad \text{for almost all } t \in [0, T].$$

Since M^α for $0 < \alpha < 1$ is a concave function of M on $[0, +\infty)$, we have

$$\left(\frac{1}{n} \sum_{k=1}^n M_{i_k}(t) \right)^\alpha \geq \frac{1}{n} \sum_{k=1}^n (M_{i_k}(t))^\alpha;$$

therefore,

$$\int_0^T \left(\frac{1}{n} \sum_{k=1}^n M_{i_k}(t) \right)^\alpha e^{-(\delta-\alpha\gamma)t} dt \geq \frac{1}{n} \sum_{k=1}^n \int_0^T (M_{i_k}(t))^\alpha e^{-(\delta-\alpha\gamma)t} dt,$$

whence

$$\lim_{n \rightarrow +\infty} \int_0^T \left(\frac{1}{n} \sum_{k=1}^n M_{i_k}(t) \right)^\alpha e^{-(\delta-\alpha\gamma)t} dt = A.$$

By the Lebesgue dominated convergence theorem,

$$\int_0^T (M_0(t))^\alpha e^{-(\delta-\alpha\gamma)t} dt = A.$$

Since $M_{i_k}(t) \geq 0$, we may assume without the loss of generality that $M_{\text{opt}}(t) \geq 0$.

The phase trajectory $\{x_{\text{opt}}(t) | t \in [0, T]\}$ corresponding to the control $\{M_{\text{opt}}(t) | t \in [0, T]\} \geq 0$ is defined as the solution of the Cauchy problem

$$\frac{dx}{dt} = S - \gamma x - \frac{1}{\theta} M_0(t) + r_D(x - M_0(t))_+ - r_L(M_0(t) - x)_+, \quad x(0) = x_0.$$

Let $x_k(t)$, $t \in [0, T]$, denote the solution of the Cauchy problem

$$\frac{dx}{dt} = S - \gamma x - \frac{1}{\theta} M_{i_k}(t) + r_D(x - M_{i_k}(t))_+ - r_L(M_{i_k}(t) - x)_+, \quad x(0) = x_0.$$

By construction, $x_k(T) \geq 0$. Set $\tilde{x}_n(t) = \frac{1}{n} \sum_{k=1}^n x_k(t)$. Obviously, $\tilde{x}_n(T) \geq 0$ and

$$\frac{d\tilde{x}_n(t)}{dt} = S - \gamma \tilde{x}_n(t) - \frac{1}{\theta n} \sum_{k=1}^n M_{i_k}(t) + \frac{1}{n} \sum_{k=1}^n \left[r_D(x_k(t) - M_{i_k}(t))_+ - r_L(M_{i_k}(t) - x_k(t))_+ \right], \quad \tilde{x}_n(0) = x_0.$$

Since $r_L > r_D$, we conclude that $r_D(y)_+ - r_L(-y)_+$ is a concave function of y ; hence,

$$\frac{d\tilde{x}_n(t)}{dt} \leq S - \gamma \tilde{x}_n(t) - \frac{1}{\theta n} \sum_{k=1}^n M_{i_k}(t) + \left[r_D \left(\tilde{x}_n(t) - \frac{1}{n} \sum_{k=1}^n M_{i_k}(t) \right)_+ - r_L \left(\frac{1}{n} \sum_{k=1}^n M_{i_k}(t) - \tilde{x}_n(t) \right)_+ \right].$$

Define $\hat{x}_n(t)$, $t \in [0, T]$, as the solution of the Cauchy problem

$$\frac{d\hat{x}_n(t)}{dt} = S - \gamma \hat{x}_n(t) - \frac{1}{\theta n} \sum_{k=1}^n M_{i_k}(t) + \left[r_D \left(\hat{x}_n(t) - \frac{1}{n} \sum_{k=1}^n M_{i_k}(t) \right)_+ - r_L \left(\frac{1}{n} \sum_{k=1}^n M_{i_k}(t) - \hat{x}_n(t) \right)_+ \right], \quad \hat{x}_n(0) = x_0.$$

Since $r_L > r_D > \gamma$, the function

$$S - \gamma z - \frac{1}{\theta n} \sum_{k=1}^n M_{i_k}(t) + \left[r_D \left(z - \frac{1}{n} \sum_{k=1}^n M_{i_k}(t) \right)_+ - r_L \left(\frac{1}{n} \sum_{k=1}^n M_{i_k}(t) - z \right)_+ \right]$$

increases monotonically in z . It follows that $\tilde{x}_n(t) \leq \hat{x}_n(t)$ and, hence, $\hat{x}_n(T) \geq \tilde{x}_n(T) \geq 0$. Thus, the control

$$\left\{ \frac{1}{n} \sum_{k=1}^n M_{i_k}(t) | t \in [0, T] \right\}$$

and the corresponding phase trajectory $\tilde{x}_n(t)$ satisfy all constraints (2)–(4) of the optimal control problem.

By the Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow +\infty} \int_0^T \left| M_{\text{opt}}(t) - \frac{1}{n} \sum_{k=1}^n M_{i_k}(t) \right| dt = 0,$$

which implies that, for any $\varepsilon > 0$, there exists $n_\varepsilon > 0$ such that, for $n \geq n_\varepsilon$, we have

$$\int_0^T \left| M_{\text{opt}}(t) - \frac{1}{n} \sum_{k=1}^n M_{i_k}(t) \right| dt < \varepsilon.$$

Note that the function $f(z, u) = S - \gamma z - \frac{1}{\theta} u + r_D(z - u)_+ - r_L(u - z)_+$ satisfies the Lipschitz condition, i.e., there exists a constant $C > 0$ such that, for any z, u, w , and v , it is true that

$$|f(z, u) - f(w, v)| \leq C(|z - w| + |u - v|).$$

Since

$$\begin{aligned} \hat{x}_n(t) &= x_0 + \int_0^t f\left(\hat{x}_n(\tau), \frac{1}{n} \sum_{k=1}^n M_{i_k}(\tau)\right) d\tau, \\ x_{\text{opt}} &= x_0 + \int_0^t f(x_{\text{opt}}(\tau), M_{\text{opt}}(\tau)) d\tau, \end{aligned}$$

the following estimate holds for $t \in [0, T]$:

$$\begin{aligned} |\hat{x}_n(t) - x_{\text{opt}}(t)| &\leq \int_0^t \left| f\left(\hat{x}_n(\tau), \frac{1}{n} \sum_{k=1}^n M_{i_k}(\tau)\right) - f(x_{\text{opt}}(\tau), M_{\text{opt}}(\tau)) \right| d\tau \\ &\leq C \left(\int_0^t |\hat{x}_n(\tau) - x_{\text{opt}}(\tau)| d\tau + \int_0^t \left| \frac{1}{n} \sum_{k=1}^n M_{i_k}(\tau) - M_{\text{opt}}(\tau) \right| d\tau \right). \end{aligned}$$

It follows that, for $t \in [0, T]$ and $n \geq n_\varepsilon$,

$$|\hat{x}_n(t) - x_{\text{opt}}(t)| \leq \varepsilon e^{Ct}$$

(see [9, p. 17, Theorem 2.1]), whence $\lim_{n \rightarrow +\infty} \hat{x}_n(T) = x_{\text{opt}}(T)$ and, hence, $x_{\text{opt}}(T) \geq 0$. Thus, the control $\{M_{\text{opt}}(t) | t \in [0, T]\} \geq 0$ and the phase trajectory $\{x_{\text{opt}}(t) | t \in [0, T]\} \geq 0$ are a solution of the optimal control problem (1)–(4). Theorem 1 is proved.

Theorem 2. *If $\{x(t), M(t) | t \in [0, T]\}$ is a solution of the optimal control problem (1)–(4), then $\{\varphi(t) | t \in [0, T]\}$ is an absolutely continuous function such that*

(i) *if $\frac{\alpha\theta}{1 + \theta r_L} > x^{1-\alpha}\varphi$, then $M = \left[\frac{\alpha\theta}{(1 + \theta r_L)\varphi} \right]^{\frac{1}{1-\alpha}}$ and*

$$\frac{dx}{dt} = S + (r_L - \gamma)x - \frac{1 + \theta r_L}{\theta} \left[\frac{\alpha\theta}{(1 + \theta r_L)\varphi} \right]^{\frac{1}{1-\alpha}}, \quad (5)$$

$$\frac{d\varphi}{dt} = [\delta + (1 - \alpha)\gamma - r_L]\varphi; \quad (6)$$

(ii) *if $\frac{\alpha\theta}{1 + \theta r_L} \leq x^{1-\alpha}\varphi \leq \left(\frac{\alpha\theta}{1 + \theta r_D} \right)$, then $M(t) = x(t)$ and*

$$\frac{dx}{dt} = S - \frac{1 + \theta\gamma}{\theta} x, \quad (7)$$

$$\frac{d\varphi}{dt} = [\delta + (1 - \alpha)\gamma - u(t)]\varphi, \quad r_D \leq u(t) \leq r_L; \quad (8)$$

(iii) if $\frac{\alpha\theta}{1+\theta r_D} < x^{1-\alpha}\varphi$, then $M = \left[\frac{\alpha\theta}{(1+\theta r_D)\varphi} \right]^{\frac{1}{1-\alpha}}$ and

$$\frac{dx}{dt} = S + (r_D - \gamma)x - \frac{1+\theta r_D}{\theta} \left[\frac{\alpha\theta}{(1+\theta r_L)\varphi} \right]^{\frac{1}{1-\alpha}}, \quad (9)$$

$$\frac{d\varphi}{dt} = [\delta + (1-\alpha)\gamma - r_D]\varphi. \quad (10)$$

Additionally, the transversality condition is satisfied:

$$x(T) = 0. \quad (11)$$

Proof. The optimal solution of the optimal control problem has to satisfy the necessary conditions of the Pontryagin maximum principle in the Clarke form (see [10, p. 115]), since the function on the right-hand side of (2) is nonsmooth.

The Hamiltonian function has the form

$$H(t, x, p) = \sup_{M \geq 0} \left\{ M^\alpha e^{-(\delta-\alpha\gamma)M} + p \left[S - \gamma x - \frac{1}{\theta} M + r_D(x - M)_+ - r_L(M - x)_+ \right] \right\},$$

the optimal control is determined by the relation

$$M^{\alpha-1} \in \frac{e^{\delta-\alpha\gamma}}{\alpha} \partial_M \left\{ p \left[S - \gamma x - \frac{1}{\theta} M + r_D(x - M)_+ - r_L(M - x)_+ \right] \right\},$$

and the adjoint variable satisfies the relations

$$\frac{dp}{dt} \in -\partial_x H(t, x, p), \quad p(T)x(T) = 0.$$

For the optimal control $M(t)$ to be finite, it is necessary that $p(t) \neq 0$ for $t \in [0, T]$. Therefore, the transversality condition implies that $x(T) = 0$. Introducing the new variable $\varphi = e^{\delta-\alpha\gamma}p$ and computing the generalized gradients ∂_M and ∂_x , we complete the proof of Theorem 2.

The phase portrait on the plane with coordinates (x, φ) consists of three domains corresponding to three different modes of the household behavior. In the first domain, $\frac{\alpha\theta}{1+\theta r_L} > x^{1-\alpha}\varphi$, which corresponds to a credit mode, and we model a household using consumer credit. In the second domain, $\frac{\alpha\theta}{1+\theta r_L} \leq x^{1-\alpha}\varphi \leq \frac{\alpha\theta}{1+\theta r_D}$, which corresponds to an autonomous mode, and we model a household that does not use consumer credit or deposit savings. In the third domain, $\frac{\alpha\theta}{1+\theta r_D} < x^{1-\alpha}\varphi$, which corresponds to a saving mode, we model a household saving in the form of deposits. The trajectory determined by the optimal solution of problem (1)–(4) must satisfy the boundary conditions $x(0) = x_0$ and $x(T) = 0$, and the dependence on the parameters r_L , r_D , γ , θ , α , and δ can involve a combination of different modes. Given the trajectory, we can construct an optimal control synthesis. In the limit as $T \rightarrow +\infty$, the optimal control synthesis is stationary, i.e., time-invariant. The control synthesis depending on the relation between the parameters can be described as follows.

Case 1 (credit mode). If $r_D < r_L < \delta - \frac{1-\alpha}{\theta}$, then the optimal control synthesis is given by the function

$$M(x; r_L, r_D, \gamma, \theta, \alpha, \delta) = \left[\frac{\theta(\delta - \alpha r_L)}{(1-\alpha)(1+\theta r_L)} \right] \left(x + \frac{S}{r_L - \gamma} \right).$$

Case 2 (combination of the credit and autonomous modes). If $r_D < \delta - \frac{1-\alpha}{\theta} < r_L < \delta + (1-\alpha)\gamma$, then the optimal control synthesis is given by the function

$$M(x; r_L, r_D, \gamma, \theta, \alpha, \delta) = \begin{cases} \left[\frac{\theta(\delta - \alpha r_L)}{(1-\alpha)(1+\theta r_L)} \right] \left(x + \frac{S}{r_L - \gamma} \right) & \text{if } x < \frac{(\delta - \alpha r_L)}{\left(r_L - \delta + \frac{1-\alpha}{\theta} \right) (r_L - \gamma)}, \\ x & \text{if } x \geq \frac{(\delta - \alpha r_L)}{\left(r_L - \delta + \frac{1-\alpha}{\theta} \right) (r_L - \gamma)}. \end{cases}$$

Case 3 (combination of the credit, autonomous, and saving modes). Suppose that $\delta - \frac{1-\alpha}{\theta} < r_D < r_L < \delta + (1-\alpha)\gamma$. The equation

$$x_R - \frac{S\theta}{1+\gamma\theta} = \frac{S\theta(1+r_L\theta)[\delta + (1-\alpha)\gamma - r_L]}{(r_L - \gamma)(1+\theta\gamma)(r_L\theta - \delta\theta + 1 - \alpha)} \left\{ \frac{(1+r_D\theta)}{(1+r_L\theta)} \left[\frac{x_R(r_L\theta - \delta\theta + 1 - \alpha)(r_L - \gamma)}{S\theta(\delta - \alpha r_L)} \right]^{1-\alpha} \right\}^{\frac{1+\gamma\theta}{\theta(\delta + (1-\alpha)\gamma - r_D)}}$$

has a solution x_R . The minimum solution on the set $\left[\frac{(\delta - \alpha r_L)}{\left(r_L - \delta + \frac{1-\alpha}{\theta} \right) (r_L - \gamma)}, +\infty \right)$ is denoted by $x_R(r_L, r_D, \gamma, \theta, \alpha, \delta)$.

If $x \in [x_R(r_L, r_D, \gamma, \theta, \alpha, \delta), +\infty)$, then the equation

$$x + \frac{S}{r_D - \gamma} = \left[\frac{(1-\alpha)(1+\theta r_D)}{\theta(\delta - \alpha r_D)} \right] M_1 + \left[\frac{\theta\delta - (1-\alpha) - \theta r_D}{\theta(\delta - \alpha r_D)} x_R(r_L, r_D, \gamma, \theta, \alpha, \delta) + \frac{S}{r_D - \gamma} \right] \left(\frac{x_R(r_L, r_D, \gamma, \theta, \alpha, \delta)}{M_1} \right)^{\frac{(1-\alpha)(r_D - \gamma)}{\delta + (1-\alpha)\gamma - r_D}}$$

has a unique solution M_1 on the set $[x_R(r_L, r_D, \gamma, \theta, \alpha, \delta), +\infty)$ that defines a differentiable implicit function $M_1(x; r_L, r_D, \gamma, \theta, \alpha, \delta)$. Then the optimal control synthesis is given by the function

$$M(x; r_L, r_D, \gamma, \theta, \alpha, \delta) = \begin{cases} \left[\frac{\theta(\delta - \alpha r_L)}{(1-\alpha)(1+\theta r_L)} \right] \left(x + \frac{S}{r_L - \gamma} \right) & \text{if } x < \frac{(\delta - \alpha r_L)}{\left(r_L - \delta + \frac{1-\alpha}{\theta} \right) (r_L - \gamma)}, \\ x & \text{if } \frac{(\delta - \alpha r_L)}{\left(r_L - \delta + \frac{1-\alpha}{\theta} \right) (r_L - \gamma)} \leq x \leq x_R(r_L, r_D, \gamma, \theta, \alpha, \delta), \\ M_1(x; r_L, r_D, \gamma, \theta, \alpha, \delta) & \text{if } x > x_R(r_L, r_D, \gamma, \theta, \alpha, \delta). \end{cases}$$

Case 4 (combination of the credit and saving modes). Suppose that $r_D < \delta + (1-\alpha)\gamma < r_L$. If $\frac{S}{\gamma - r_L} < x < \frac{S\theta}{1+\gamma\theta}$, then the equation

$$x + \frac{S}{r_L - \gamma} = \frac{S(1+\theta r_L)[\delta + (1-\alpha)\gamma - r_L]}{(\delta - \alpha r_L)(1+\gamma\theta)(r_L - \gamma)} \left[\frac{M_2(1+\gamma\theta)}{S\theta} \right]^{\frac{(1-\alpha)(\gamma - r_L)}{(\delta + (1-\alpha)\gamma - r_L)}} + \left[\frac{(1-\alpha)(1+\theta r_L)}{\theta(\delta - \alpha r_L)} \right] M_2 \quad (12)$$

has two positive solutions for M_2 . We define $M_2(x; r_L, r_D, \gamma, \theta, \alpha, \delta)$ as the minimum solution of Eq. (12). By the implicit function theorem, $M_2(x; r_L, r_D, \gamma, \theta, \alpha, \delta)$ is a differentiable function.

If $x > \frac{S\theta}{1 + \gamma\theta}$, then the equation

$$x + \frac{S}{r_D - \gamma} = \frac{S(1 + \theta r_D)[\delta + (1 - \alpha)\gamma - r_D]}{(\delta - \alpha r_D)(1 + \gamma\theta)(r_D - \gamma)} \left[\frac{M_3(1 + \gamma\theta)}{S\theta} \right]^{\frac{(1 - \alpha)(\gamma - r_D)}{(\delta + (1 - \alpha)\gamma - r_D)}} + \left[\frac{(1 - \alpha)(1 + \theta r_D)}{\theta(\delta - \alpha r_D)} \right] M_3 \quad (13)$$

has two positive solutions for M_3 . Let $M_3(x; r_L, r_D, \gamma, \theta, \alpha, \delta)$ be defined as the maximum solution of Eq. (13). By the implicit function theorem, $M_3(x; r_L, r_D, \gamma, \theta, \alpha, \delta)$ is a differentiable function for $x \in \left(\frac{S\theta}{1 + \gamma\theta}, +\infty \right)$.

Then the optimal control synthesis is given by the function

$$M(x; r_L, r_D, \gamma, \theta, \alpha, \delta) = \begin{cases} M_2(x; r_L, r_D, \gamma, \theta, \alpha, \delta) & \text{if } x < \frac{S\theta}{1 + \gamma\theta}, \\ M_3(x; r_L, r_D, \gamma, \theta, \alpha, \delta) & \text{if } x \geq \frac{S\theta}{1 + \gamma\theta}. \end{cases}$$

Case 5 (combination of crediting and saving regimes). If $\delta + (1 - \alpha)\gamma < r_D < r_L$, then the optimal control synthesis is given by the function

$$M(x; r_L, r_D, \gamma, \theta, \alpha, \delta) = \begin{cases} M_2(x; r_L, r_D, \gamma, \theta, \alpha, \delta) & \text{if } x < \frac{S\theta}{1 + \gamma\theta}, \\ \frac{\theta(\delta - \alpha r_D)}{(1 - \alpha)(1 + \theta r_D)} \left(x + \frac{S}{r_D - \gamma} \right) & \text{if } x \geq \frac{S\theta}{1 + \gamma\theta}. \end{cases}$$

Assume that a rational representative household makes a decision taking into account the economic environment and the possibility of its variation in terms of its behavioral parameters (the income discount factor δ , the risk aversion coefficient $(1 - \alpha)$, and the velocity of money $1/\theta$). Given the dynamics of the interest rates $r_L(t)$ and $r_D(t)$ and the household income growth rate $\gamma(t)$, the household's economic behavior can be modeled using the constructed optimal control synthesis $M(x; r_L, r_D, \gamma, \theta, \alpha, \delta)$. For this purpose, we need to solve the Cauchy problem for the differential equation

$$\begin{aligned} \frac{dx(t)}{dt} = & S - \gamma(t)x(t) - \frac{1}{\theta} M(x(t); r_L(t), r_D(t), \gamma(t), \theta, \alpha, \delta) \\ & + r_D(t)(x(t) - M(x(t); r_L(t), r_D(t), \gamma(t), \theta, \alpha, \delta))_+ \\ & - r_L(t)(M(x(t); r_L(t), r_D(t), \gamma(t), \theta, \alpha, \delta) - x(t))_+, \quad x(0) = x_0, \end{aligned}$$

whose solution determines the dynamics of the household financial position $x(t)$. Note that the right-hand side of the equation is a Lipschitz function of x . Given the household financial position $x(t)$, we can determine the household consumption expenditures $\frac{1}{\theta} M(x(t); r_L(t), r_D(t), \gamma(t), \theta, \alpha, \delta)$, the consumer debt $L(t) = (M(x(t); r_L(t), r_D(t), \gamma(t), \theta, \alpha, \delta) - x(t))_+$, and savings in the form of deposits in commercial banks $D(t) = (x(t) - M(x(t); r_L(t), r_D(t), \gamma(t), \theta, \alpha, \delta))_+$.

When the model is identified on long time intervals (about ten years), the behavioral household characteristics θ , δ , and α may vary due to variations in the economic environment.

3. IDENTIFICATION AND VERIFICATION OF THE MODEL AGAINST RUSSIAN STATISTICAL DATA

The above-described model of household economic behavior is nonclosed. It can be treated as a mathematical model for medium-range analysis and forecasting of the dynamics of basic Russian economy indicators. The input variables of the model are the deposit interest rate r_D , the consumer credit interest rate r_L , the household income growth rate γ , and the consumer price inflation rate j . The output parameters are the money supply M_0 (the amount of cash in households and demand deposit accounts), household consumption expenditures, deposits, and consumer debt. The latent parameters of the model, which are determined in the model identification, are the discount factor δ , the utility function parameter α (risk aversion coefficient), and velocity of money $1/\theta$.

Table 1. Group of rich regions

Characteristics of social group	$L > 0$ (59.6%)	$L > 0$ (40.4%)	$L = 0, D = 0$	$L = 0, D > 0$	Total
Number of households	70 (17.24%)	30 (7.39%)	159 (39.16%)	147 (36.21%)	406 (100%)
Number of people	241 (24.82%)	76 (7.83%)	364 (37.49%)	290 (29.86%)	971 (100%)
Average number of people per household	3.45	2.55	2.29	1.97	2.39
Average income per individual in group, rubles	19118	41787	20224	27185	23708
Average consumption per individual in group, rubles	22181	56994	15643	22724	22636
Group's aggregate income based on RLMS sample, rubles	4.61×10^6 (20.03%)	3.18×10^6 (13.81%)	7.35×10^6 (31.93%)	7.88×10^6 (34.23%)	23.02×10^6 (100%)
Group's aggregate consumption based on RLMS sample, rubles	5.35×10^6 (24.34%)	4.35×10^6 (19.79%)	5.69×10^6 (25.89%)	6.59×10^6 (29.98%)	21.98×10^6 (100%)
Population segment	1	2	3	4	

Table 2. Group of poor regions

Characteristics of social group	$L > 0$ (52.3%)	$L > 0$ (47.7%)	$L = 0, D = 0$	$L = 0, D > 0$	Total
Number of households	553 (25.3%)	237 (10.84%)	952 (43.55%)	444 (20.31%)	2186 (100%)
Number of people	1816 (32.83%)	601 (10.87%)	2194 (39.67%)	920 (16.63%)	5531 (100%)
Average number of people per household	3.28	2.54	2.3	2.07	2.53
Average income per individual in group, rubles	12497	31583	14221	19599	16440
Average consumption per individual in group, rubles	15099	33391	12138	17139	16254
Group's aggregate income based on RLMS sample, rubles	22.7×10^6 (20.03%)	18.99×10^6 (20.88%)	31.2×10^6 (34.31%)	18.04×10^6 (19.85%)	90.93×10^6 (100%)
Group's aggregate consumption based on RLMS sample, rubles	27.42×10^6 (30.05%)	20.08×10^6 (22.34%)	26.63×10^6 (29.62%)	15.77×10^6 (17.54%)	89.9×10^6 (100%)
Population segment	1	2	3	4	

The model of household economic behavior was calibrated using statistical data from the Russia Longitudinal Monitoring Survey (RLMS) conducted by the National Research University “Higher School of Economics” and Rosstat data. Based on the analysis of RLMS data, the regions of Russia were divided into two groups, rich and poor, depending on the features of the debt burden distribution over deciles, poverty indicators, and purchasing power per capita income. Moscow, Moscow region, St. Petersburg, Kazan, and New Moscow were ranked as rich. The other regions of the Russian Federation were assigned to the group of poor regions. Population segments with high and low levels of income were identified in each group. Population segments with low income were divided into three groups: borrowers of unsecured credit (segment 1), nonusers of bank services (segment 3), and people saving in the form of deposits (segment 4). Note that the levels of income and expenditure in low-income segments do not differ widely. The population segment with high income (segment 2) is characterized by high expenditure. Households within this segment are classified as borrowers of secured credit.

Tables 1 and 2 present characteristics describing population segments in two groups of regions. RLMS data over the period 2015–2018 were used (see [11]). In RLMS, households were surveyed once a year. To obtain a high-quality classification, we chose households that participated in the survey for four years (2015–2018). The questions in RLMS data included how much they pay back on credit last month and whether they were able to save last month. Since the survey is annual, population segments were classified according to the following principle: households that paid on credit at least once over the considered

Fig. 2. Actual statistics of consumption, debt, deposit, and money supply recovered by the model in the first (left) and second (right) groups of regions: actual data (dotted), modeled values (solid), and predictions (dashed). D are deposits, M is the quantity of money in circulation, L is the consumer debt, and C is the consumption of the population. All quantities are given in trillion rubles.

period were regarded as credit borrowers. Among the rest of the households, those that saved at least once were classified as households saving in the form of deposits. The other households were assigned to the population segment not interacting with banks.

Note the typical household composition in various population segments: the largest family size is observed in unsecured credit borrowers. On the contrary, the smallest family size is observed in the saving population segment. It can be assumed that unsecured credit borrowers are young families with children, while the saving population segment consists of older people. It is possible that households assigned to different population segments have family relations. This circumstance implies the possibility of transfers between the saving and credited segments when the expenditures of the latter fall below the living wage.

The model was initially calibrated using time series of income, consumption, savings, and money supply in the various groups of regions from April 2009 to January 2019. The latent parameters of the model were identified by solving inverse problems. It was assumed that a behavioral characteristic, such as the discount factor δ , could vary with the economic environment (inflation rate of consumer prices, interest rates, and the income growth rate). The constructed econometric regressions are given in Appendix A. Note that basic regressor of the discount factor is the credit interest rate for borrowers and the deposit interest rate for the saving population segment. Each population segment is characterized by its own behavioral parameters, so it was modeled separately. For the group of rich regions, we used the following parameters: $\theta_1 = 1$, $\theta_2 = 2$, $\theta_4 = 4$, $\alpha_1 = 0.88$, $\alpha_2 = 0.75$, and $\alpha_4 = 0.765$ (here and below, the subscript characterizes a population segment). For the group of poor regions, the parameters were specified as $\theta_1 = 0.7$, $\theta_2 = 2$, $\theta_4 = 3$, $\alpha_1 = 0.9$, $\alpha_2 = 0.8$, and $\alpha_4 = 0.775$. Note that the risk aversion coefficient for unsecured credit borrowers is larger than that for secured credit borrowers. Moreover, it can be seen that the risk aversion in borrowers of the poor group is higher than in borrowers of the rich group. The same is true of the inverse of the velocity of money. Segment 3 (no consumer credit or savings) is significant in number. The balance of income and expenditure in segment 3 and the model of the influence exerted by the economic environment on its behavioral characteristic θ close the description of the household economic behavior. As a result, the general dynamics of household expenditure and money supply can be qualitatively reproduced.

The model was verified by constructing forecast on the time interval from February 2019 to February 2020, so that the suitability of the model was qualitatively assessed. In Fig. 2, the statistical data reproduced by the model are compared with the forecast over February 2019 to February 2020. It can be seen that the model is able to qualitatively reproduce data describing the demand for credit, money supply, consumption, and amounts of savings in the households.

Fig. 3. Actual and regression-based values of (a) overdue debt and (b) interest rate.

4. ANALYSIS OF COVID-19 IMPACT ON THE HOUSEHOLD ECONOMY

To make analysis and predictions based on the constructed model of household credit behavior, we need to specify scenarios describing the dynamics of population income and the credit interest rate. The COVID-19 pandemic and the quarantine measures introduced in Russia in the late March 2020 led to a significant decrease in the actual income of the population. For example, according to Rosstat data, the seasonally adjusted nominal income of the population in the second quarter of 2020 declined by 6.6% compared to the first quarter of 2020. The reduction in the nominal income was caused by a decline in wage labor income, which had reduced by 4% compared to the first quarter of 2020. A significant portion of the income decline was caused by the contraction of business income, which had reduced by 42% compared to the first quarter in seasonally adjusted terms. Under these conditions, it seems plausible that the credit overdue would grow due to the inability of some of the population to pay on credits. The growth of credit overdue, in turn, can lead to an increase in interest rates if banks decide to transfer the risk of crediting to the credit cost. To take into account this possibility, we performed regression modeling of the credit interest rate depending on the overdue consumer debt and the Bank of Russia key interest rate. At the same time, the amount of overdue debt is directly affected by the credit cost expressed in terms of the credit interest rate, as well as by the dynamics of unsecured credits, which was taken into account by constructing an additional regression. The results of retrospective modeling are shown in Fig. 3.

Under conditions of uncertainty in the future trajectory of population income, we consider several scenarios. In all of them, the credit interest rates are modeled using the regressions described in Appendix A.

4.1. Scenario 1: No COVID-19 Pandemic

In this scenario, the nominal population income grows over the entire forecast horizon at an average rate of 3.5% compared to the corresponding period of last year and the key interest rate is preserved at a level of 6% starting in February 2020. In the no-pandemic scenario, due to the insignificant growth of the population income, the debt grows in both groups of regions (Fig. 4). In the group of poor regions, the unsecured credit debt grows more pronouncedly (Fig. 4b).

The bold dots in Fig. 4b show the time periods where the state constraint $x \geq -\frac{S}{r_L - \gamma}$ is violated. The simulation results show that, with increasing nominal income in the absence of quarantine measures, the growth rate of unsecured debt in the group of rich regions would slow down. At the same time, a proportional decrease in the growth rate in the secured credit segment would lead to stagnation of secured consumer debt in the population.

For the group of poor regions, the consumer debt dynamics modeled without income decline differs from the results obtained for the rich group. In the case of growing nominal income, the unsecured consumer debt would increase by nearly three times on the forecast time horizon. Under the stagnation of

Fig. 4. Dynamics of unsecured (blue) and secured (red) consumer debts in the groups of (a) rich and (b) poor regions in the absence of quarantine measures and nominal income growth.

Fig. 5. Dynamics of overdue consumer debt in the case of no pandemic.

actual income, such dynamics of unsecured debt in the group of poor regions represents, in fact, a debt bubble.

In 2019 the Minister of Economic Development M.S. Oreshkin talked about risks of unbounded growth of consumer credit. According to his statement (see [4]), the fast growth rate of unsecured credit would lead to the critical growth of overindebted population by 2021, which would give rise to recession. The simulation results confirm the conclusions of the former minister.

In addition to the growth of debt in the population, it would be interesting to analyze the dynamics of overdue debt and the income of banks in no pandemic conditions. The bank income was calculated using the formula

$$B(t) = \frac{\ln(1 + r_L(t))}{12} L(t) \left(1 - \frac{\xi_{\%}(t)}{100} \right) - (\xi(t) - \xi(t-1)).$$

Fig. 6. Dynamics of bank income in the case of no pandemic.

Fig. 7. Scenarios of population income decline due to the COVID-19 pandemic.

Here, $\xi_{\%}(t)$ is the percentage of overdue debt in the entire debt and $\xi(t)$ is the regression value of the overdue debt (see Appendix A). In other words, the bank income consists of payoffs on overdue credits minus the gain in the overdue debt.

A gradual increase in the overdue debt toward the end of the predicted period (Fig. 5) is caused by an increase in the debt burden due to the growth of interest rates, which leads to a significant decrease in the bank income from consumer credit at the end of the predicted period (Fig. 6).

4.2. Scenario 2: Population's Income Decline at a Key Interest Rate of 4.25%

Scenario 1 does not take into account the income decline in the population caused by introducing the quarantine in April–May 2020. Scenarios involving population income decline were based primarily on various dynamics of the unemployment growth rate and on available estimates of production decline in

Fig. 8. Dynamics of the credit interest rate under various income decline scenarios in the case of a varying key interest rate.

Fig. 9. Dynamics of consumer debt in the groups of (a) rich and (b) poor regions under various income decline scenarios in the case of a constant key interest rate: total debt (left) and unsecured and secured credit debts (right).

Fig. 10. Dynamics of overdue consumer debt in the case of a constant key interest rate.

Fig. 11. Dynamics of bank income under various income decline scenarios in the case of a constant key interest rate.

the Russian economy during the quarantine. The spectrum of unemployment growth included 5, 10, and 15% increases in the unemployment rate.

Presented in Fig. 7, the different dynamics of population income exert a significant impact on the value of taken credits and, hence, on the value of overdue debt in the future. In turn, overdue debt has a direct effect on the degree of risk associated with bank credit. As a result, various income dynamics imply variations in the trajectory of credit interest rates, which is reflected in Fig. 8.

With falling incomes, the growth of consumer credit is observed in the group of poor regions (Fig. 9b), where the demand for consumer credit considerably exceeds the crediting capabilities of commercial banks. The group of rich regions is capable of avoiding a significant increase in consumer debt (Fig. 9a).

Income decline, which is manifested in reduced ability of households to pay back on credit, leads to a sharp decrease in the rate of secured crediting. At the same time, the growth rate of unsecured credit accelerates toward the end of the predicted period in all scenarios, which is explained by the desire of households to maintain the existing consumption standards, as well as by the necessity of restructuring previously taken credits in the conditions of income decline. The high rates of income decline under pandemic

Fig. 12. Sharp decrease in the key interest rate to 3%.

Fig. 13. Dynamics of consumer debt in the groups of (a) rich and (b) poor regions under various income decline scenarios in the case of a key interest rate sharply decreasing to 3%: total debt (left) and unsecured and secured credit debts (right).

Fig. 14. Dynamics of overdue consumer debt in the case of a key interest rate sharply decreasing to 3%.

Fig. 15. Dynamics of the bank income under various income decline scenarios in the case of a key interest rate sharply decreasing to 3%.

conditions are manifested in more rapid growth of unsecured debt, which is expressed in the future dynamics of debt in the group of poor regions. Unsecured credit debt in the group of poor regions can double by the end of 2021 and make up 9.8 trillion rubles under the worst-case scenario. Income decline in poor segment 2 leads to a significant decrease in the ability of households to pay back on credit, which is also manifested in accelerating growth of overdue debt (Fig. 10). Note that, despite the attempts of banks to preserve their profitability by increasing interest rates, the poorer quality of borrowers leads to a faster decrease in bank income from consumer credit (Fig. 11).

4.3. Scenario 3: Sharp Decrease in the Key Interest Rate to 3% under Population Income Decline

Consider a scenario with an interest rate decreasing sharply to 3% in the third quarter of 2020 (Fig. 12). In this scenario, the consumer debt in the group of rich regions ceases to grow (Fig. 13a) and, as compared with scenario 2, the growth of consumer debt in the group of poor regions slows down (Fig. 13b). Although the population income declines, the rate of crediting slows down significantly in both rich and poor

Fig. 16. Gradual decrease in the key interest rate to 2%.

Fig. 17. Dynamics of consumer debt in the groups of (a) rich and (b) poor regions under various income decline scenarios in the case of a key interest rate gradually decreasing to 2%: total debt (left) and unsecured and secured credit debts (right).

Fig. 18. Dynamics of overdue consumer debt in the case of a key interest rate gradually decreasing to 2%.

Fig. 19. Dynamics of bank income under various income decline scenarios in the case of a key interest rate gradually decreasing to 2%.

groups. In the group of poor regions, the unsecured credit debt grows to 6.89, rather than to 9.8 trillion rubles, as in the preceding scenario. Overdue debt (Fig. 14) and the profitability of the banking system (Fig. 15) also demonstrate more positive dynamics as compared with scenario 2.

4.4. Scenario 4: Gradual Decrease in the Key Interest Rate to 2% under Population Income Decline

Consider a scenario with a key interest rate decreasing gradually to 2% by the end 2021 of (Fig. 16). Under this scenario, positive tendencies are exhibited not as strongly as in the scenario with a key interest rate decreasing sharply to 3%. As before, the debt of households grows along a lower trajectory as compared with the case of a constant key interest rate, but the reduction in the rate of crediting is less pronounced (Fig. 17).

The level of overdue debt and the income of commercial banks from consumer credit (Figs. 18, 19) also improve as compared to scenario 2 with a constant key interest rate of 4.25%.

5. CONCLUSIONS

An optimal control synthesis was constructed in the model of a rational household. As a result, we described variations in the household economic behavior under varying economic environment. Representative types of households were identified using RLMS data produced at the National Research University “Higher School of Economics” (see [11]). The model was identified using statistical data on income, expenditure, consumer credit, and savings of Russian households from April 2009 to January 2019. Statistical data from February 2019 to February 2020 were used for model verification. The model was used to analyze the important problem of security of consumer credits and associated risks for commercial banks. This problem had been vigorously discussed in the economic block of the Russian Federation Government in the mid-2019 (see [4]). The model computations showed that the concern about solvency of borrowers in some regions of the Russian Federation was well founded. A model-based analysis of the influence exerted by the COVID-19 pandemic revealed that the problem of borrowers' solvency aggravates significantly due to the decline in the population income. In addition to income decline, a major source leading to the growth of debt burden on the population is an increasing interest rate. In contrast to population income, which is primarily formed by production activities in the economy, the interest rate can be quickly adjusted by the Government by varying the Bank of Russia key interest rate, which is the starting point for banks to compute interest rates to be used to credit the economy. The model computations showed that a cutting of the Bank of Russia key interest rate reduces the debt burden on households and the fraction of insolvent borrowers. The effectiveness of cutting the key interest rate depends substantially on the cutting dynamics. Despite the clear advantages of the scenario with a sharply cut key interest rate, it should be noted that this strategy can lead to unexpected negative effects, such as a sharp weakening of the ruble exchange rate or a significant increase in the inflation rate.

APPENDIX A

REGRESSIONS

The regression used to model the dynamics of overdue debt has the form

$$\xi(t) \approx -1.78736 + 0.0936723r_{L,\%}(t-3) + 0.253206L_1(t-3),$$

where the regression parameters are the credit interest rate and the computed dynamics of consumer debt in population segment 1 with a delay of 3 months (variation in the parameters over 1 month is used as a model step).

The dependence of the interest rate on the risks of crediting, which are expressed in terms of overdue debt, and on the Bank of Russia key interest rate is expressed by the regression

$$r_L(t) \approx 0.0128 + 0.007213KIR_{\%}(t) + 0.0118\xi(t) + 0.4259r_L(t-1),$$

where $KIR_{\%}(t)$ is the key interest rate (in percent) fixed by the Central Bank of Russia and $\xi(t)$ is the regression overdue debt presented above.

The following regressors were used to construct regression data for model identification: $r_L(t)$ is the credit interest rate, $r_D(t)$ is the deposit interest rate, $r_{D,curr}(t)$ is the currency deposit interest rate, $j_m(t)$ is the monthly inflation rate, $j_q(t)$ is the quarterly inflation rate, $j_y(t)$ is the annual inflation rate, $\gamma_m(t)$ is the monthly income growth rate, $\gamma_q(t)$ is the quarterly income growth rate, and $\gamma_y(t)$ is the annual income growth rate. In the regressions given below, the subscript characterizes membership in household segment i , $i \in \{1, 2, 3, 4\}$. Note that the function $v(t)$ describes the income dynamics of the saving population segment (segment 4), so the dynamics of population deposits can be reproduced more accurately.

Group of rich regions:

1. $\delta_1(t) \approx 0.0278 + 0.8177r_L(t) + 0.0383j_q(t) - 0.0666j_y(t) + 0.0059\gamma_y(t) + 0.213\delta_1(t-1),$
2. $\delta_2(t) \approx 0.0054 + 0.786r_L(t) + 0.0408j_q(t) - 0.0477j_y(t) + 0.1114\gamma_q(t) - 0.0108\gamma_y(t) + 0.2333\delta_2(t-1),$
3. $\delta_4(t) \approx -0.0296 + 0.6658r_D(t) - 0.0995r_{D,curr}(t) + 0.0313j_q(t) + 0.2921\delta_4(t-1),$
4. $v(t) \approx 0.2013 + 0.5303r_D(t) + 0.7147\gamma_q(t) - 0.3343\gamma_y(t) - 0.1845j_q(t) + 0.67r_{D,curr}(t) + 0.8249v(t-1),$
5. $\theta_3(t) \approx 3.2387 - 11.9275r_D(t) + 0.1845\gamma_q(t) - 2.9587\gamma_y(t) - 11.9042j_q(t) + 9.4019j_y(t) + 0.9798\theta_3(t-1),$

$$6. C_3(t) \approx -0.3108 - 0.0492\gamma_q(t) + 0.1119\gamma_y(t) + 0.2841j_m(t) + 0.3977j_q(t) - 0.3448j_y(t) + 0.9374C_3(t-1).$$

Group of poor regions:

$$1. \delta_1(t) \approx 0.0282 + 0.7956r_L(t) + 0.0374j_q(t) - 0.0658j_y(t) + 0.016\gamma_y(t) + 0.2253\delta_1(t-1),$$

$$2. \delta_2(t) \approx 0.0524 + 0.7324r_L(t) + 0.0113j_q(t) - 0.0656j_y(t) + 0.1128\gamma_q(t) + 0.2936\delta_2(t-1),$$

$$3. \delta_4(t) \approx -0.0392 + 0.7181r_D(t) - 0.0993r_{D,curr}(t) + 0.0405j_q(t) + 0.2433\delta_4(t-1),$$

$$4. v(t) \approx -0.1848 + 0.3674r_D(t) - 0.1095\gamma_q(t) + 0.0168\gamma_y(t) + 0.1787j_q(t) + 0.0235j_y(t) + 0.723v(t-1),$$

$$5. \theta_3(t) \approx 1.9163 - 3.0933r_D(t) - 0.4745\gamma_q(t) - 1.2085j_q(t) - 0.3759j_y(t) + 0.9348\theta_3(t-1),$$

$$6. C_3(t) \approx -0.7029 + 0.4254\gamma_q(t) - 0.0048\gamma_y(t) + 0.2154j_m(t) + 0.9445j_q(t) - 0.4237j_y(t) + 0.9844C_3(t-1),$$

$$7. C_{l,add}(t) \approx -9.921 + 11.7407j_m(t) - 1.8498j_q(t) + 0.0177j_y(t) - 4.4094\gamma_m(t) + 1.07\gamma_q(t) - 0.0264\gamma_y(t) + 0.7834C_{l,add}(t-1). \text{ This regression implies that some of the expenses of population segment 1 are taken by segment 4.}$$

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REFERENCES

1. N. A. Krupenskii and O. E. Kuzina, "Over-indebtedness of Russians: Myth or reality," *Vopr. Ekon.* **11**, 85–104 (2018).
2. *World Bank. Household Over-Indebtedness in Russia* (World Bank, Washington, DC, 2020).
3. N. V. Zubarevich and S. G. Safronov, "People and money: Incomes, consumption, and financial behavior of the population of Russian regions in 2000–2017," *Reg. Res. Russ.* **9**, 359–369 (2019).
4. E. Bazanova, "Ministry of Economic Development of the Russian Federation is ready to support highly indebted borrowers," *Vedomosti* (2019). www.vedomosti.ru/economics/articles/2019/07/2.
5. F. P. Ramsey, "A mathematical theory of savings," *Econ. J.* **152** (38), 543–559 (1928).
6. A. V. Rudeva and A. A. Shananin, "Control synthesis in a modified Ramsey model with a liquidity constraint," *Differ. Equations* **45** (12), 1835–1839 (2009).
7. I. F. Gimaltdinov, "Study of demand for consumer credit and money in cash," *Mat. Model.* **24** (2), 84–98 (2012).
8. J. Komlós, "A generalization of a problem of Steinhaus," *Acta Math. Acad. Sci. Hung.* **18**, 217–229 (1967).
9. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955).
10. F. Clarke, *Optimization and Nonsmooth Analysis* (Wiley, New York, 1983).
11. *Russia Longitudinal Monitoring Survey, RLMS-HSE*, conducted by National Research University "Higher School of Economics" and OOO "Demoscope" together with Carolina Population Center, University of North Carolina at Chapel Hill and the Institute of Sociology of the Federal Center of Theoretical and Applied Sociology of the Russian Academy of Sciences.

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